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Nilsson, Christer

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ON NONLOCAL PLASTICITY, STRAIN SOFTENING AND LOCALIZATION

CHRISTER NILSSON

To
the memory of
my father

Preface

About eight or nine years ago, Professor Hans Petersson, head of the Division of Structural Mechanics at Lund University, called my attention to the mechanics of distributed cracking, damage and strain localization in solids. Upon learning about my interest in theories of generalized continua he suggested, somewhat vaguely, 'nonlocal theory' to be applied to problems of strain localization in fracture mechanics, at the time a subject almost unknown to me.

During the last four or five years I have been involved in research projects on constitutive modelling of strain softening and localization in heterogeneous materials, supported by the Swedish National Board for Technical Development and later by the Swedish Research Council for Engineering Sciences. Their assistance is gratefully acknowledged.

The thesis is based on preliminary studies commenced upon my arrival at the Division of Structural Mechanics. The manuscript of the first three chapters was completed in early 1992; while it has been thorougly revised since then, the basic ideas and the principal contents are unchanged.

I wish to thank those at the Division of Structural Mechanics who have helped me to understand what strain softening really is about, in particular Professor Petersson and Dr. Per Johan Gustafsson. Thanks also to Mr. Per-Erik Austrell and Dr. Göran Sandberg for their assistance with the computational work with regard to the analytical solutions presented in Section 4.7.1.

I am indebted to Professor Kenneth Runesson for many valuable remarks on various topics related to this work, and to Miss Lena Strömberg for implementing some of the proposed numerical algorithms in Chapter 4 into computer codes.

Thanks are also due to Mr. Peter Nilsson for typing the manuscript, to Mr. Bo Zadig for his assistance with the preparing of the final manuscript prior to printing, and to Associate Professor Richard Fisher for assistance with my written English.

Finally I express my gratitude to the members of my family for their patience during the last five or six weeks. In particular I am grateful to Nickan, who welcomed me home and joined me for supper at three or four o'clock in the morning during these last weeks of intensive work.

Lund, November 1, 1994

Christer Nilsson

Abstract

Plasticity theory represents a fundamental continuum approach to the study of a great variety of phenomena in the mechanics of inelastic solids. Basic to the theory is the appearance of permanent deformation and its association with the phenomenological concept of plastic deformation. The origin of plastic deformation in a solid may be looked upon as the result of a complex interference of its microstructure with its macrostructure. In a crystalline material it is the defects in its structure which cause the permanent deformation, and especially, as far as metals are concerned, it is dislocations which act as carriers of plastic deformation. Thus, in a general phenomenological approach there must be an interplay between microstructural and macrostructural scales. This leads us to the conclusion that plastic deformation is nonlocal in character and hence a general theory of plasticity should be nonlocal.

The present work on nonlocal plasticity is based on a strain space formulation where plastic strain is regarded as a primitive variable, characterized by an appropriate constitutive equation for its rate. Nonlocal constitutive variables are constructed from a set of basic state functions, constituted by total (kinematical) strain, plastic strain and a scalar measure of strain hardening. A rate-independent theory is formulated where stress is assumed to be a function of the nonlocal variables.

A yield function in strain space is introduced, where the same set of independent variables occurs as in the case of the stress response function. This is fundamental for the theory. We recall that in classical plasticity the yield condition implies that whether a state is elastic or plastic depends only on the plastic strain at the actual stress point and not on plastic strain at neighbouring points, hence excluding any dependence on gradients of plastic strain. From a physical view, however, it is hard to find any support for rejecting long-range interactions in the yield criterion, keeping in mind the complex interplay between microstructure and macrostructure with regard to plastic deformation.

Yield criteria, flow rules, and loading conditions are formulated. The loading conditions in strain space give rise to associated conditions in stress space of quite different form (constrasting with local theory). This difference between stress space and strain space is seen to favour the choice of strains over stresses as primary variables in nonlocal plasticity.

Numerical techniques are developed for integrating the rate equations, subject to the constraints implied by the consistency condition, in nonlocal plasticity, being represented by an integral equation defined throughout the region of plastic loading.

A strain softening problem is investigated by finite element analysis. Solutions are obtained which converge properly and also show computational objectivity.

Keywords: nonlocal plasticity, plastic dissipation, strain softening, localization, integration of elastic-plastic equations, finite element analysis

Contents

Chapter 1

Introduction

This work[1.0−1] deals with nonlocal plasticity, the ultimate purpose being to investigate strain softening and localization in solids. It is well known that conventional continuum models have essential deficiencies if used to describe strain softening behaviour. However, theories of nonlocal continua are seen to embrace excellent possibilities to overcome these drawbacks, as will be discussed in Section 1.2 below.

Preliminarily we draw attention to the fact that the concept of strain softening may afford a macroscopic approach to distributed cracking in a wide variety of solid materials, where in the special case of uniaxial stress, strain softening is simply understood as the decline of stress at increasing strain. Further aspects of strain softening phenomena are discussed in Section 1.1.

Nonlocal theories are briefly discussed in Section 1.3, and some general features of such theories are outlined in Appendix A. For the moment we merely notice that in nonlocal continuum theories the principle of local action is not valid, i.e. the stress is not only affected by an infinitesimal neighbourhood about the actual stress point but by the entire body under consideration.

In Section 1.4 classical plasticity theory is shortly reviewed, with an emphasis on features which become significant when the theory is generalized to include nonlocal interactions. The result of such generalizations is presented in Chapter 2, which deals with general nonlocal thermodynamic plasticity, and in Chapter 3, where the focus is on a purely mechanical theory.

In Chapter 4 the main interest is devoted to numerical techniques in nonlocal plasticity. A brief survey is given in Section 1.5 below.

[1.0−1]The content of the first three chapters is based on Nilsson (1992).

1.1 Strain softening and localization phenomena

In experiments, a variety of engineering materials shows the characteristic feature of increasing loss of stiffness and strength with increasing loading. This is the case for certain classes of granular materials including concrete and rocks, and probably also for ceramics, powder metals and many other materials. The micromechanical mechanism of stiffness degradation is not fully understood, being the result of complicated microstructural processes involving nucleation and growth of microcracks and microvoids. To indicate precisely one or another phenomenological effect on the macroscale to a certain microstructural source is of course not possible in general. It is common, however, to consider stiffness degradation as damage and yield degradation (strain softening) as an effect of inelastic deformation. This deteroriation of the material will eventually lead to failure, which is then characterized as an accumulation of microcracks (localization) into a fracture region rather then the result of the propagation of one single crack. Since the crack distribution involves a large system of microcracks randomly located and oriented, it seems natural to assume that the material can be treated as a continuum, which leads to the concept of strain softening. In the special case of uniaxial stress, as already mentioned, strain softening is recognized as the decline of stress at increasing strain. Several difficulties arise, however, when conventional continuum mechanics is used - from a computational as well as a phenomenological point of view.

Here we have thought of localization as accumulation of cracking, but more generally we understand that localization in solid materials is progressive deformation into a narrow zone. Even if localization is a striking feature of strain softening, it should be remembered that localization phenomena also occur in a hardening elastic-plastic solid, exposing two different aspects of the relationship between localization and material instability. In fact, questions about instabilities in the softening region have cast doubt on the substance of the concept of strain softening. The question whether softening behaviour represents a material property or a structural property has been intensely discussed by researchers in the field for some years, but no general agreement has been achieved. The reader is referred to discussions by Sandler and Wright (1984), Read and Hegemier (1984), Nemat-Nasser (1985) and Chen (1988). Some objectives in this context have also been addressed in relation to the issue of ill-posedness of initial-boundary value problems. The point that elastic waves cannot propagate in the softening region of a rate-independent material is not completely relevant, however. It is true that the wave velocity becomes imaginary and the differential equation of motion changes type if the tangent moduli matrix ceases to be positive definite, but it should be remembered that the unloading modulus remains positive also in the softening range, so during unloading from an elastic-plastic state it may be possible for elastic waves to propagate even in the softening region. It is important to note that the discussion here concerns the phenomonological basis for strain softening, whereas from a mathematical point of view a precise definition of the softening concept can be uniquely given within the general theory of plasticity. It should also be pointed out that even if it can be claimed that strain softening does not exist at a micromechanical level, it is clear that the softening concept is useful at a macromechanical level, as confirmed from an extensive number of theoretical as well as numerical investigations.

1.2 Strain softening models and finite element solutions

Localization in a strain softening rate-independent material is associated with material instability, as already mentioned, and loss of ellipticity in quasi-static problems. From a computational point of view this appears to be a crucial problem due to numerical instability and mesh sensitivity. When conventional continuum models are applied finite element solutions show the feature of non-objectivity with respect to the mesh when standard finite elements are used. As a consequence, the strain softening zone localizes into a region of vanishing volume under zero energy dissipation, when the elements tend to be infinitely small. Objectivity can be achieved by different approaches. One technique commonly used, probably first by Pietruszczak and Mroz (1981), modifies the constitutive relation and makes it depend on the mesh size. Another approach takes advantage of the concept of localization limiters, which force the localized zone to have a certain minimum finite size (see Belytschko and Lasry 1989 for a recent survey). These different methods can be combined, a possibility frequently used. The fictitious crack model by Hillerborg et al. (1976) and the crack band model by Bazant and Oh (1983) represent simple examples of the use of localization limiters. A general form of localization limiters may be utilized within the theory of a nonlocal continuum. (Nonlocal theories are briefly discussed in Section 1.3 below.) In fact a nonlocal approach is well adapted to interpret strain softening as smeared distributed cracking. It provides in a natural way for the introduction of a characteristic length as a material parameter, and by the definition of a representative volume of a heterogeneous medium it appears that the nonlocal concept represents a generalized form of a localization limiter. It is important to note that the problem of achieving computational objectivity seems to be inherently solved by the nonlocal approach. The work on strain softening phenomena by Bazant and Lin (1988) within the theory of nonlocal plasticity seems to give accurate results, but takes advantage of assumptions which are not consistent with a general theory.

Rate-dependent models have been proposed to eliminate the problems due to change of type of differential equations[1.2-1], e.g. by Sandler and Wright (1984) and Needleman (1988). The inclusion of rate effects in the constitutive model causes stability in the sense that no change of type of differential equation occurs. However, such models

[1.2-1] See Section 1.1.

probably cannot be applied efficiently for static (or nearly static) loading conditions. Strain softening phenomena can be described within continuum models for materials with gradient effects[1.2-2] and for oriented media (micropolar theory). Various types of discrete micromechanical models have also been proposed for the analysis of fracture and localization in strain softening materials. The idea of using computer generated discrete models to investigate the microstructural processes associated with localization and fracture is important. Such models are usually based on concepts of probabilistic networks (see e.g. the pioneering work of Burt and Dougill 1971), but fractal concepts have also been utilized (Hermann and Roux 1990).

1.3 Nonlocal theories

It is evident that physical models or theories have a certain domain of applicability outside of which they fail to predict relevant physical phenomena with reasonable accuracy. This domain of applicability of the theory is a function of internal scales of the medium to which it applies. When these scales are sufficiently small compared to corresponding external scales, classical continuum theories give acceptable results. On the other hand, the existence of characteristic length scales of the medium, common in many areas of mechanics, disproves local theories.

Classical continuum mechanics is based on the principle of local action, and on the assumption that the equations of balance are valid for every part of the body, however small it may be. The principle of local action is not valid in nonlocal theories, that is the stress at a point $\mathbf{X}$ is not only affected by the infinitesimal neighbourhood about the same point $\mathbf{X}$, but by the entire body under consideration. Thus long-range interactions between a particle at $\mathbf{X}$ and a particle at $\mathbf{Z}$ may contribute to the stress at $\mathbf{X}$. Continuum theories including long-range interactions have been of considerable interest for over twenty years (see e.g. Kunin 1982, 1983), Gurtin and Williams 1971 and Edelen 1976). A theory based on Edelen and Laws (1971) and Edelen (1976) is outlined in Appendix A. There are different approaches to the problem of describing nonlocal interactions and they do not yield identical theories. It is common, however, in nonlocal continuum theories that global balance equations are postulated for the entire body and not for an arbitrary part of it. Local equations can still be obtained, but they then contain nonlocal residuals, which take the long-range interactions into account.

[1.2-2] Some authors characterize materials with gradient effects as nonlocal. This terminology is not adopted here; cf. Appendix A, Section 1.

1.4 Plasticity theories

Plasticity theory represents a fundamental continuum approach to the study of a great variety of phenomena in the mechanics of inelastic solids. Basic to the theory is the appearance of permanent deformation and its association with the phenomenological concept of plastic deformation. Most researchers in the field introduce some measure of plastic strain as an internal variable, but unfortunately there is no general agreement as to how this measure should be defined and be introduced into a theory of finite plasticity. No objections, however, can be raised against the idea of the origin of plastic deformation in a solid as the result of a complex interference of its microstructure with its macrostructure. In a crystalline material it is the defects in its structure which cause the permanent deformation, and especially, as far as metals are concerned, it is dislocations which act as carriers of plastic deformation. Thus in a general phenomenological approach there must be an interplay between microstructural and macrostructural scales. This leads us to the conclusion that plastic deformation is nonlocal in character and hence a general theory of plasticity should be nonlocal. In this context it is worth mentioning that the principle of local action is a restriction of a much more general concept (the principle of determinism)[1.4-1], and consequently theories of continuous media are basically nonlocal. The validity of corresponding local theories is then a question of applicability as discussed in Section 1.3. We also draw attention to the fact that extensive subregions of an elastic-plastic body during loading may often respond elastically, while substantial plastic deformation occurs in the neighbourhood or far away - a behaviour which in general also calls for a nonlocal theory.

Usually strain softening models intended for three-dimensional problems during general loading conditions are extended from physical understanding and mathematical representation in one dimension. Such extensions cannot generally be effected unambiguously and are far from obvious. Consequently there is a need for a general framework for the treatment of strain softening phenomena. Now the concept of softening is well established in plasticity theory - at least mathematically. Hence it is logical to try the possibility of treating strain softening in heterogeneous materials within a general theory of plasticity. Taking into consideration the correspondence between nonlocality and localization limiters, it surely can be argued that a theory of nonlocal softening plasticity will be capable of describing the essential features of strain softening, including that of localization.

In the present work we intend to construct a rather general nonlocal theory for finitely deformable elastic-plastic materials, in which viscous effects can be neglected. In Chapter 2 a thermomechanical theory is briefly outlined, while in Chapter 3 a purely

[1.4-1] The principle of determinism merely states that past and present configurations given by the motion of the material points in a body determine the stress field in its present configuration.

mechanical theory is derived. The nonlocal mechanical theory is based on works on local finite plasticity by Naghdi and co-workers (see e.g. Casey and Naghdi 1984). Yield criteria are introduced with reference to strain space as well as stress space, the yield function in strain space taken as primary. Stress space and strain space formulations are different in concept, and with regard to nonlocal theory a strain space formulation turns out to be the natural choice. It is also noted that for computational purposes it is convenient to use strains and not stresses as primary variables.

In this context we will draw attention to three rather recent papers in which plasticity is critically reviewed, namely those of Drucker (1988), Cleja Tigoiu (1990) and Naghdi (1990), representing different schools of plasticity. Additional references on plasticity will be given in Chapters 2 and 3.

1.5 Numerical techniques in nonlocal plasticity

In engineering mechanics, numerical implementation of the elastic-plastic equations is usually accomplished within the realm of the finite element method. The literature attributed to this issue is extensive - as far as local plasticity is concerned. In the case of nonlocal plasticity not much has been done, and no general concepts are known to exist.

In Chapter 4 various numerical techniques commonly used in local plasticity are extended to comply with the nonlocal concept. Thus, based on a weak form of the equilibrium equation, a finite element formulation is derived, capable of treating nonlocal constitutive assumptions. Also addressed is the intricate problem of integrating the rate equations subject to the constraints implied by the consistency condition - in nonlocal plasticity expressed by an integral equation defined throughout the region of loading points. It turns out that the rate equations in general cannot be integrated pointwise as in local theory - in fact the integration procedure has to be performed simultaneously for the whole set of loading points.

The potential of the nonlocal formulation is numerically demonstrated by the analysis of a strain softening bar. The material is isotropic, stress is assumed to depend linearly on strain and to be independent of the strain hardening variable, and the yield function is of von Mises type with nonlocal hardening/softening. Finite element solutions converge properly and are computationally objective, and the size of the localized zone depends, as expected, essentially on the characteristic length of the material.

Chapter 2

On general nonlocal plasticity

2.1 Introduction

We recall that the main purpose of this work is to derive a purely mechanical nonlocal theory of a finitely deforming elastic-plastic body, and to deduce that such a theory may serve as a natural basis for the description of strain softening materials. However, in order to understand the origin of the basic differences between local and nonlocal theory, it is instructive to start with a general thermodynamic theory. The purely mechanical theory is then dealt with in the next chapter.

In Chapter 1 we have in a general sense discussed when (and why) it may be preferred to use nonlocal continuum theory instead of discrete or classical field theories. More specifically, plastic deformation caused by nonlocal interactions has been somewhat investigated in relation to the discussion of the concept of strain softening. These arguments will not be repeated here, but again it is emphasized that plastic deformations indeed generally are of nonlocal character, due to the micromechanical structure of yielding materials. As may be understood from the brief discussion in Section 1.4 there is no unequivocal definition of the concept of plastic strain. This is, at least partly, due to the fact that plastic strain cannot be defined on a purely kinematical basis - not even in the infinitesimal theory of plasticity - but must be characterized through some constitutive framework. Many theories of finite plasticity use an intermediate stress-free (relaxed) configuration in order to introduce elastic and plastic deformation. The deformation gradient $\mathbf{F}$ is then decomposed as a product of an elastic part $\mathbf{F}^e$ and a plastic part $\mathbf{F}^p$ in the form $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$. Total strain $\mathbf{E}$ (Lagrangian strain) is defined in terms of $\mathbf{F}$, while plastic strain $\mathbf{E}^p$ and elastic strain $\mathbf{E}^e$ are defined in terms of $\mathbf{F}^e$ and $\mathbf{F}^p$. Whereas $\mathbf{E}$ is unambiguously defined, there is no general agreement as to exactly how the measures of plastic and elastic strain should be defined, nor with regard to how total strain should be decomposed into elastic and plastic parts. It should be noted that while in the theory of infinitesimal plasticity the decomposition $\mathbf{E} = \mathbf{E}^e + \mathbf{E}^p$ is valid, this is not necessarily true in the case of finite plasticity. It is to be emphasized that generally neither $\mathbf{F}^e$ nor $\mathbf{F}^p$ is the gradient of a deformation field, nor do they necessarily satisfy any compatibility conditions.

An extensive literature exists concerning issues with relation to plasticity models with intermediate relaxed configurations. In addition to the basic problems of existence and uniqueness of the multiplicative decomposition, it can be noticed that the issue of invariance properties of $\mathbf{F}^e$ and $\mathbf{F}^p$ under a change of frame has not been entirely resolved. For a discussion we refer to the reviews of Naghdi (1990) and Cleja-Tigoiu (1990) mentioned in Section 1.4. References to original papers, e.g. those of Kröner (1960), Lee and Liu (1967) and Mandel (1973) are also found in these review articles.

Because of the difficulties attached to theories using relaxed configurations, many authors of papers on plasticity avoid introducing $\mathbf{F}^p$, preferring to introduce a plastic strain variable $\mathbf{E}^p$ as a primitive quantity with certain prescribed characteristics (e.g. being a symmetric second order tensor with the same invariance properties as $\mathbf{E}$). This will certainly circumvent the problems related to the relaxed configuration concept, but on the other hand it leaves the identification of $\mathbf{E}^p$ with some ambiguity. It should also be mentioned that the issue concerning the relationship between these two different approaches has been intensely debated in the literature for many years, but has not yet been satisfactorily resolved.

In this attempt to formulate a nonlocal plasticity theory we will regard plastic strain as a primitive variable, characterized by an appropriate constitutive equation for its rate. This was the approach in the papers of Green and Naghdi (1965, 1966). Also thermodynamic arguments used in this chapter are similar to those in these papers of Green and Naghdi, while the nonlocal formulation is based essentially on the works of Edelen and Laws (1971) and Edelen, Green and Laws (1971). Whereas Green and Naghdi used a stress space formulation, the treatment here is based on a strain space formulation. In Section 3.1 we discuss some aspects of the main differences between formulations in stress space and strain space.

In the case of nonlocal theory, it can additionally be noted that in a strain space formulation it is not in general necessary to require that stress-strain relations be invertible, a fact which is important in the case of unrestricted nonlocality as will be discussed below. In this context it can be mentioned that theories of nonlocal plasticity are developed in Eringen (1981) by use of a strain space formulation, and in Eringen (1983) by use of a stress space formulation.

It is common in theories of elastic-plastic materials to represent strain hardening by one single scalar function. The list of inelastic variables may however be extended to include an arbitrary number of scalar and tensor functions. However, we will not add to the theory a complexity which would obscure the basic features of nonlocality, without exposing anything fundamentally new in the theory of plasticity. Hence in the development which follows, strain hardening is basically represented by a scalar function.

In Section 2.2 below, we will introduce strain measures, consisting of the Lagrangian strain tensor and a plastic strain tensor together with a scalar strain hardening variable. This scalar and these tensors are basic state functions of the theory, from which will be constructed nonlocal quantitites to be used as independent variables in the constitutive theory. Balance laws and the nonlocal Clausius-Duhem inequality are presented in Section 2.3, while Section 2.4 deals with the formulation of the constitutive equations.

Finally, as our objective is to make the treatment reasonably self-contained, some arguments will be repeated in what follows, albeit they can be found elsewhere in the literature.

2.2 Kinematics

The motion of a material point of a body B is referred to a fixed reference configuration of the body. The position of the material point in the present configuration at time t is designated by $\mathbf{x} = \chi(\mathbf{X}, t)$, where $\mathbf{X}$ is the position of the same material point in the fixed reference configuration[2.2−1].

As strain measure we adopt the symmetric Lagrangian strain tensor, defined by

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1}), \qquad (2.2-1)$$

where $\mathbf{F} = \mathit{Grad}\ \mathbf{x} = \partial\chi/\partial\mathbf{X}$ is the deformation gradient tensor and $\mathbf{1}$ denotes the second order identity tensor.

With reference to the discussion in Section 2.1 we assume the existence of a plastic strain tensor, which is a symmetric[2.2−2] second order tensor-valued function $\mathbf{E}^p = \mathbf{E}^p(\mathbf{X}, t)$, and a measure of strain hardening, which is a scalar-valued function $\kappa = \kappa(\mathbf{X}, t)$. As for plastic strain, it is further assumed that $\mathbf{E}^p$ has the same invariance properties as $\mathbf{E}$, being unaltered under a change of frame. Again it is emphasized that $\mathbf{E}^p$ is not derivable from the displacement field. As in local theory $\dot{\mathbf{E}}^p$ as well as $\dot{\kappa}$ will be described by constitutive equations.

The basic *functions* which are assumed to constitute the *thermodynamic state* are represented by $\mathcal{U}(\mathbf{X}, t)$, where $\mathcal{U}$ is a collection of functions defined by

[2.2−1] Bold lower and upper case letters are usually used to denote vectors and second order tensors, respectively, but some exceptions will occur. Standard vector and tensor notations are used as far as possible. In most cases equations are written in coordinate-free forms, but it is understood that all entities are defined with reference to a fixed system of rectangular Cartesian axes. Superposed dots denote material time derivatives.

[2.2−2] It is to be noted that the symmetry of $\mathbf{E}^p$ is an assumed property, which cannot be proved in a general theory of plasticity.

$$\mathcal{U}(\mathbf{X},t) = \{\mathbf{E}(\mathbf{X},t),\ \mathbf{E}^p(\mathbf{X},t),\ \kappa(\mathbf{X},t), \theta(\mathbf{X},t)\}, \tag{2.2-2}$$

θ being the absolute temperature.

This assumption manifests the nonlocal character of the theory in that we require, in contrast to local theory, the thermodynamic state functions — not only their values at $\mathbf{X}$— to specify the dependent variables. If we take the Helmholtz free energy ψ as the basic thermodynamic quantity, this means that the value of ψ at $\mathbf{X}$ is determined by the values of the thermodynamic state functions all over the body. In order to provide for such dependence we construct quantities

$$\left.\begin{aligned}
<\mathbf{E}>(\mathbf{X},t) = &\ \textstyle\int_B \boldsymbol{\alpha}^e\{\mathbf{X},\mathbf{Z},\mathcal{U}(\mathbf{X},t),\mathcal{U}(\mathbf{Z},t)\}dV(\mathbf{Z}),\\
<\mathbf{E}^p>(\mathbf{X},t) = &\ \textstyle\int_B \boldsymbol{\alpha}^p\{\mathbf{X},\mathbf{Z},\mathcal{U}(\mathbf{X},t),\mathcal{U}(\mathbf{Z},t)\}dV(\mathbf{Z}),\\
<\kappa>(\mathbf{X},t) = &\ \textstyle\int_B \alpha^h\{\mathbf{X},\mathbf{Z},\mathcal{U}(\mathbf{X},t),\mathcal{U}(\mathbf{Z},t)\}dV(\mathbf{Z}),\\
<\theta>(\mathbf{X},t) = &\ \textstyle\int_B \alpha^\theta\{\mathbf{X},\mathbf{Z},\mathcal{U}(\mathbf{X},t),\mathcal{U}(\mathbf{Z},t)\}dV(\mathbf{Z}),
\end{aligned}\right\} \tag{2.2-3}$$

or

$$<\mathcal{U}>(\mathbf{X},t) = \int_B \alpha[\mathbf{X},\mathbf{Z},\mathcal{U}(\mathbf{X},t),\mathcal{U}(\mathbf{Z},t)]dV(\mathbf{Z}), \tag{2.2-4}$$

where

$$\alpha = \{\boldsymbol{\alpha}^e,\ \boldsymbol{\alpha}^p,\ \alpha^h,\ \alpha^\theta\} \tag{2.2-5}$$

is a collection of prescribed functions of the arguments indicated. It is assumed that α preserves symmetry, i.e. $<\mathbf{E}>$ and $<\mathbf{E}^p>$ are symmetric second order tensors. For convenience the explicit dependency on $\mathbf{X}$ and $\mathbf{Z}$ is henceforth suppressed.

Remark 2.1. We use $\mathbf{Z}$ to indicate functional relationship in the sense that $\mathbf{Z}$ represents all points of the body, while $\mathbf{X}$ represents an arbitrarily distinguished point. Henceforth, when confusion does not arise, we will omit the dependence on $\mathbf{X}$ and t in the arguments.□

Remark 2.2. Note that $<\mathcal{U}>$ is a function symbol; the pointed brackets should not be confused with the concept of averaging a function. □

Remark 2.3. If we take $\alpha = \mathcal{U}(\mathbf{X},t)/V(B)$, where $V(B)$ is the volume of the body in its referential configuration, we recover the original state functions, i.e.

$$<\mathcal{U}>=\mathcal{U},\ \alpha = \mathcal{U}(\mathbf{X},t)/V(B). \tag{2.2-6}$$

Before specifying constitutive assumptions we will briefly discuss the equations of balance in the next section.□

2.3 Equations of balance

For a locally mass closed body (cf. Appendix A) with negligible long range gravitational effects, we assume that the localization residuals for linear as well as rotational momentum vanish. Then with $\hat{\rho} = 0$, $\hat{f} = 0$ and $\hat{\mathbf{M}} = \mathbf{0}$ we find from (A-12), (A-18) and (A-26) that the equation of conservation of mass and the equations of balance of linear and rotational momentum read

$$\dot{\rho} + \rho \div \dot{\mathbf{x}} = 0, \qquad (2.3-1)$$

$$\div \ \mathbf{T} + \rho(\mathbf{f} - \ddot{\mathbf{x}}) = \mathbf{0}, \qquad (2.3-2)$$

$$\mathbf{T} = \mathbf{T}^{\mathrm{T}}, \qquad (2.3-3)$$

where it is recognized that ρ is the mass density, $\mathbf{T}$ the Cauchy stress and $\mathbf{f}$ the specific body force.

The reduced global nonlocal Clausius-Duhem inequality (A-42) takes the form

$$\int_{B(t)} \{-\rho(\dot{\psi} + \dot{\theta}\eta) + \mathbf{T} \cdot \mathrm{grad}\ \dot{\mathbf{x}} + \frac{1}{\theta}\mathbf{q} \cdot \mathrm{grad}\ \theta\} dv \geq 0, \qquad (2.3-4)$$

where ψ is the Helmholtz free energy, η specific entropy, θ absolute temperature, $\mathbf{q}$ the external heat flux vector, and where $B(t)$ denotes the image of the body B under the motion $\chi(\mathbf{X}, t)$ defined by (A-1), (also cf. the beginning of Section 2.2).

2.4 Constitutive assumptions

To provide for a material description we introduce the symmetric second Piola- Kirchhoff stress tensor $\mathbf{S}$ defined by

$$\mathbf{T} = (\det \mathbf{F})^{-1}\ \mathbf{F}\ \mathbf{S}\ \mathbf{F}^{\mathrm{T}}, \qquad (2.4-1)$$

and the material external heat flux vector $\mathbf{Q}$ defined by

$$\mathbf{q} = (\det \mathbf{F})^{-1}\ \mathbf{F}\ \mathbf{Q}, \qquad (2.4-2)$$

where

$$\mathbf{F} = \text{Grad } \boldsymbol{\chi}(\mathbf{X}, t) \tag{2.4 – 3}$$

is the deformation gradient.

Using (2.3-4), (2.4-1), (2.4-2), (2.2-1) and $(A\text{-}8)_2$ we conclude that the reduced global nonlocal Clausius-Duhem inequality in material form reads

$$\int_B \{-\rho_0(\dot{\psi} + \dot{\theta}\eta) + \mathbf{S} \cdot \dot{\mathbf{E}} + \frac{1}{\theta}\mathbf{Q} \cdot \text{Grad } \theta\} dV \geq 0, \tag{2.4 – 4}$$

where now $\mathbf{X}$ replaces $\mathbf{x}$ as independent variable, Grad is the gradient with respect to $\mathbf{X}$ and ρ_0 is the mass density in the referential configuration.

As constitutive variables we choose $< \mathcal{U} >$ defined by (2.2-4), and assume that

$$\left.\begin{aligned} \mathbf{S} &= \tilde{\mathbf{S}}(< \mathcal{U} >), \\ \psi &= \tilde{\psi}(< \mathcal{U} >), \\ \eta &= \tilde{\eta}(< \mathcal{U} >), \end{aligned}\right\} \tag{2.4 – 5}$$

where the explicit dependency on $\mathbf{X}$ is suppressed. The inequality (2.4-4) and the constitutive equations (2.4-5) represent a nonlocal thermodynamic system. To evaluate the restrictions on the functions in (2.4-5) imposed by the inequality (2.4-4) we need the material time derivative of the Helmholtz free energy. Differentiating $(2.4\text{-}5)_2$, we obtain

$$\rho_0 \dot{\psi} = \rho_0 \frac{\partial \tilde{\psi}}{\partial < \mathcal{U} >} < \dot{\mathcal{U}} >$$

$$= \rho_0 \frac{\partial \tilde{\psi}}{\partial < \mathcal{U} >} \int_B \{\frac{\partial \alpha}{\partial \mathcal{U}(\mathbf{X}, t)} \dot{\mathcal{U}}(\mathbf{X}, t) + \frac{\partial \alpha}{\partial \mathcal{U}(\mathbf{Z}, t)} \dot{\mathcal{U}}(\mathbf{Z}, t)\} dV(\mathbf{Z}), \tag{2.4 – 6}$$

where (2.2-4) has been used. Here ρ_0 means $\rho_0(\mathbf{X})$ and $\partial\tilde{\psi}/\partial < \mathcal{U} >$ is some function of $< \mathcal{U} > (\mathbf{X}, t)$, say

$$\frac{\partial \tilde{\psi}}{\partial < \mathcal{U} >} = F[< \mathcal{U} > (\mathbf{X}, t)]. \tag{2.4 – 7}$$

Now define $\partial\tilde{\psi}(\mathbf{Z})/\partial < \mathcal{U} >$ by

$$\frac{\partial \tilde{\psi}}{\partial < \mathcal{U} >}(\mathbf{Z}) = \mathbf{F}[< \mathcal{U} > (\mathbf{Z}, t)] \tag{2.4 – 8}$$

and α^* by the relations

$$\left.\begin{aligned} \alpha = &\quad \alpha\{\mathcal{U}(\mathbf{X},t),\ \mathcal{U}(\mathbf{Z},t)\}, \\ \alpha^* = &\quad \alpha\{\mathcal{U}(\mathbf{Z},t),\ \mathcal{U}(\mathbf{X},t)\}. \end{aligned}\right\} \tag{2.4-9}$$

Thus we get α^* by interchanging $\mathbf{X}$ and $\mathbf{Z}$ in the arguments of α.

With these definitions (2.4-6) can be rearranged and expressed in the form

$$\rho_0\dot{\psi} = \int_B \{\rho_0 \frac{\partial\tilde{\psi}}{\partial < \mathcal{U} >} \frac{\partial\alpha}{\partial\mathcal{U}(\mathbf{X},t)} + \rho_0(\mathbf{Z})\frac{\partial\tilde{\psi}}{\partial < \mathcal{U} >}(\mathbf{Z})\frac{\partial\alpha^*}{\partial\mathcal{U}(\mathbf{X},t)}\}\dot{\mathcal{U}}(\mathbf{X},t)dV(\mathbf{Z})$$

$$+ \int_B \{\rho_0 \frac{\partial\tilde{\psi}}{\partial < \mathcal{U} >} \frac{\partial\alpha}{\partial\mathcal{U}(\mathbf{Z},t)}\ \dot{\mathcal{U}}(\mathbf{Z},t) - \rho_0(\mathbf{Z})\frac{\partial\tilde{\psi}}{\partial < \mathcal{U} >}(\mathbf{Z})\frac{\partial\alpha^*}{\partial\mathcal{U}(\mathbf{X},t)}\}\dot{\mathcal{U}}(\mathbf{X},t)dV(\mathbf{Z})$$

$$= \int_B \{\rho_0 \frac{\partial\tilde{\psi}}{\partial < \mathcal{U} >} \frac{\partial\alpha}{\partial\mathcal{U}(\mathbf{X},t)}\}$$

$$+ \rho_0(\mathbf{Z})\frac{\partial\tilde{\psi}}{\partial < \mathcal{U} >}(\mathbf{Z})\frac{\partial\alpha^*}{\partial\mathcal{U}(\mathbf{X},t)}\}\ \dot{\mathcal{U}}(\mathbf{X},t)dV(\mathbf{Z}) + \mathcal{H}_\psi \tag{2.4-10}$$

where

$$\mathcal{H}_\psi = \int_B \{\rho_0 \frac{\partial\tilde{\psi}}{\partial < \mathcal{U} >} \frac{\partial\alpha}{\partial\mathcal{U}(\mathbf{Z},t)}\dot{\mathcal{U}}(\mathbf{Z},t)$$

$$- \rho_0(\mathbf{Z})\frac{\partial\tilde{\psi}}{\partial < \mathcal{U} >}(\mathbf{Z})\frac{\partial\alpha^*}{\partial\mathcal{U}(\mathbf{X},t)}\}\dot{\mathcal{U}}(\mathbf{X},t)\ dV(\mathbf{Z}) \tag{2.4-11}$$

By interchanging $\mathbf{X}$ and $\mathbf{Z}$ and reversing the order of integration, we note that the functional $\mathcal{H}_\psi$ identically satisfies the equation

$$\int_B \mathcal{H}_\psi\ dV(\mathbf{X}) = 0 \tag{2.4-12}$$

for arbitrary differentiable functions $\tilde{\psi}$ and α.

Remark 4.1. If the body B constitutes a local system, then

$$\frac{\partial \alpha}{\partial \mathcal{U}(\mathbf{Z},t)} = 0, \qquad (2.4-13)$$

and hence (2.4-9) is reduced to

$$\begin{aligned} \alpha = &\ \alpha\{\mathcal{U}(\mathbf{X},t)\}, \\ \alpha^* = &\ \alpha\{\mathcal{U}(\mathbf{Z},t)\}, \end{aligned} \qquad (2.4-14)$$

from which it follows that

$$\frac{\partial \alpha^*}{\partial \mathcal{U}(\mathbf{X},t)} = 0. \qquad (2.4-15)$$

Thus from (2.4-13), (2.4-15) and (2.4-11) we conclude that $\mathcal{H}_\psi$ vanishes at each point of B if the system is local. Likewise $\mathcal{H}_\psi$ is identically zero in every static motion ($\dot{\mathcal{U}} = 0$).□

If we define a quantity $(\rho_0\tilde{\psi})^*$ by the expression

$$(\rho_0\tilde{\psi})^* = \int_B \rho_0(\mathbf{Z}) \frac{\partial \tilde{\psi}}{\partial < \mathcal{U} >}(\mathbf{Z})\alpha^* dV(\mathbf{Z}) \qquad (2.4-16)$$

and note that

$$\frac{\partial(\rho_0\ \tilde{\psi})}{\partial \mathcal{U}} = \rho_0 \frac{\partial \tilde{\psi}}{\partial < \mathcal{U} >} \int_B \frac{\partial \alpha}{\partial \mathcal{U}(\mathbf{X},t)}\ dV(\mathbf{Z}), \qquad (2.4-17)$$

we can write (2.4-10) in the form

$$\rho_0\dot{\psi} = \frac{\partial < \rho_0\tilde{\psi} >}{\partial \mathcal{U}}\dot{\mathcal{U}} + \mathcal{H}_\psi \qquad (2.4-18)$$

where we also have introduced the notation

$$< \rho_0\tilde{\psi} > = \rho_0\tilde{\psi} + (\rho_0\tilde{\psi})^*. \qquad (2.4-19)$$

When arguments are omitted it should be remembered that we understand $\mathbf{X}$ and t.

Remark 4.2. In Chapter 3, time derivatives of stress and yield functions in strain space and stress space, respectively, will be decomposed in the same way as the Helmholtz free energy, as described in (2.4-18). Since the nonlocal quantity $\partial < \rho_0\tilde{\psi} >$

$/\partial\mathcal{U}$ does not depend on rates, it is seen that the rate dependence appears *locally* (and linearly) in the first term of the right-hand side of (2.4-18). Accordingly, we call $(1/\rho_0)(\partial < \rho_0\tilde{\psi} > /\partial\mathcal{U})\dot{\mathcal{U}}$ the *quasi-local rate* of the Helmholtz free energy at $\mathbf{X}$, while we will refer to $\mathcal{H}_\psi/\rho_0$, defined by (2.4-11), as a *functional rate.*□

It is convenient to make the decomposition

$$\left.\begin{aligned} \mathcal{U} = & \ \{\mathbf{E}, \mathcal{U}', \theta\}, \\ \alpha = & \ \{\boldsymbol{\alpha}^e, \alpha', \ \alpha^\theta\}, \end{aligned}\right\} \qquad (2.4-20)$$

where in view of (2.2-2) and (2.2-5)

$$\left.\begin{aligned} \mathcal{U}' = & \ \{\mathbf{E}^p, \kappa\}, \\ \alpha' = & \ \{\boldsymbol{\alpha}^p, \alpha^h\}. \end{aligned}\right\} \qquad (2.4-21)$$

Upon substituting (2.4-18) into the reduced nonlocal Clausius-Duhem inequality (2.4-4), using the decomposition (2.4-20) and the condition (2.4-12), we obtain

$$\int_B \{-(\frac{\partial < \rho_0\tilde{\psi} >}{\partial\theta} + \rho_0\eta)\dot{\theta} + (\mathbf{S} - \frac{\partial < \rho_0\tilde{\psi} >}{\partial\mathbf{E}}) \cdot \dot{\mathbf{E}}$$

$$- \frac{\partial < \rho_0\tilde{\psi} >}{\partial\mathcal{U}'}\dot{\mathcal{U}}' + \frac{\mathbf{Q} \cdot Grad\ \theta}{\theta}\} dV(\mathbf{X}) \leq 0. \qquad (2.4-22)$$

Though (2.4-22) is general in concept we recall that B is an elastic-plastic body, and hence the members of $\mathcal{U}$ are not completely independent. However, the inequality must hold during loading as well as unloading. Now unloading corresponds to $\dot{\mathcal{U}}' = 0$ with any value of θ and $\mathbf{E}$ inside some bounding surface, and since the inequality is linear in $\dot{\theta}$ and $\dot{\mathbf{E}}$, it follows that

$$\left.\begin{aligned} \rho_0\eta = & \ -\frac{\partial < \rho_0\tilde{\psi} >}{\partial\theta}, \\ \mathbf{S} = & \ \frac{\partial < \rho_0\tilde{\psi} >}{\partial\mathbf{E}}, \end{aligned}\right\} \qquad (2.4-23)$$

and that

$$\int_B \frac{\mathbf{Q} \cdot Grad\ \theta}{\theta} dV(\mathbf{X}) \geq 0 \qquad (2.4-24)$$

during unloading (or neutral loading).

We have tacitly assumed that expressions of the forms given by (2.4-5) hold during unloading as well as loading. Hence, since η and $\mathbf{S}$ are independent of $\dot{\mathcal{U}}'$, the results given by (2.4-23) remain valid also during loading. (Note that η and $\mathbf{S}$ in (2.4-23) are evaluated at fixed but arbitrary values of $\mathcal{U}'$, i.e. each of Eqs. (2.4-23) is valid for every $\mathbf{E}^p$ and κ). Substituting (2.4-23) back into (2.4-22) yields

$$\int_B \{-\frac{\partial < \rho_0 \tilde{\psi} >}{\partial \mathcal{U}'} \dot{\mathcal{U}}' + \frac{\mathbf{Q} \cdot Grad\ \theta}{\theta}\} dV(\mathbf{X}) \geq 0. \qquad (2.4-25)$$

By considering an arbitrary homogeneous temperature motion we conclude from (2.4-25) that during loading

$$\int_B -\frac{\partial < \rho_0 \tilde{\psi} >}{\partial \mathcal{U}'} \dot{\mathcal{U}}'\ dV(\mathbf{X}) \geq 0, \qquad (2.4-26)$$

an inequality which can be looked upon as an expression for *plastic dissipation.* If $\mathbf{Q}$ does not depend on $Grad\,\theta$ we note from (2.4-24) or (2.4-25) that $\mathbf{Q}$ vanishes identically.

Remark 4.3. In a local theory, using (2.4-15), (2.4-16) and (2.4-19), we obtain that

$$\frac{\partial < \rho_0 \tilde{\psi} >}{\partial \mathcal{U}} = \rho_0 \frac{\partial \tilde{\psi}}{\partial \mathcal{U}}, \qquad (2.4-27)$$

and hence (2.4-23) reduces to

$$\left.\begin{aligned} \eta &= -\frac{\partial \tilde{\psi}}{\partial \theta}, \\ \mathbf{S} &= \rho_0 \frac{\partial \tilde{\psi}}{\partial \mathbf{E}}, \end{aligned}\right\} \qquad (2.4-28)$$

Further, the corresponding local forms of (2.4-24) and (2.4-26) become

$$\frac{\mathbf{Q} \cdot Grad\ \theta}{\theta} \geq 0, \qquad (2.4-29)$$

$$-\frac{\partial \tilde{\psi}}{\partial \mathbf{E}^p} \cdot \dot{\mathbf{E}}^p - \frac{\partial \tilde{\psi}}{\partial \kappa} \dot{\kappa} \geq 0. \qquad (2.4-30)$$

Finally if we take $\alpha = \mathcal{U}(\mathbf{X}, t)/V(B)$, which is in agreement with (2.4-14), we conclude by virtue of (2.2-6) that

$$\psi = \tilde{\psi}(\mathbf{E},\ \mathbf{E}^p,\ \kappa,\ \theta). \qquad (2.4-31)$$

Then (2.4-28)-(2.4-31) constitute classical thermodynamical relations for elastic-plastic continua, similar to those of Green and Naghdi (1966).□

Remark 4.4. By including a set of *damage* variables in $\mathcal{U}'$ the set of equations (2.4-23)-(2.4-26) constitutes the base for a general nonlocal thermodynamic damage-plasticity theory.□

Remark 4.5. It is easy to prove that e.g. $(2.4\text{-}5)_2$ may be replaced by a relation of the form

$$\psi = \tilde{\psi}(\mathcal{U}, < \mathcal{U} >). \qquad (2.4-32)$$

Define namely

$$\bar{\mathcal{U}} = \{\mathcal{U}, \mathcal{V}\}, \qquad (2.4-33)$$

$$< \mathcal{V} > (\mathbf{X}) = \int_B \beta\{\mathcal{V}(\mathbf{X}), \mathcal{V}(\mathbf{Z})\} dV(\mathbf{Z}), \qquad (2.4-34)$$

and assume that

$$\psi = \hat{\psi}(< \bar{\mathcal{U}} >). \qquad (2.4-35)$$

Then choose

$$\beta = \frac{\mathcal{V}}{V(B)}, \qquad (2.4-36)$$

i.e.

$$< \mathcal{V} >= \mathcal{V}, \qquad (2.4-37)$$

in view of (2.2-6). Hence

$$\psi = \hat{\psi}(\mathcal{V}, < \mathcal{U} >), \qquad (2.4-38)$$

and by the choice $\mathcal{V} = \mathcal{U}$ (2.4-32) is obtained.□

We will not go on to derive a general thermodynamic theory here, since, as mentioned already, our main interest concerns a purely mechanical theory of plasticity.

Thus we postpone to the next chapter a discussion of questions concerning loading and unloading criteria, the definition of a yield function, and the evolution equations for the plastic strain and the strain hardening function.

Though we are not making extensive use of the equations (2.4-23), (2.4-24) and (2.4-26) resulting from the Clausius-Duhem inequality, we will take advantage of the notions and the definitions introduced here, when we continue with the derivation of a purely mechanical nonlocal plasticity theory in the next chapter. It is to be emphasized that the thermodynamic statement in the form of the local Clausius-Duhem inequality is not equivalent to the work postulates used in classical plasticity theory, e.g. those of Drucker (1952), Il'yushin (1961) or Naghdi and Trapp (1975b). Cf. Section 3.4 in the next chapter.

Chapter 3

Nonlocal rate-independent plasticity

3.1 Introduction

Classical mechanical plasticity theory must not be looked upon as merely a restriction from a thermodynamic theory to a corresponding formulation at constant temperature. This should be obvious since none of the various work postulates used in mechanical theories of plasticity follows directly from any statement of the *Second Law of Thermodynamics*, as e.g. expressed in the form of the Clausius-Duhem inequality.

The nonlocal, purely mechanical and rate independent theory, which will now be derived for an elastic-plastic body undergoing finite deformations, is, where applicable, similar to a corresponding local formulation used by Casey and Naghdi (1984), which in turn is based on theories of Naghdi and Trapp (1975a,b) and Green and Naghdi (1965, 1966).

In our development, strains and not stresses are taken as primary, as was the case with the thermodynamic formulation in Chapter 2. Historically, theories of infinitesimal plasticity have been derived using stress space formulations (yield surfaces defined in stress space with loading criteria expressed in terms of stresses). The reason for this is presumably that material behaviour appears to be easier to understand in terms of stresses (applied loads) than in terms of corresponding strains. Most certainly strain space and stress space formulations are equally permissible, but they are not entirely equivalent and may not be equally convenient for all kinds of applications.

A strain space formulation within a general theory of finite plasticity was first presented by Naghdi and Trapp (1975a,b). In the opinion of Naghdi and Trapp the strain space and the stress space formulations are not equivalent, and this nonequivalence has been discussed by several authors (see e.g. the review article of Naghdi 1990). As a main reason for a strain space formulation, authors on the subject usually refer to the fact that it is always possible to relate plastic strain rate to the rate of strain but not to the rate of stress, as is the case with perfect plasticity where plastic strain does not affect the stress at all. Other aspects of the difference between the stress space and the strain space formulation (with regard to nonlocality) will be discussed below. Also cf. Section 2.1.

In fact the theory of Casey and Naghdi (1984) takes advantage of stress space as well as strain space in order to characterize strain hardening behaviour (hardening, softening and perfectly plastic behaviour). Explicitly a yield function with reference to strain space is first introduced and then, by use of a constitutive equation for strain, a corresponding yield function in stress space is calculated. Loading criteria are defined in strain space and associated conditions in stress space are derived. As in classical infinitesimal plasticity, the theory allows for simple geometrical interpretations - with respect to strain space as well as stress space.

In a corresponding nonlocal formulation it is possible to similarly introduce a yield function and loading criteria in strain space, but it turns out that there is no unequivocal definition of the associated loading conditions in stress space. Moreover, geometrical interpretations are not as obvious as in local theory.

The scope of this part of the work is indicated by the table of contents. The chapter is divided into six sections, this introduction being Section 3.1. In Section 3.2 we introduce the basic state functions of the theory. Strains and not stresses are primary variables, and plastic strain is regarded as a primitive variable, characterized by a constitutive equation for its rate. Nonlocal constitutive variables are constructed from a set of the basic state functions, constituted by total strain, plastic strain and a measure of strain hardening. Nonlocal yield functions in strain space as well as stress space are introduced.

In Section 3.3 two different types of elastic-plastic response functions are derived, one general nonlocal and one which will be referred to as quasi-local. A nonlocal function Φ is defined, which characterizes the material response during loading, such that the material is hardening if $\Phi > 0$, softening if $\Phi < 0$ and is exhibiting perfectly plastic behaviour if $\Phi = 0$. It is shown that Φ equals the determinant of the general nonlocal response function, which (like its quasi-local counterpart) is a fourth order tensor. It appears that the general nonlocal theory can easily be restricted to describe purely local behaviour, and that the resulting corresponding local formulation is in agreement with classical theory.

Various work assumptions used in local plasticity, as e.g. the postulates of Drucker (1952) and of Il'iushin (1961) and the work assumption of Naghdi and Trapp (1975b) are treated in Section 3.4. A generalization to nonlocal plasticity of the classical principle of maximum dissipation is also discussed. A plastic potential function is defined and the elastic-plastic response function is expressed in terms of the plastic potential. The special case of associated plasticity is treated and comparison with local theory is made. Results for perfectly plastic behaviour are derived and the relationship between loading directions and strain hardening behaviour is discussed.

In Section 3.5 we introduce an equivalent set of constitutive variables. It is used to obtain results for a restricted class of elastic-plastic materials, for which the stress is independent of the strain hardening measure.

In Section 3.6 the theory is illustrated by considering a special class of familiar elastic-plastic materials, which belongs to the type of materials discussed in the previous section.

3.2 The strain space formulation

In this section will be presented the basic features of the nonlocal mechanical rate independent theory. As in classical plasticity, we understand that rate independence requires that constitutive equations be invariant under time rescaling.

The basic state functions are now $\mathbf{E} = \mathbf{E}(\mathbf{X}, t)$, $\mathbf{E}^p = \mathbf{E}^p(\mathbf{X}, t)$ and $\kappa = \kappa(\mathbf{X}, t)$, each of which is discussed in Chapter 2. To simplify wordings we call these variables *local*, while we refer to the corresponding quantities $<\mathbf{E}>$, $<\mathbf{E}^p>$ and $<\kappa>$ as *nonlocal*. Thus e.g. $\mathbf{E}^p$ is local plastic strain and $<\mathbf{E}^p>$ is nonlocal plastic strain. However, it should be observed that these notations are somewhat misleading since the domain of e.g. $<\mathbf{E}^p>$ contains not only $\mathbf{E}^p(\mathbf{Z}, t)$ but also $\mathbf{E}(\mathbf{Z}, t)$ and $\kappa(\mathbf{Z}, t)$.

In Subsection 3.2.1 a constitutive equation for stress is assumed. The notions of unrestricted nonlocality and restricted nonlocality are introduced, and in that context the requirement that the stress-strain relation be invertible and its consequences are discussed.

In Subsection 3.2.2 a yield function g with reference to strain space is introduced. It is important for the development of the theory that g is a function of *nonlocal* variables (the same set of variables as the stress function), and not merely local variables, as is the case e.g. in the paper of Eringen (1981) on nonlocal plasticity. Loading conditions are defined and evolution equations (flow rules) for the plastic strain function $\mathbf{E}^p$ and the strain hardening function κ are established. In the end of the subsection an explicit expression for a nonlocal consistency condition is derived.

A yield function in stress space is calculated in Subsection 3.2.3, and a relationship between the loading conditions in strain space and the associated criteria in stress space is derived.

3.2.1 Constitutive assumptions

In a purely mechanical theory, temperature drops out as constitutive variable and $(2.4\text{-}5)_1$ is replaced by

$$\tilde{\mathbf{S}}(<\mathcal{U}>) = \tilde{\mathbf{S}}(<\mathbf{E}>, <\mathcal{U}'>), \qquad (3.2-1)$$

where $<\mathcal{U}>$ in the first equation is defined by (2.2-2) and (2.2-4), while $(2.4\text{-}20)_1$ has been used to obtain the second equation.

Looking back at (2.2-4) we note that if $\partial\alpha/\partial\mathcal{U}(Z,t)$ vanishes for each member of α, it follows that each member of $<\mathcal{U}>$ is an ordinary function of $\mathcal{U}(X,t)$, in which case (2.4-5) reduces to the classical constitutive assumption. Hence we conclude that the stress in (3.2-1) is *local* if

$$\frac{\partial\alpha}{\partial\mathbf{E}(\mathbf{Z},t)} \equiv 0, \quad \frac{\partial\alpha}{\partial\mathcal{U}'(\mathbf{Z},t)} \equiv 0, \qquad (3.2-2)$$

for each member of the set (2.2-5). If neither $(3.2\text{-}2)_1$ nor $(3.2\text{-}2)_2$ holds true, we say that the elastic-plastic material possesses *unrestricted nonlocality.* If $(3.2\text{-}2)_1$ is identically satisfied, but $(3.2\text{-}2)_2$ is not, the material possesses *restricted nonlocality*[3.2-1]. Thus an elastic-plastic body possesses restricted nonlocality if the total strain enters into the stress response function (3.2-1) only in terms of its value at the stress point $\mathbf{X}$.

As will be apparent in Subsection 3.2.3, it is essential that stress rather than strain may be used as an independent state function. Now, if we refer to an elastic-plastic body with unrestricted nonlocality, it is in a general case extremely difficult to invert the stress-strain relation in order to establish the strain at the actual stress point as a functional of the stress distribution of the body. If however, we refer to an elastic-plastic body with restricted nonlocality (i.e. $(3.2\text{-}2)_1$ holds but $(3.2\text{-}2)_2$ does not), the assumption that the stress-strain relation under some mild conditions be invertible is reasonable. This point comprises a conclusive argument for turning our interest to the case of restricted nonlocality. If we take (cf. (2.2-6))

$$\alpha^e = \mathbf{E}(\mathbf{X},t)/V(B) \qquad (3.2-3)$$

in agreement with $(3.2\text{-}2)_1$, we obtain by $(2.2\text{-}3)_1$

$$<\mathbf{E}> = \mathbf{E}, \qquad (3.2-4)$$

[3.2-1] Edelen and Laws (1971) introduced the notion of restricted nonlocality for a thermodynamical system. Here it is used in a slightly different sense.

and the constitutive assumption (3.2-1) then reads

$$\mathbf{S} = \tilde{\mathbf{S}}(\mathbf{E} <\mathcal{U}'>). \qquad (3.2-5)$$

We recall that the basic state functions are $\mathbf{E}(\mathbf{X},t)$, $\mathbf{E}^p(\mathbf{X},t)$ and $\kappa(\mathbf{X},t)$, each of them discussed in Section 2.2. As before, we use the short hand notation (2.4-21), i.e. $\mathcal{U}' = \{\mathbf{E}^p,\ \kappa\}$ and $\alpha' = \{\boldsymbol{\alpha}^p,\ \alpha^h\}$, but the definitions of the members of the $<\mathcal{U}'>$ now differ from $(2.2\text{-}3)_{2,3}$ and read

$$\left.\begin{aligned} <\mathbf{E}^p> (\mathbf{X},t) &= \textstyle\int_B \boldsymbol{\alpha}^p\{\mathcal{U}'(\mathbf{X},t),\ \mathcal{U}'(\mathbf{Z},t)\}\ dV(\mathbf{Z}), \\ <\kappa> (\mathbf{X},t) &= \textstyle\int_B \alpha^h\{\mathcal{U}'(\mathbf{X},t)\ \mathcal{U}'(\mathbf{Z},t)\}\ dV(\mathbf{Z}). \end{aligned}\right\} \qquad (3.2-6)$$

Note that we have retained the same function symbols $\boldsymbol{\alpha}^p$ and α^h as in $(2.2\text{-}3)_{2,3}$, but now with different meanings.

The fact that α' does not depend on $\mathbf{E}$ legitimates our assumption that $\tilde{\mathbf{S}}$ in some range possesses an inverse of the form

$$\mathbf{E} = \tilde{\mathbf{E}}(\mathbf{S},\ <\mathcal{U}'>), \qquad (3.2-7)$$

where the range is defined by a prescribed yield criterion, as discussed later on.

Apparently we have now taken a decision about level of generality, since the choice (3.2-3) seriously restricts the theory. If we look at the case of purely elastic response, we note that (3.2-3) is the condition for the stress to be local. In the elastic-plastic case the notion of plastic strain is of course fundamental, and evidently the quantity $\partial\boldsymbol{\alpha}^p/\partial\mathbf{E}^p(\mathbf{Z},t)$ will not vanish in a theory of nonlocal plasticity. Hence, with respect to the microstructural differences between elastic and plastic deformation (cf. Section 1.4), it is not unreasonable to assume, somewhat vaguely, that for a certain range of application elastic response is local, whereas plastic response is nonlocal. This point does not really justify the condition (3.2-3) (since $\mathbf{E}$ is total strain and not 'elastic strain'), so (3.2-5) is questionable even in the case when local elastic response is assumed[3.2-2]. No matter how, (3.2-5) represents a wide class of nonlocal elastic-plastic materials, and it is believed general enough to be the starting-point of a theory within which it will be possible to master a variety of essential problems concerning strain softening phenomena in heterogeneous media, as discussed in Chapter 1.

Thus we will adopt (3.2-3) and its consequence (3.2-5). In what follows we will see why it is essential to make this restriction of the general theory.

[3.2-2]Note that in finite plasticity $\mathbf{E} - \mathbf{E}^p$ is *not* elastic strain, so the problem cannot adequately be resolved by replacing $\mathbf{E}$ in the constitutive equation (3.2-5) by an elastic strain function $\mathbf{E}^e(\mathbf{X},t)$ with $\partial\boldsymbol{\alpha}^e/\partial\mathbf{E}^e(\mathbf{Z},t)$ being zero.

3.2.2 Yield criterion and flow rules

We assume the existence of a sufficiently smooth yield (or loading) function g, which is nonpositive for all admissible values of the constitutive variables,

$$g(\mathbf{E},\ <\mathcal{U}'>) \leq 0, \tag{3.2-8}$$

with $<\mathcal{U}'>$ defined by (2.4-21) and (3.2-6). For fixed values of $<\mathcal{U}'>$ the equation $g(\mathbf{E},\ <\mathcal{U}'>) = 0$ defines an open region $\mathcal{E}$ ($g < 0$) of six-dimensional strain space, called the elastic region. Its boundary $\partial\mathcal{E}$ ($g = 0$) is called the yield surface. A motion defined by $\mathbf{x} = \chi(\mathbf{X}, t)$ generates for each $\mathbf{X}$ a strain trajectory in strain space. Then for each value of $<\mathbf{E}^p>$ *and* $<\kappa>$, states $(\mathbf{E},\ <\mathcal{U}'>)$ with $g < 0$ are elastic, while states with $g = 0$ are elastic-plastic.

We introduce a scalar function $\hat{g}$, defined by

$$\hat{g} = \frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}}. \tag{3.2-9}$$

At a regular point of $\partial\mathcal{E}$, $\partial g/\partial\mathbf{E}$ is not zero and hence (3.2-9) can be interpreted as the inner product between the outward normal to $\partial\mathcal{E}$ and the tangent vector to a strain trajectory in strain space. It is then evident that (3.2-9) affords definitions of *unloading* from an elastic-plastic state ($g = 0$, $\hat{g} < 0$; $\dot{\mathbf{E}}$ directed into $\partial\mathcal{E}$), *neutral loading* from an elastic-plastic state ($g = 0$, $\hat{g} = 0$; $\dot{\mathbf{E}}$ tangent to $\partial\mathcal{E}$) and *loading* from an elastic-plastic state ($g = 0$, $\hat{g} > 0$; $\dot{\mathbf{E}}$ directed out of $\partial\mathcal{E}$).

Remark 2.1. Note that in (3.2-8) we have retained the same set of independent variables as in (3.2-5). In fact the same list of independent variables will be employed in all constitutive equations to appear in the subsequent derivation of the theory (i.e. the *principle of equipresence* of continuum mechanics is invoked). It is our position that each independent variable in (3.2-8) is admissible, in the sense that its presence does not violate any fundamental principle of continuum mechanics. Nor from a physical point of view can well-founded objections be raised against the choice of independent variables in (3.2-8). Hence in a nonlocal plasticity theory it should be inappropriate to replace the nonlocal variables in (3.2-8) with corresponding local ones, rendering the yield criterion to the same form as in local plasticity. We recall that in classical plasticity the yield condition implies that whether a state is elastic or plastic depends only on the inelastic state at the actual stress point and not on corresponding states at neighbouring points, hence excluding any dependence on gradients of inelastic variables. However, considering the complex interplay between microstructure and macrostructure with regard to plastic deformation, it is hard to find any support for rejecting long range interactions in the yield criterion. Thus there is no reason for replacing (3.2-8) with a local formulation. This is in contradiction to results found in Eringen's paper on nonlocal plasticity (1981), where it is argued that dependence on such long-range interactions can in fact be discarded in the yield function. However, Eringen's argument is rejected in recent works on plastic localization, e.g. by Mühlhaus and Aifantis (1991) and Vardoulakis and Aifantis (1991), where second order derivatives of plastic parameters appear in the yield functions. We also draw attention to a paper by Kratochvil (1988) with a discussion of nonlocality and the microstructural origin of plastic deformation.□

Along a strain trajectory we assume constitutive equations for the rates of $\mathbf{E}^p$ and κ of the form

$$\dot{\mathbf{E}}^p = \left\{ \begin{array}{ll} \mathcal{A}(\mathbf{E},\ <\mathcal{U}'>)\dot{\mathbf{E}} & (g=0,\ \hat{g}>0, \\ \mathbf{0} & (\text{otherwise}), \end{array} \right\} \qquad (3.2-10)$$

and

$$\dot{\kappa} = \left\{ \begin{array}{ll} \mathbf{B}(\mathbf{E},\ <\mathcal{U}'>)\cdot\dot{\mathbf{E}} & (g=0,\ \hat{g}>0, \\ 0 & (\text{otherwise}), \end{array} \right\} \qquad (3.2-11)$$

where $\mathcal{A}$ is a fourth order tensor-valued function possessing minor symmetry ($\mathcal{A}_{KLMN} = \mathcal{A}_{LKMN} = \mathcal{A}_{LKNM}$) and $\mathbf{B}$ is a second order symmetric tensor-valued function[3.2-3].

Reduced forms of $(3.2\text{-}10)_1$ and $(3.2\text{-}11)_1$ are easily obtained if we assume that $\dot{\mathbf{E}}^p$ and $\dot{\kappa}$ are continuous functions of $\dot{\mathbf{E}}$ at each point of the yield surface. Beginning with $\dot{\mathbf{E}}^p$ we conclude by the continuity requirement that $\mathcal{A}\dot{\mathbf{E}} = \mathbf{0}$ for every $\dot{\mathbf{E}}$ with $\hat{g} = 0$ (neutral loading), i.e. whenever $\dot{\mathbf{E}}$ lies in the tangent plane of the surface $g = 0$. By an argument used e.g. by Green and Naghdi (1965) and repeated in a similar way below, it follows that

$$\mathcal{A} = \pi\mathbf{R}\otimes\frac{\partial g}{\partial \mathbf{E}}, \qquad (3.2-12)$$

where $\mathbf{R}(\mathbf{E},\ <\mathcal{U}'>)$ is a symmetric second order tensor, and where the scalar function $\pi = \pi(\mathbf{E}, <\mathcal{U}'>)$ has been introduced for convenience. Trivially the result (3.2-12) is sufficient for continuity. To see that this is also necessary we note that continuity requires that

$$\mathcal{A}\dot{\mathbf{E}} - \pi\mathbf{R}\hat{g} = \mathbf{0}, \qquad (3.2-13)$$

where the unknown *Lagrange multiplier* $\pi\mathbf{R}$ is a symmetric second order tensor. Now (3.2-13) can be written

[3.2-3] Script capitals are used to denote fourth order tensors and, as before, boldface capitals to denote symmetric second order tensors. We use the notation $\mathbf{U}\cdot\mathbf{V} = tr(\mathbf{U}\ \mathbf{V}^{\mathrm{T}})$ for inner product, i.e. for the scalar $U_{KL}V_{KL}$. By $\mathcal{A}\mathbf{V}$ we understand the symmetric second order tensor with components $\mathcal{A}_{KLMN}V_{MN}$, while $\mathbf{U}\otimes\mathbf{V}$ and $\mathcal{AB}$ denote fourth order tensors with components $U_{KL}V_{MN}$ and $\mathcal{A}_{KLPQ}\mathcal{B}_{PQMN}$, respectively.

$$(\mathcal{A} - \pi \mathbf{R} \otimes \frac{\partial g}{\partial \mathbf{E}})\dot{\mathbf{E}} = \mathbf{0}, \qquad (3.2-14)$$

and must hold for all $\dot{\mathbf{E}}$ in the tangent plane. Since the fourth order tensor operating on $\dot{\mathbf{E}}$ is independent of $\dot{\mathbf{E}}$ we then arrive at (3.2-12).

By the same argument we obtain

$$\mathbf{B} = \pi r \frac{\partial g}{\partial \mathbf{E}}, \qquad (3.2-15)$$

where r is a scalar function of $\mathbf{E}$ and $<\mathcal{U}'>$.

Substituting (3.2-12) and (3.2-15) into (3.2-10) and (3.2-11), respectively, gives us the reduced *flow rules*

$$\dot{\mathbf{E}}^p = \left\{ \begin{array}{ll} \pi \hat{g}\, \mathbf{R} & (g = 0,\ \hat{g} \geq 0), \\ \mathbf{0} & \text{(otherwise)}, \end{array} \right\} \qquad (3.2-16)$$

and

$$\dot{\kappa} = \left\{ \begin{array}{ll} \pi \hat{g}\, r & (g = 0,\ \hat{g} \geq 0), \\ 0 & \text{(otherwise)}. \end{array} \right\} \qquad (3.2-17)$$

Using the notation

$$\Lambda' = \{\mathbf{R},\ r\}, \qquad (3.2-18)$$

the flow rules can be written in the form

$$\dot{\mathcal{U}}' = \left\{ \begin{array}{ll} \pi \hat{g}\, \Lambda' & (g = 0,\ \hat{g} \geq 0), \\ 0 & \text{(otherwise)}. \end{array} \right\} \qquad (3.2-19)$$

Remark 2.2. Recall that in a corresponding *local* theory $\partial\mathcal{E}$ is stationary when $\dot{\mathcal{U}}'(\mathbf{X}, t) = 0$ (i.e. for fixed values of $\mathbf{E}^p(\mathbf{X}, t)$ and $\kappa(\mathbf{X}, t)$). From $(3.2\text{-}19)_2$ then follows that $\partial\mathcal{E}$ is stationary in an elastic state. Our nonlocal formulation does not afford that simple geometrical interpretation, due to the fact that, while in an elastic state $\dot{\mathcal{U}}'(\mathbf{X}, t)$ still vanishes (as seen from $(3.2\text{-}19)_2$), $<\dot{\mathcal{U}}'>$ $(\mathbf{X}, t)$ in general does not. Thus $\partial\mathcal{E}$ is not necessarily stationary at $\mathbf{X}$, but may change due to plastic deformation at other parts of the body (i.e. $\dot{\mathcal{U}}(\mathbf{Z}, t) \neq 0$ in some finite region).□

For future use we record below the explicit expression for the material time derivative of the yield function g. We recall that in an elastic state ($g < 0$) the value of $\dot{g}$ may be of any sign, while in a plastic state ($g = 0$) the inequality (3.2-8) requires that $\dot{g}$ is negative during unloading and zero during neutral loading or loading. This comprises the *consistency* condition

$$\dot{g} = 0, \ (g = 0, \ \hat{g} \geq 0), \qquad (3.2-20)$$

i.e. loading from a plastic state results in a new plastic state.

Looking back at the calculation of the material derivative of the Helmholz free energy, we arrive at results similar to those of (2.4-6) and (2.4-18),

$$\rho_0 \dot{g} = \rho_0 \hat{g} + \rho_0 \frac{\partial g}{\partial <\mathcal{U}'>} \int_B \{ \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{X},t)} \dot{\mathcal{U}}'(\mathbf{X},t)$$

$$+ \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z},t)} \dot{\mathcal{U}} \ (Z,t) \} dV(\mathbf{Z}) = \rho_0 \hat{g} + \frac{\partial <\rho_0 g>}{\partial \mathcal{U}'} \dot{\mathcal{U}}' + \mathcal{H}_g, \qquad (3.2-21)$$

where $\hat{g} + (1/\rho_0)(\partial <\rho_0 g> / \partial \mathcal{U}') \dot{\mathcal{U}}'$ is recognized as the *quasi-local* and $\mathcal{H}_g / \rho_0$ as the *functional yield rate* (cf. *Remark 4.2* in Section 2.4). To obtain (3.2-21) we have used the following definitions

$$<\rho_0 \ g> = \rho_0 \ g + (\rho_0 \ g)^*, \qquad (3.2-22)$$

$$(\rho_0 \ g)^* = \int_B \rho_0(\mathbf{Z}) \frac{\partial g}{\partial <\mathcal{U}'>} (\mathbf{Z}) (\alpha')^* dV(\mathbf{Z}), \qquad (3.2-23)$$

and

$$\mathcal{H}_g = \int_B \{ \rho_0 \frac{\partial g}{\partial <\mathcal{U}'>} \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z},t)} \dot{\mathcal{U}}'(\mathbf{Z},t)$$

$$- \rho_0(\mathbf{Z}) \frac{\partial g}{\partial <\mathcal{U}'>} (\mathbf{Z}) \frac{\partial (\alpha')^*}{\partial \mathcal{U}'(\mathbf{X},t)} \dot{\mathcal{U}}'(\mathbf{X},t) \} dV(\mathbf{Z}), \qquad (3.2-24)$$

with

$$\int_B \mathcal{H}_g \ dV(\mathbf{X}) = 0. \qquad (3.2-25)$$

The consistency condition implies that

$$\int_B \dot{g} \ dV(\mathbf{X}) = 0, \qquad (3.2-26)$$

which with (3.2-21) and (3.2-25) gives

$$\int_B \{\hat{g} + \frac{1}{\rho_0}\frac{\partial <\rho_0 g>}{\partial \mathcal{U}'}\dot{\mathcal{U}}'\} dV(\mathbf{X}) = 0, \quad (g = 0,\ \hat{g} \geq 0). \qquad (3.2-27)$$

Explicitly, using (3.2-21), the consistency condition (3.2-20) has the form

$$\hat{g} + \frac{1}{\rho_0}\frac{\partial <\rho_0 g>}{\partial \mathbf{E}^p}\cdot\dot{\mathbf{E}}^p + \frac{1}{\rho_0}\frac{\partial <\rho_0 g>}{\partial \kappa}\dot{\kappa} + \frac{1}{\rho_0}\mathcal{H}_g = 0,\ (g = 0,\ \hat{g} \geq 0) \qquad (3.2-28)$$

or during loading, in view of (3.2-16) and (3.2-17),

$$1 + \pi(\frac{1}{\rho_0}\frac{\partial <\rho_0 g>}{\partial \mathbf{E}^p}\cdot\mathbf{R} + \frac{1}{\rho_0}\frac{\partial <\rho_0 g>}{\partial \kappa} r) + \frac{1}{\rho_0\hat{g}}\mathcal{H}_g = 0,\ (g = 0,\ \hat{g} > 0), \qquad (3.2-29)$$

where it is assumed that (3.2-19) has been used to express (3.2-24) in the form (the time dependence suppressed)

$$\mathcal{H}_g = \int_B \{\rho_0 \frac{\partial g}{\partial <\mathcal{U}'>}\frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z})}\pi(\mathbf{Z})\Lambda'(\mathbf{Z})\hat{g}(\mathbf{Z})$$

$$- \rho_0(\mathbf{Z})\frac{\partial g}{\partial <\mathcal{U}'>}(\mathbf{Z})\frac{\partial (\alpha')^*}{\partial \mathcal{U}'}\pi\Lambda'\hat{g}\}\ dV(\mathbf{Z}), \qquad (3.2-30)$$

$\pi(\mathbf{Z}) = 0$ at non-loading points ($g < 0$ or $g = 0$, $\hat{g} \leq 0$). (Note that $\Lambda(\mathbf{Z})$ in general is assumed to be defined for all $\mathbf{Z}$ in B).

In concise form (3.2-29) has the form

$$1 + \frac{1}{\rho_0}\frac{\partial <\rho_0 g>}{\partial \mathcal{U}'}\pi\Lambda' + \frac{1}{\rho_0\hat{g}}\mathcal{H}_g = 0,\ (g = 0,\ \hat{g} > 0), \qquad (3.2-31)$$

with $\mathcal{H}_g$ given by (3.2-30). In the sequel we will refer to (3.2-31) in combination with (3.2-30) as the general strain space statement of the consistency condition.

We note that, according to our basic assumptions, a nonlocal elastic-plastic body is completely specified by $\tilde{\mathbf{S}}$, Λ', π and g as prescribed functions of $\mathbf{E}$ and $<\mathcal{U}'>$, when in addition α' is a given function of $\mathcal{U}'(\mathbf{X})$ and $\mathcal{U}'(\mathbf{Z})$. The functions g, π and Λ' are subjected to the restriction (3.2-31) (or (3.2-29)). It should be noted that the consistency condition in general cannot be used to solve for π analytically (as is the case in local theory), since (3.2-31), due to the appearance of $\mathcal{H}_g$, is an integral equation with regard to the function π. Of course the possibility remains at loading directions for which $\mathcal{H}_g$ vanishes (or in such a special case where π does not depend on $\mathbf{Z}$). Since $\mathcal{H}_g$ is linear in $\dot{\mathcal{U}}'(\mathbf{X})$ and $\dot{\mathcal{U}}'(\mathbf{Z})$ we finally note that the quotient $\mathcal{H}_g/\hat{g}$ is rate independent, which is also evident from (3.2.31) or (3.2-29).□

Remark 2.3. During loading ($g = 0, \hat{g} > 0$) it follows from (3.2-31) that the function π necessarily must satisfy the global condition

$$\int_{B_L} (1 + \frac{1}{\rho_0} \frac{\partial < \rho_0\, g >}{\partial \mathcal{U}'} \pi\ \Lambda')\rho_0\ \hat{g}\ dV = 0, \qquad (3.2-32)$$

where B_L indicates the region at loading[3.2-4]. Clearly (and contrasting with local theory) we may not conclude that π cannot vanish at $\mathbf{X}$, i.e. plastic loading at $\mathbf{X}$ does not imply $\pi(\mathbf{X}) \neq 0$, or, since $\hat{g} > 0$, that $\dot{\mathbf{E}}(\mathbf{X})$ and $\dot{\kappa}(\mathbf{X})$ are non-vanishing. (Of course π cannot vanish identically.) Consequently an inequality of the type $\pi > 0$ is *ad hoc* and does not follow from a purely mechanical theory. In Section 3.4 we will use a generalization of the principle of maximum dissipation to prove that π in fact cannot vanish during loading. For the time being, however, there is no need for restrictions on π beyond what is imposed by (3.2-32).□

3.2.3 Yield function in stress space

The flow rules (3.2-16) and (3.2-17) (or equivalently (3.2-19)), together with (3.2-6) constitute a system of integro-differential equations for the plastic strain and strain hardening variables. For a given motion and an associated strain trajectory at each point $\mathbf{X}$ in an assigned body B, this system of equations may be solved - appropriate initial conditions prescribed - to obtain the inelastic functions $\mathcal{U}'(\mathbf{X}, t)$ for all $\mathbf{X}$ in B. Then, by (3.2-5) and (3.2-6), corresponding stress trajectories in stress space are obtained. Furthermore, for a given yield function g, in view of (3.2-5) and (3.2-7), we define a corresponding yield function f in stress space by the relation

$$g(\mathbf{E},\ <\mathcal{U}'>) = g(\tilde{\mathbf{E}}(\mathbf{S},\ <\mathcal{U}'>),\ <\mathcal{U}'>) = f(\mathbf{S},\ <\mathcal{U}'>). \qquad (3.2-33)$$

It follows from (3.2-8) and (3.2-33) (sufficient smoothness conditions assumed) that

$$f(\mathbf{S},\ <\mathcal{U}'>) \leq 0 \qquad (3.2-34)$$

for all admissible states. For fixed values of $<\mathcal{U}'>$ the equation $f(\mathbf{S}, <\mathcal{U}'>) = 0$ defines an open region $\mathcal{S}$ ($f < 0$) of six-dimensional stress space, with boundary $\partial\mathcal{S}$ ($f = 0$). It is clear that the interior of $\mathcal{S}$ defines the elastic region in stress space, and that $\partial\mathcal{S}$

[3.2-4] In view of (3.2-30) and (3.2-25) it follows that $\int_B \mathcal{H}_g\ dV = \int_{B_L} \mathcal{H}_g\ dV = 0$.

defines the yield surface in stress space. By (3.2-34), every stress trajectory must lie inside $\mathcal{S}$ or on $\partial\mathcal{S}$; positive values of f are impossible to reach.

In the next subsection the relationship between the loading conditions in strain space and the associated conditions in stress space will be discussed. Preparatory to that discussion a set of fundamental relations will be derived from (3.2-5), (3.2-7) and (3.2-33).

As a first step we calculate the material time derivatives of the stress response $\mathbf{S}$ and the yield function f. Using the same arguments as in the derivation of (2.4-6) and (2.4-18), we obtain an expression for $\rho_0 \dot{f}$ of the same form as that of $\rho_0 \dot{g}$ (cf. (3.2-21), namely

$$\rho_0 \dot{f} = \rho_0 \hat{f} + \frac{\partial < \rho_0 f >}{\partial \mathcal{U}'} \dot{\mathcal{U}}' + \mathcal{H}_f. \tag{3.2-35}$$

Here $\hat{f}$ is given by the relation

$$\hat{f} = \frac{\partial f}{\partial \mathbf{S}} \cdot \dot{\mathbf{S}} \ , \tag{3.2-36}$$

whereas $< \rho_0 \ f >$ is defined as $< \rho_0 \ g >$ in (3.2-22) and (3.2-23) with g replaced by f, and similarly, $\mathcal{H}_f$ is defined as $\mathcal{H}_g$ in (3.2-24). In (3.2-35) $\hat{f} + (1/\rho_0) \ (\partial < \rho_0 f > / \partial \mathcal{U}') \dot{\mathcal{U}}'$ is identified as the *quasi-local yield rate* (in stress space), while $\mathcal{H}_f / \rho_0$ is the functional yield rate (in stress space).

Differentiation of stress yields

$$\dot{\mathbf{S}} = \mathcal{L} \dot{\mathbf{E}} + \frac{1}{\rho_0} \frac{\partial < \rho_0 \tilde{\mathbf{S}} >}{\partial \mathcal{U}'} \dot{\mathcal{U}}' + \frac{1}{\rho_0} \boldsymbol{\mathcal{H}}_s, \tag{3.2-37}$$

where $\mathcal{L}$ is the fourth order tensor-valued function defined by $\mathcal{L} = \partial \tilde{\mathbf{S}} / \partial \mathbf{E}$, with symmetries $\mathcal{L}_{MNKL} = \mathcal{L}_{NMKL} = \mathcal{L}_{MNLK}$, while the functionals $< \rho_0 \tilde{\mathbf{S}} >$ and $\boldsymbol{\mathcal{H}}_s$ are defined similar to $< \rho_0 \ g >$ and $\mathcal{H}_g$. The sum of the first two terms in (3.2-37) represents the *quasi-local stress rate*, while $\boldsymbol{\mathcal{H}}_s / \rho_0$ is the *functional stress rate.*

For the sake of clarity we give the explicit forms of $\mathcal{H}_f$ and $\boldsymbol{\mathcal{H}}_s$,

$$\mathcal{H}_f = \int_B \{ \rho_0 \frac{\partial f}{\partial < \mathcal{U}' >} \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z})} \dot{\mathcal{U}}'(\mathbf{Z})$$

$$- \rho_0(\mathbf{Z}) \frac{\partial f}{\partial < \mathcal{U}' >}(\mathbf{Z}) \frac{\partial (\alpha')^*}{\partial \mathcal{U}'} \dot{\mathcal{U}}' \} \ dV(\mathbf{Z}) \tag{3.2-38}$$

and

$$\boldsymbol{\mathcal{H}}_s = \int_B \{\rho_0 \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>} \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z})} \dot{\mathcal{U}}'(\mathbf{Z})$$

$$- \rho_0(\mathbf{Z}) \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>}(\mathbf{Z}) \frac{\partial (\alpha')^*}{\partial \mathcal{U}'} \dot{\mathcal{U}}'\}\ dV(\mathbf{Z}), \qquad (3.2-39)$$

where $\mathcal{H}_f$ and $\boldsymbol{\mathcal{H}}_s$ satisfy

$$\int_B \mathcal{H}_f\ dV(\mathbf{X}) = 0 \qquad (3.2-40)$$

and

$$\int_B \boldsymbol{\mathcal{H}}_s\ dV(\mathbf{X}) = \mathbf{0}, \qquad (3.2-41)$$

respectively.

Next we note, by the chain rule of differentiation, that

$$\frac{\partial g}{\partial <\mathcal{U}'>} - \frac{\partial f}{\partial <\mathcal{U}'>} = \frac{\partial f}{\partial \mathbf{S}} \frac{\partial \mathbf{S}}{\partial <\mathcal{U}'>} \qquad (3.2-42)$$

holds separately for the members $<\mathbf{E}^p>$ and $<\kappa>$ of $<\mathcal{U}'>$. To get (3.2-42) we have used (3.2-33), (3.2-5) and (3.2-7).

We want to establish a result corresponding to (3.2-42), but involving differentiation with respect to the basic functions of the set $\mathcal{U}'$ instead of those of $<\mathcal{U}'>$. Looking back at (3.2-21) we recall that

$$\frac{\partial <\rho_0 g>}{\partial \mathcal{U}'} \dot{\mathcal{U}}' = (\int_B \{\rho_0 \frac{\partial g}{\partial <\mathcal{U}'>} \frac{\partial \alpha'}{\partial \mathcal{U}'}$$

$$+ \rho_0(\mathbf{Z}) \frac{\partial g}{\partial <\mathcal{U}'>}(\mathbf{Z}) \frac{\partial (\alpha')^*}{\partial \mathcal{U}'}\}\ dV(\mathbf{Z}))\dot{\mathcal{U}}', \qquad (3.2-43)$$

and note that the same expression also applies when g is replaced by f. Then by use of (3.2-24), (3.2-38), (3.2-39), (3.2-42) and (3.2-43) it follows that

$$\frac{\partial}{\partial \mathcal{U}'}(<\rho_0\, g> - <\rho_0\, f>)\dot{\mathcal{U}}' + \mathcal{H}_g - \mathcal{H}_f - \frac{\partial f}{\partial \mathbf{S}}\mathcal{H}_s$$

$$= \int_B \{\rho_0 \frac{\partial f}{\partial \mathbf{S}} \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>} \frac{\partial \alpha'}{\partial \mathcal{U}'} + \rho_0(\mathbf{Z})\frac{\partial f}{\partial \mathbf{S}}(\mathbf{Z})\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>}\frac{\partial (\alpha')^*}{\partial \mathcal{U}'}\}\; dV(\mathbf{Z})\, \dot{\mathcal{U}}'$$

$$+ \int_B \{\rho_0 \frac{\partial f}{\partial \mathbf{S}} \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>} \frac{\partial \alpha'}{\partial \mathcal{U}'}\dot{\mathcal{U}}'(\mathbf{Z}) - \rho_0(\mathbf{Z})\frac{\partial f}{\partial \mathbf{S}}(\mathbf{Z})\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>}(\mathbf{Z})\frac{\partial (\alpha')^*}{\partial \mathcal{U}'}\dot{\mathcal{U}}'\}\; dV(\mathbf{Z})$$

$$-\frac{\partial f}{\partial \mathbf{S}} \int_B \{\rho_0 \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>} \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z})}\dot{\mathcal{U}}'(\mathbf{Z}) - \rho_0(\mathbf{Z})\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>}(\mathbf{Z})\frac{\partial (\alpha')^*}{\partial \mathcal{U}'}\dot{\mathcal{U}}'\}\; dV(\mathbf{Z})$$

$$= \frac{\partial f}{\partial \mathbf{S}} \int_B \{\rho_0 \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>} \frac{\partial \alpha'}{\partial \mathcal{U}'} + \rho_0(\mathbf{Z})\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>}(\mathbf{Z})\frac{\partial (\alpha')^*}{\partial \mathcal{U}'}\; dV(\mathbf{Z})\}\dot{\mathcal{U}}' \,. \qquad (3.2-44)$$

Thus we have shown that

$$\frac{\partial}{\partial \mathcal{U}'}(<\rho_0\, g> - <\rho_0\, f> \dot{\mathcal{U}}' + \mathcal{H}_g - \mathcal{H}_f = \frac{\partial f}{\partial \mathbf{S}}(\frac{\partial <\rho_0 \tilde{\mathbf{S}}>}{\partial \mathcal{U}'}\dot{\mathcal{U}}' + \mathcal{H}_s). \qquad (3.2-45)$$

The relation (3.2-45) can alternatively be derived by using the time derivatives of g and f as the starting point. Accordingly, by aid of (3.2-21) and (3.2-35) it then follows that

$$\rho_0(\hat{g} - \hat{f}) + \frac{\partial}{\partial \mathcal{U}'}(<\rho_0\, g> - <\rho_0\, f>)\dot{\mathcal{U}}' + \mathcal{H}_g - \mathcal{H}_f = 0, \qquad (3.2-46)$$

where we have also used $\dot{g} = \dot{f}$ in view of (3.2-33). Further by (3.2-42) and by direct calculation (without introducing $\mathcal{H}_g$ and $\mathcal{H}_f$) it easily follows that

$$\rho_0(\hat{g} - \hat{f}) + \frac{\partial f}{\partial \mathbf{S}}(\frac{\partial <\rho_0 \tilde{\mathbf{S}}>}{\partial \mathcal{U}'}\dot{\mathcal{U}}' + \mathcal{H}_s) = 0. \qquad (3.2-47)$$

Thus (3.2-45) follows from (3.2-46) and (3.2-47). Conversely each of (3.2-46) and (3.2-47) follows from (3.2-45) with the aid of (3.2-21) and (3.2-35). Consequently (3.2-45) and (3.2-46) together with (3.2-47) are equivalent statements. It is emphasized that each of (3.2-45), (3.2-46) or (3.2-47) holds in elastic as well as plastic states, irrespective of the type of loading conditions.

As an illustration of the signification of the fundamental relation (3.2-45), we look at the difference $(\mathcal{H}_g - \mathcal{H}_f)$. From the definitions (3.2-24) and (3.2-38) and by using (3.2-42) we obtain explicitly

$$\mathcal{H}_g - \mathcal{H}_f = \int_B \{\rho_0 \frac{\partial f}{\partial \mathbf{S}} \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>} \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z})} \dot{\mathcal{U}}'(\mathbf{Z})$$

$$- \rho_0(\mathbf{Z}) \frac{\partial f}{\partial \mathbf{S}}(\mathbf{Z}) \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>}(\mathbf{Z}) \frac{\partial (\alpha')^*}{\partial \mathcal{U}'} \dot{\mathcal{U}}'\}\, dV(\mathbf{Z}). \qquad (3.2-48)$$

If $\partial f / \partial \mathbf{S}$ does not depend on $\mathbf{Z}$ we note by (3.2-39) that (3.2-48) can be written in the simple form[3.2-5]

$$\mathcal{H}_g - \mathcal{H}_f = \boldsymbol{\mathcal{H}}_s \cdot \frac{\partial f}{\partial \mathbf{S}}, \qquad (3.2-49)$$

while (3.2-45) is reduced to

$$\frac{\partial}{\partial \mathcal{U}'}(<\rho_0\, g> - <\rho_0\, f>)\dot{\mathcal{U}}' = \frac{\partial f}{\partial \mathbf{S}} \frac{\partial <\rho_0 \tilde{\mathbf{S}}>}{\partial \mathcal{U}'} \dot{\mathcal{U}}'. \qquad (3.2-50)$$

It is obvious that the case when $\partial f / \partial \mathbf{S}$ is constant in space is too restrictive to embrace a useful theory. Consequently (3.2-49) or (3.2-50) is of minor relevance, and it is recalled that it is (3.2-45) which constitutes the general relationship between the functionals $\mathcal{H}_g$, $\mathcal{H}_f$ and $\boldsymbol{\mathcal{H}}_s$.

We can also use (3.2-45) to obtain the stress space analogue of (3.2-31), which correspondingly will be referred to as the general statement of the consistency condition in stress space. Thus

$$1 + \frac{1}{\rho_0} \frac{\partial <\rho_0 f>}{\partial \mathcal{U}'} \pi \Lambda' + \frac{1}{\rho_0 \hat{g}} \mathcal{H}_f$$

$$+ (\frac{1}{\rho_0} \frac{\partial <\rho_0 \tilde{\mathbf{S}}>}{\partial \mathcal{U}'} \pi \Lambda' + \frac{1}{\rho_0 \hat{g}} \boldsymbol{\mathcal{H}}_s) \frac{\partial f}{\partial \mathbf{S}} = 0, \; g = 0, \; \hat{g} > 0, \qquad (3.2-51)$$

where it is assumed that (3.2-19) has been used in combination with (3.2-38) and (3.2-39), respectively, to express $\mathcal{H}_f$ as well as $\boldsymbol{\mathcal{H}}_s$ in a form similar to that of $\mathcal{H}_g$ in (3.2-30). Alternatively, (3.2-51) may be derived with the expression for $\dot{f}$ in (3.2-35) as the starting point. By use of (3.2-19), (3.2-20), (3.2-33) and (3.2-47), then (3.2-51) follows.

[3.2-5] Note that (3.2-49) also applies at unloading points for general yield functions, as seen from (3.2-45).

Before ending this subsection we will make some additional comments on notations. In (3.2-49) it has been emphasized (by the occurrence of the dot symbol) that $\mathcal{H}_s$ and $\partial f/\partial \mathbf{S}$ are second order tensors, $\mathcal{H}_s \cdot \partial f/\partial \mathbf{S}$ being their inner product. Usually however, in order to avoid unnecessary complications, operation symbols are omitted, leaving to the reader to understand the precise meaning of the combinations of tensors, vectors and scalars appearing in the equations. As an example, the second order tensor $(\partial f/\partial \mathbf{S})(\partial \tilde{\mathbf{S}}/\partial <\mathbf{E}^p>)$ is the result of a fourth order tensor operating on a tensor of second order. Explicitly in compact notation

$$\frac{\partial f}{\partial \mathbf{S}}\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathbf{E}^p>} = \left(\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathbf{E}^p>}\right)^{\mathrm{T}} \frac{\partial f}{\partial \mathbf{S}}, \qquad (3.2-52)$$

or in tensor component notations

$$\left(\frac{\partial f}{\partial \mathbf{S}}\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathbf{E}^p>}\right)_{KL} = \frac{\partial f}{\partial \tilde{S}_{MN}}\frac{\partial \tilde{S}_{MN}}{\partial <E^p>_{KL}} = \left(\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathbf{E}^p>}\right)^{\mathrm{T}}_{KLMN} \left(\frac{\partial f}{\partial \mathbf{S}}\right)_{MN}. \qquad (3.2-53)$$

Notice that the summation convention is always implied. For instance note the double summation in the expression $(\partial \mathbf{S}/\partial <\mathcal{U}'>)(\partial \alpha'/\partial \mathcal{U}'(\mathbf{Z}))\dot{\mathcal{U}}'(\mathbf{Z})$, which explicitly reads

$$\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathcal{U}'>}\frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z})}\dot{\mathcal{U}}'(\mathbf{Z})$$

$$= \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathbf{E}^p>}\{\frac{\partial \boldsymbol{\alpha}^p}{\partial \mathbf{E}^p(\mathbf{Z})}\dot{\mathbf{E}}^p(\mathbf{Z}) + \frac{\partial \boldsymbol{\alpha}^p}{\partial \kappa(\mathbf{Z})}\dot{\kappa}(\mathbf{Z})\}$$

$$+ \frac{\partial \tilde{\mathbf{S}}}{\partial <\kappa>}\{\frac{\partial \alpha^h}{\partial \mathbf{E}^p(\mathbf{Z})} \cdot \dot{\mathbf{E}}^p(\mathbf{Z}) + \frac{\partial \alpha^h}{\partial \kappa(\mathbf{Z})}\dot{\kappa}(\mathbf{Z})\}. \qquad (3.2-54)$$

3.3 Classification of strain hardening behaviour

In this section we will derive an elastic-plastic response function $\mathcal{K}$, which turns out to be of fundamental importance in the theory of nonlocal elastic-plastic materials. The function $\mathcal{K}$ is a fourth order tensor with a determinant Φ which classifies strain hardening behaviour. This and other characteristic properties of $\mathcal{K}$ and Φ are discussed below.

3.3.1 On elastic-plastic response functions

If we use the notation $\check{\mathbf{S}}$ for the quasi-local stress rate, i.e.

$$\check{\mathbf{S}} = \mathcal{L}\,\dot{\mathbf{E}} + \frac{1}{\rho_0}\frac{\partial < \rho_0 \tilde{\mathbf{S}} >}{\partial \mathcal{U}'}\dot{\mathcal{U}}', \qquad (3.3-1)$$

we can write (3.2-37) in the form

$$\dot{\mathbf{S}} = \check{\mathbf{S}} + \frac{1}{\rho_0}\mathcal{H}_s. \qquad (3.3-2)$$

Recall that the rate dependence in the right-hand side of (3.3-1) is *local* (and linear), and that, in view of (3.2-9) and (3.2-19), $\check{\mathbf{S}} = \mathbf{0}$ whenever $\dot{\mathbf{E}} = \mathbf{0}$.

Remark 3.1. By use of (3.2-41) we conclude that

$$\int_B \rho_0(\dot{\mathbf{S}} - \check{\mathbf{S}})dV = \mathbf{0}. \qquad (3.3-3)$$

Hence we may assert that the actual stress rate and the quasi-local stress rate coincide in an average sense. Also note that the introduction of $\check{\mathbf{S}}$ affords an interpretation of the functional $\mathcal{H}_s$ as is seen by the observation that $\mathcal{H}_s/\rho_0$ represents the deviation of the quasi-local stress rate from the actual stress state.□

We recall that $\mathcal{H}_s$ in general will not vanish at a point which is elastic, unloading or loading neutrally, due to the first term in the integrand in (3.2-39). Only if the body as a whole behaves, say for example elastic, will $\mathcal{H}_s$ *identically* become zero. Thus (3.2-37) in general reduces to

$$\dot{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}} + \frac{1}{\rho_0}\mathcal{H}_s \qquad (3.3-4)$$

when loading does not occur at the point $\mathbf{X}$ (nonvanishing $\mathcal{H}_s$), while (3.3-1) reduces to

$$\check{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}}. \qquad (3.3-5)$$

Note that $\mathcal{L}$ in (3.3-4) and (3.3-5) is not reduced to a function of $\mathbf{E}$ only (as in a corresponding local theory). This is obviously so, since in fact $\mathcal{L}$ depends on the members of $< \mathcal{U}' >$, which in general are not constants when $\dot{\mathcal{U}}' = 0$.

Remark 3.2. Notice the nonlocal character of (3.3-5). Zero strain at $\mathbf{X}$ does not imply that the stress rate vanishes at $\mathbf{X}$, as is the case in local rate independent plasticity theories. From the condition $\dot{\mathbf{E}} = \mathbf{0}$ it follows that $\check{\mathbf{S}} = \mathbf{0}$, or from (3.3-4) that

$$\dot{\mathbf{S}} - \frac{1}{\rho_0}\mathcal{H}_s = \mathbf{0}. \qquad (3.3-6)$$

In view of (3.2-39) we thus conclude that $\dot{\mathbf{S}}$ in general does not vanish at $\mathbf{X}$, because of the occurrence of plastic deformation at points $\mathbf{Z}$ throughout the body. Only special loading directions at $\mathbf{Z}$ will force $\mathcal{H}_s$ to vanish, and leave the point $\mathbf{X}$ with zero stress rate. It should be noted that the function symbol $\mathcal{L}$ has been used with different meanings in (3.3-5) and (3.3-1), since in (3.3-5) $\mathcal{L}$ is restricted by the condition that $\dot{\mathcal{U}}'$ vanishes at $\mathbf{X}$, while in (3.3-1) there is no such restriction.□

From (3.2-19) follows that (3.3-1) can be written in the form

$$\check{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}} + \pi \; \hat{g} \; \check{\check{\boldsymbol{\sigma}}}, \qquad (3.3-7)$$

or by use of (3.2-9)

$$\check{\mathbf{S}} = (\mathcal{L} + \pi \; \check{\check{\boldsymbol{\sigma}}} \otimes \frac{\partial g}{\partial \mathbf{E}})\dot{\mathbf{E}}, \qquad (3.3-8)$$

where the symmetric second order tensor $\check{\check{\boldsymbol{\sigma}}}$ is defined by

$$\check{\check{\boldsymbol{\sigma}}} = \frac{1}{\rho_0}\frac{\partial < \rho_0 \tilde{\mathbf{S}} >}{\partial \mathcal{U}'}\Lambda'. \qquad (3.3-9)$$

In view of (3.2-33) we have

$$\frac{\partial g}{\partial \mathbf{E}} = \mathcal{L}^T \frac{\partial f}{\partial \mathbf{S}}, \qquad (3.3-10)$$

and noting that $\partial g / \partial \mathbf{E}$ does not vanish on the loading surface $\partial \mathcal{E}$ in strain space, we conclude that then $\partial f / \partial \mathbf{S}$ does not vanish on the corresponding loading surface $\partial \mathcal{S}$ in stress space.

We introduce a fourth order tensor-valued function $\check{\mathcal{K}}$ defined by

$$\check{\mathcal{K}} = \mathcal{J} + \pi \, \check{\tilde{\boldsymbol{\sigma}}} \otimes \frac{\partial f}{\partial \mathbf{S}} \,, \qquad (3.3-11)$$

where $\mathcal{J}$ is a fourth order unit tensor[3.3-1] with (major) symmetries,

$$\mathcal{J}_{KLPQ} = \mathcal{J}_{LKPQ} = \mathcal{J}_{KLQP} = \mathcal{J}_{PQKL} \,. \qquad (3.3-12)$$

Upon substituting (3.3-10) into (3.3-8) and using (3.3-11) we obtain[3.3-2]

$$\check{\mathbf{S}} = \check{\mathcal{K}} \, \mathcal{L} \, \dot{\mathbf{E}}. \qquad (3.3-13)$$

Using the relationship between the actual stress rate and the quasi-local stress rate, given by (3.2-2), together with (3.3-13) we may write the stress rate in the form

$$\dot{\mathbf{S}} = \check{\mathcal{K}} \, \mathcal{L} \, \dot{\mathbf{E}} + \frac{1}{\rho_0} \mathcal{H}_s \,, \qquad (3.3-14)$$

or, if loading $(g = 0, \hat{g} > 0)$ is presupposed,

$$\begin{aligned} \dot{\mathbf{S}} &= \check{\mathcal{K}} \, \mathcal{L} \, \dot{\mathbf{E}} + \frac{1}{\rho_0 \hat{g}} \mathcal{L}^T \frac{\partial f}{\partial \mathbf{S}} \cdot \dot{\mathbf{E}} \, \mathcal{H}_s \\ &= (\check{\mathcal{K}} + \frac{1}{\rho_0 \hat{g}} \mathcal{H}_s \otimes \frac{\partial f}{\partial \mathbf{S}}) \mathcal{L} \dot{\mathbf{E}} \,, \end{aligned} \qquad (3.3\text{- }15)$$

where again (3.2-9) and (3.3-10) have been used.

Hence

$$\dot{\mathbf{S}} = \mathcal{K} \mathcal{L} \dot{\mathbf{E}} \,, \qquad (3.3-16)$$

with $\mathcal{K}$ defined by

$$\mathcal{K} = \check{\mathcal{K}} + \frac{1}{\rho_0 \hat{g}} \mathcal{H}_s \otimes \frac{\partial f}{\partial \mathbf{S}} \,. \qquad (3.3-17)$$

[3.3-1] Defined by the relation $\mathcal{J} = 1/2(\delta_{KP}\delta_{LQ} + \delta_{KQ}\delta_{LP})$, where the Kronecker symbol δ_{IJ} denotes the components of the second order unit tensor.

[3.3-2] If $\mathcal{D}$ is a fourth order tensor and if $\mathcal{D}^T$ as before is defined by the relation $\mathcal{D}^T_{KLMN} = \mathcal{D}_{MNKL}$, then note that $\mathbf{A} \otimes \mathbf{B} \mathcal{D} = \mathbf{A} \otimes \mathcal{D}^T \mathbf{B}$ for all second order tensors $\mathbf{A}$ and $\mathbf{B}$.

We will refer to $\mathcal{K}$ as an *elastic-plastic response function* and to $\check{\mathcal{K}}$ - because of its connection to $\check{\mathbf{S}}$ - as a *quasi-local elastic-plastic response function.* There are fundamental differences between $\mathcal{K}$ and $\check{\mathcal{K}}$, as is seen from (3.3-17) and from the definitions of $\check{\mathcal{K}}$ and $\check{\tilde{\boldsymbol{\sigma}}}$ given by (3.3-11) and (3.3-9), respectively. The tensor $\check{\mathcal{K}}$ is clearly rate independent and involves all constitutive functions presented, namely $\mathbf{S}$, Λ', f and the prescribed functions α' defined by (2.4-21) and (2.2-5). As for $\mathcal{K}$ it is seen from (3.3-17) that it depends on the rate dependent functional $\mathcal{H}_s$.

It is assumed (since loading at $\mathbf{X}$ is presupposed) that $(3.2\text{-}19)_1$ is used in the expression for $\mathcal{H}_s$ in (3.3-17). Hence, the second order tensor $\mathcal{H}_s/(\rho_0 g)$ in the right-hand side of (3.3-17), using (3.2-39), explicitly becomes

$$
\begin{aligned}
\frac{1}{\rho_0 \hat{g}}\mathcal{H}_s &= \int_B \{ \frac{\partial \tilde{\mathbf{S}}}{\partial < \mathcal{U}' >} \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z})} \pi(\mathbf{Z}) \Lambda'(\mathbf{Z}) \frac{\hat{g}(\mathbf{Z})}{\hat{g}} \\
&- \frac{\rho_0(\mathbf{Z})}{\rho_0} \frac{\partial \tilde{\mathbf{S}}}{\partial < \mathcal{U}' >}(\mathbf{Z}) \frac{\partial (\alpha')^*}{\partial \mathcal{U}} \pi \Lambda' \} dV(\mathbf{Z}), \qquad (3.3\text{-}18)
\end{aligned}
$$

where it is understood that $\pi(\mathbf{Z}) = 0$ at non-loading points.

Clearly (3.3-18) is rate independent but only in a, say, weak sense since the strain rates in the first term of the integrand cannot be cancelled out (unless $\hat{g}$ is uniform). This means that $\mathcal{K}$ cannot be determined completely without knowledge of strain rates in advance, which of course is not in agreement with properties normally expected from an elastic-plastic response function (and contrasting with $\check{\mathcal{K}}$). On the other hand, the simplicity of (3.3-13) should not obscure the fact that the actual stress rate $\dot{\mathbf{S}}$ cannot be determined from (3.3-13) unless the functional $\mathcal{H}_s$ is also known.

It is important to remember that loading is presupposed in the definition (3.3-17) of the function $\mathcal{K}$, which is left undefined when $\hat{g} \leq 0$, and therefore does not equal the unity tensor $\mathcal{J}$ during corresponding loading conditions, which obviously is the case for $\check{\mathcal{K}}$ in (3.3-13). During elastic behaviour, unloading or neutral loading from a plastic state, (3.3-16) consequently reduces to (3.3-4), while (3.3-13) reduces to (3.3-5) as discussed in the beginning of the subsection.

3.3.2 Hardening, softening and perfectly plastic behaviour

A correspondence between the loading criteria of the strain space formulation and the associated stress space conditions will now be established and, as in local theory, three types of material response, namely hardening, softening and perfectly plastic behaviour will be defined. To simplify, these distinct types of response will be collectively referred to as strain hardening behaviour.

In view of (3.2-47) we conclude that

$$\hat{f} - \frac{1}{\rho_0}\mathcal{H}_s \cdot \frac{\partial f}{\partial \mathbf{S}} = \hat{g} \;\; \text{if} \; \dot{\mathcal{U}}' = 0. \tag{3.3-19}$$

The function $\hat{f}$, defined in (3.2-36), is geometrically interpreted as the inner product of the tangent vector $\dot{\mathbf{S}}$ to the stress trajectory passing through the stress point in the six-dimensional stress space and the normal vector $\partial f / \partial \mathbf{S}$.

We define a quantity $\check{f}$ by the relation

$$\check{f} = \hat{f} - \frac{1}{\rho_0}\mathcal{H}_s \cdot \frac{\partial f}{\partial \mathbf{S}}, \tag{3.3-20}$$

and conclude with the aid of (3.2-36) and (3.3-2) that (3.3-20) may be written as

$$\check{f} = \frac{\partial f}{\partial \mathbf{S}} \cdot \check{\mathbf{S}}, \tag{3.3-21}$$

where $\check{\mathbf{S}}$ is the quasi-local stress rate defined by (3.3-1). Substituting (3.3-20) into (3.3-19) yields

$$\check{f} = \hat{g} \;\; \text{if} \; \mathcal{U}' = 0. \tag{3.3-22}$$

Now, in an elastic state $f = g < 0$. Then by (3.2-20) $\mathcal{U}' = 0$, and hence $\check{f} = \hat{g}$ in view of (3.3-22). Due to (3.2-19) and (3.3-22) it is seen that in an elastic state, during unloading or neutral loading from a plastic state, the strain space conditions imply the following corresponding stress space conditions

$$\left.\begin{array}{lll} g < 0 & \Rightarrow & f < 0 \\ g = 0, \; \hat{g} \leq 0 & \Rightarrow & f = 0, \; \check{f} \leq 0. \end{array}\right\} \tag{3.3-23}$$

In view of (3.3-20) and (3.3-23) we alternatively conclude that the strain space conditions $(3.3\text{-}23)_2$ imply associated stress space conditions of the form

$$g = 0, \;\; \hat{g} \leq 0 \Rightarrow f = 0 \;, \; \hat{f} \leq \frac{1}{\rho_0}\mathcal{H}_s \cdot \frac{\partial f}{\partial \mathbf{S}}, \tag{3.3-24}$$

where the second inequality in view of (3.2-39) explicitly reads (note that $\mathcal{H}_s$ is restricted by the condition $\dot{\mathcal{U}}' = 0$),

$$\hat{f} \leq \frac{\partial f}{\partial \mathbf{S}} \int_B \{ \frac{\partial \tilde{\mathbf{S}}}{\partial < \mathcal{U}' >} \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z})} \dot{\mathcal{U}}'(\mathbf{Z}) \} dV(\mathbf{Z}). \tag{3.3-25}$$

As seen from (3.3-25), it is clear that the inner product $\mathcal{H}_s \cdot \partial f/\partial \mathbf{S}$ may be of any sign, and hence no statement can be made about the sign of $\hat{f}$, which geometrically establishes the orientation of the stress rate $\dot{\mathbf{S}}$ in relation to the tangent plane to the yield surface in stress space. Thus a trajectory in strain space intersects the yield surface $\partial\mathcal{E}$, and is moving either in an inward direction ($g = 0$, $\hat{g} < 0$) or is tangent to $\partial\mathcal{E}$ ($g = 0$, $\hat{g} = 0$)[3.3-3]. This, however does not apply to the corresponding trajectory in stress space, since the function $\hat{f}$ may be positive and the stress trajectory directed outwards, causing the yield surface $\partial\mathcal{S}$ to move locally outwards.

It remains to discuss the case of loading from a plastic state ($g = 0$, $\hat{g} > 0$). For that purpose we define a dimensionless, rate independent function $\check{\Phi}$ by

$$\check{\Phi} = 1 + \frac{1}{\rho_0}\frac{\partial < \rho_0\tilde{\mathbf{S}} >}{\partial \mathcal{U}'}\pi\Lambda'\frac{\partial f}{\partial \mathbf{S}}, \qquad (3.3-26)$$

where Λ' is defined by (3.2-18), and note in view of (3.3-9) that

$$\check{\Phi} = 1 + \pi\check{\boldsymbol{\sigma}} \cdot \frac{\partial f}{\partial \mathbf{S}}. \qquad (3.3-27)$$

Comparison with (3.3-11) shows that

$$\det \check{\mathcal{K}} = \check{\Phi}. \qquad (3.3-28)$$

The significance of the function $\check{\Phi}$ will be discussed in Subsection 3.3.3. First however, assuming loading ($g = 0$, $\hat{g} > 0$), define a function Φ by the relation

$$\Phi = \check{\Phi} + \frac{1}{\rho_0\hat{g}}\mathcal{H}_s \cdot \frac{\partial f}{\partial \mathbf{S}}\ , \qquad (3.3-29)$$

and note from (3.3-17), (3.3-11) and (3.3-26) that

$$\det \mathcal{K} = \Phi. \qquad (3.3-30)$$

What was previously said about fundamental differences between $\check{\mathcal{K}}$ and $\mathcal{K}$ obviously also applies for their determinants, i.e. Φ being rate independent only in a weak sense.

From (3.3-29), (3.3-26) and (3.2-47) it follows that

$$\Phi = \frac{\hat{f}}{\hat{g}} \quad (g = 0,\ \hat{g} > 0). \qquad (3.3-31)$$

[3.3-3] Note that $\partial\mathcal{S}$ in general is not stationary, as was discussed in Section 3.2.2, *Remark 2.*

The function Φ, called the *nonlocal strain hardening modulus*, is used to characterize the material response for an elastic-plastic body during loading. We state that the material is

$$\left.\begin{array}{ll} a) & hardening\ precisely\ if\ \Phi > 0, \\ b) & softening\ precisely\ if\ \Phi < 0, \\ c) & exhibiting\ perfectly\ plastic\ behaviour\ if\ \Phi = 0. \end{array}\right\} \qquad (3.3-32)$$

In view of (3.3-24), (3.3-31) and (3.3-32) the difference between stress space and strain space formulations is highlighted, the loading criteria in strain space giving rise to associated conditions in stress space of basically different form. We note that the stress space conditions do not unambiguously imply those of strain space. For example, the condition $f = 0$, $\hat{f} > 0$ (and then necessarily $1/\rho_0 \mathcal{H}_s \cdot \partial f/\partial \mathbf{S} > 0$ in view of (3.3-24)), may represent unloading from an elastic-plastic state ($g = 0$, $\hat{g} < 0$) as well as hardening behaviour during loading ($g = 0$, $\hat{g} > 0$). Hence the presence of the quantity $\mathcal{H}_s \cdot \partial f/\partial \mathbf{S}$ leads to ambiguity in the associated stress space formulation, not only in the softening region (as in local theory), but also in the hardening regime[3.3−4].

Remark 3.3. Alternatively it is possible to use $\hat{f}$ to characterize the different types of material response appearing in (3.3-32). However, it seems to be preferable in general to use Φ and not $\hat{f}$, since the former is rate independent (in a weak sense), while the latter is not.□

Other expressions for Φ may also be useful. For example, using (3.2-51) and (3.3-81) we conclude that

$$\Phi = -\left(\frac{1}{\rho_0}\frac{\partial < \rho_0 f >}{\partial \mathcal{U}'}\pi\Lambda' + \frac{1}{\rho_0 \hat{g}}\mathcal{H}_f\right), \qquad (3.3-33)$$

where it is understood that $\mathcal{H}_f$ should be expressed in a form similar to that of $\mathcal{H}_g$ in (3.2-30). (Cf. the related discussion in Section 3.2.3.)

Further, in view of (3.3-31) and (3.3-32), we conclude that during loading ($g = 0$, $\hat{g} > 0$) in the hardening or softening regime the reduced flow rules $(3.2\text{-}19)_1$ in terms of $\hat{g}$ are given by

$$\dot{\mathcal{U}}' = \pi\frac{\hat{f}}{\Phi}\Lambda'. \qquad (3.3-34)$$

[3.3−4] Again it is emphasized that this nonequivalence does not mean that stress space should be ruled out as a permissible base for the definition of the yield function, but obviously, as far as nonlocal plasticity is concerned, the loading criteria must accordingly be defined with regard to the direction of loading. (Note that $\hat{g}$ by use of (3.3-5) may be written $\hat{g} = \mathcal{L}^T\ \partial f/\partial \mathbf{S} \cdot \dot{\mathbf{E}}$.) Also cf. the discussion in Section 3.1.)

In a region of perfectly plastic behaviour, it is of course not possible to express $\dot{\mathcal{U}}'$ in terms of $\hat{g}$: (3.2-19) must be used. The set of equations given by (3.3-34) should be looked upon as flow rules for loading in stress space, to be employed e.g. in cases where tractions and not displacements are prescribed as boundary values.

Returning to the definition of the elastic-plastic response function $\mathcal{K}$ we first note by (3.3-11) and (3.3-27) that

$$\check{\mathcal{K}}^T \frac{\partial f}{\partial \mathbf{S}} = \check{\Phi} \frac{\partial f}{\partial \mathbf{S}}, \qquad (3.3-35)$$

and hence by (3.3-17) and (3.3-29) that

$$\mathcal{K}^T \frac{\partial f}{\partial \mathbf{S}} = \Phi \frac{\partial f}{\partial \mathbf{S}}\,. \qquad (3.3-36)$$

Thus the normal to the yield surface in stress space is an eigenvector of $\check{\mathcal{K}}^T$ as well as $\mathcal{K}^T$ with eigenvalues $\check{\Phi}$ and Φ, respectively.

From (3.3-36) we conclude that

$$\Phi = \frac{\dfrac{\partial f}{\partial \mathbf{S}} \cdot \mathcal{K} \dfrac{\partial f}{\partial \mathbf{S}}}{\dfrac{\partial f}{\partial \mathbf{S}} \cdot \dfrac{\partial f}{\partial \mathbf{S}}}, \qquad (3.3-37)$$

from which it is clear that the scalar $\partial f/\partial \mathbf{S} \cdot \mathcal{K} \partial f/\partial \mathbf{S}$ is positive when the material is hardening, negative when it is softening and zero when it is behaving perfectly plastic. For the case of perfectly plastic behaviour we alternatively note that

$$\mathcal{K}^T \frac{\partial f}{\partial \mathbf{S}} = \mathbf{0}, \qquad (3.3-38)$$

as a direct result from (3.3-36).

3.3.3 An equivalent formulation using quasi-local quantities

In the beginning of this section we defined the quasi-local stress rate $\check{\mathbf{S}}$, and the corresponding quasi-local response function $\check{\mathcal{K}}$ (see (3.3-1) and (3.3-11)). It is important to note that (3.3-13) is merely an alternative way of expressing the stress rate - strain rate relation (3.3-16) as is seen by imposing the conditions (3.3-2) and (3.3-17). Similarily, through (3.3-29), we can use $\check{\Phi}$ (defined by (3.3-26)) instead of the strain hardening modulus Φ to define the different types of strain hardening. Because of its definition and its connection to Φ we call $\check{\Phi}$ the *quasi-local strain hardening modulus.* We also refer to $\check{f}$ - defined by (3.3-20) - as a quasi-local quantity, and note that we may provide a geometrical interpretation of $\check{f}$ similar to that of $\hat{f}$ given in Section 3.2.4. From (3.3-21) namely, it is clear that $\check{f}$ establishes the orientation of the quasi-local stress rate relative to the tangent plane to the yield surface ∂S in stress space.

The quasi-local quantities mentioned above and their nonlocal counterparts are related to each other through simple relationships. For convenience they are repeated below:

$$\left.\begin{aligned}
\check{\mathbf{S}} &= \dot{\mathbf{S}} - \frac{1}{\rho_0}\,\mathcal{H}_s, && [(3.3\text{-}2)]\\
\check{\mathcal{K}} &= \mathcal{K} - \frac{1}{\rho_0 \hat{g}}\,\mathcal{H}_s \otimes \frac{\partial f}{\partial \mathbf{S}}, && [(3.3\text{-}17)]\\
\check{f} &= \hat{f} - \frac{1}{\rho_0}\,\mathcal{H}_s \cdot \frac{\partial f}{\partial \mathbf{S}}, && [(3.3\text{-}20)]\\
\check{\Phi} &= \Phi - \frac{1}{\rho_0 \hat{g}}\,\mathcal{H}_s \cdot \frac{\partial f}{\partial \mathbf{S}}, && [(3.3\text{-}29)]
\end{aligned}\right\} \qquad (3.3-39)$$

From $(3.3\text{-}39)_3$, $(3.3\text{-}39)_4$ and (3.3-31) it is seen that

$$\check{\Phi} = \frac{\check{f}}{\hat{g}}, \qquad (3.3-40)$$

loading ($g = 0$, $\hat{g} > 0$) presupposed. It appears, in view of (3.3-23), that during unloading it is the use of $\check{f}$ (and not $\hat{f}$) that maintains a local structure of the relationship between strain space conditions and corresponding stress space conditions. During loading, however, this is not true. For example $(3.3\text{-}32)_1$ should be replaced by

$$\check{f} + \frac{1}{\rho_0}\mathcal{H}_s \cdot \frac{\partial f}{\partial \mathbf{S}} > 0, \qquad (3.3-41)$$

as seen from (3.3-31) and $(3.3\text{-}39)_2$ (or from (3.3-40) and $(3.3\text{-}39)_4$).

For certain loading directions in elastic-plastic materials of special constitution it may happen that the scalar product $\mathcal{H}_s \cdot \partial f / \partial \mathbf{S}$ vanishes, which by $(3.3\text{-}39)_1$ happens if

$$(\dot{\mathbf{S}} - \check{\mathbf{S}}) \cdot \frac{\partial f}{\partial \mathbf{S}} = 0. \qquad (3.3-42)$$

Then $\check{f} = \hat{f}$, $\check{\Phi} = \Phi$ and the apparent local structure of the relationship between strain space conditions and stress space conditions is preserved during unloading as well as loading.

Though extremely convenient, enforcing (3.3-42) would restrict the theory far beyond what is reasonable and will not be done here[3.3-5]. However, in view of (3.2-39) and (3.2-41), $\mathcal{H}_s$ may be regarded as some measure of the deviation of actual rates from average ones due to plastic deformation throughout the body. If the components of this tensorial measure are small, then each of the quasi-local quantities coincides with its nonlocal counterpart in an approximate sense[3.3-6]. The corresponding *approximate* formulation with $\dot{\mathbf{S}}$, $\mathcal{K}$, $\hat{f}$ and Φ simply replaced by $\check{\mathbf{S}}$, $\check{\mathcal{K}}$, $\check{f}$ and $\check{\Phi}$ will be referred to as quasi-local. Notice that a quasi-local formulation in general is not local, not only because of the nonlocal character of each of the quasi-local functions (as ascertained e.g. by (3.3-1)), but also due to the fact that $\mathcal{H}_s = \mathbf{0}$ does not imply that $\mathcal{H}_g$ or $\mathcal{H}_f$ will vanish. (Recall e.g. how the appearance of a nonvanishing functional $\mathcal{H}_g$ in (3.2-31) highlights the nonlocal character of the consistency condition.)

Remark 3.4. If $\mathcal{H}_s$ happens to vanish identically, the material response is of a quasi-local nature, but nevertheless the formulation is exact. A trivial case for which $\mathcal{H}_s$ vanishes identically, is when the stress-strain relation is of purely local form, as e.g. when

$$\mathbf{S} = \mathcal{L}(\mathbf{E} - \mathbf{E}^p), \qquad (3.3-43)$$

$\mathcal{L}$ being constant. Here (3.3-43) represents a new type of restricted nonlocality (cf. Section 3.2.1), where not only kinematical strain but also plastic strain appears in local form, leaving the strain hardening function as the only nonlocal variable to be present in the constitutive expression for the yield function. The idea of treating yielding as nonlocal but to keep stress local has recently been used in gradient plasticity theories; see e.g. de Borst and Mühlhaus (1992). (It is recalled that gradient materials are not really nonlocal according to our terminology, cf. Section 1.2).□

3.3.4 Inverse relations

In the preceding subsection we derived an expression for $\check{\mathbf{S}}$, the quasi-local rate of stress tensor, in terms of the rate of strain tensor (see (3.3-13)). Here we address our interest to the question of the existence of an inverse relation, i.e. the rate of strain expressed in terms of the quasi-local rate of stress. (Note that the stress rate - strain rate relation (3.3-16) cannot be strictly inverted, since the response function $\mathcal{K}$ is independent of $\dot{\mathbf{E}}$ only in a weak sense.)

We conclude from (3.3-5) that, in an elastic state and during unloading or neutral

[3.3-5]For the class of general nonlocal materials considered in Section 3.6, (3.3-42) is satisfied (with exceptions of no practical interest) only for homogeneous motions.

[3.3-6]It is basically assumed that the components of $\mathcal{H}_s$ are bounded for all loading directions throughout B.

loading from a plastic state,

$$\dot{\mathbf{E}} = \mathcal{M}\check{\mathbf{S}}, \tag{3.3-44}$$

where the fourth order tensor $\mathcal{M}$ satisfies

$$\mathcal{L}\,\mathcal{M} = \mathcal{M}\,\mathcal{L} = \mathcal{J}, \tag{3.3-45}$$

with $\mathcal{J}$ defined by (3.3-12). Of course $\mathcal{M}$, like $\mathcal{L}$ in (3.3-5), is restricted by the condition that $\dot{\mathcal{U}}'$ vanishes at $\mathbf{X}$.

During loading, if the quasi-local strain hardening modulus $\check{\Phi}$ does not vanish, it follows from (3.3-8), (3.3-40) and (3.2-20) that

$$\check{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}} + \frac{\pi}{\check{\Phi}}\,\check{\boldsymbol{\sigma}}\,\frac{\partial f}{\partial \mathbf{S}} \cdot \check{\mathbf{S}}, \tag{3.3-46}$$

or

$$\mathcal{L}\dot{\mathbf{E}} = (\mathcal{J} - \frac{\pi}{\check{\Phi}}\,\check{\boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \mathbf{S}})\check{\mathbf{S}}. \tag{3.3-47}$$

Hence, using (3.3-45), we can write (3.3-47) in the form

$$\dot{\mathbf{E}} = \mathcal{M}\check{\mathcal{N}}\check{\mathbf{S}}, \tag{3.3-48}$$

where

$$\check{\mathcal{N}} = \mathcal{J} - \frac{\pi}{\check{\Phi}}\,\check{\boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \mathbf{S}}. \tag{3.3-49}$$

A simple calculation, using (3.3-49), (3.3-11) and (3.3-27), shows that

$$\check{\mathcal{K}}\check{\mathcal{N}} = \check{\mathcal{N}}\check{\mathcal{K}} = \mathcal{J}. \tag{3.3-50}$$

In correspondance with (3.3-35), we have

$$\check{\mathcal{N}}^T \frac{\partial f}{\partial \mathbf{S}} = \frac{1}{\check{\Phi}}\frac{\partial f}{\partial \mathbf{S}}, \tag{3.3-51}$$

which follows immediately from (3.3-35) and (3.3-50) (or directly from (3.3-49) and (3.3-27)). Thus $\partial f/\partial \mathbf{S}$ is an eigenvector of the transpose of $\check{\mathcal{N}}$ with eigenvalue $1/\check{\Phi}$.

From (3.3-48) we note that the strain rate vanishes with zero quasi-local stress rate. Consequently, as long as (3.3.48) is valid, there is no direction of loading for which the quasi-local stress rate vanishes. Conversely, if there indeed is a direction of loading which results in zero quasi-local stress rate, then the material response at $\mathbf{X}$ necessarily must correspond to zero quasi-local strain hardening modulus $\check{\Phi}$, as is also evident in view of (3.3-21) and (3.3-40). It is emphasized that these statements apply to the quasi-local stress rate. It might also be possible to find a loading direction at $\mathbf{X}$, such that the actual stress rate indeed vanishes at this point. Then, it is seen from (3.2-36) and (3.3-31) that the material response during loading ($g = 0$, $\hat{g} > 0$) necessarily must be perfectly plastic. In view of (3.3-2) such a loading direction exists exactly if the strain rate field satisfies the integral equation (assuming (3.3-48) to be valid)

$$\dot{\mathbf{E}} + \frac{1}{\rho_0} \mathcal{M} \, \check{\mathcal{N}} \, \mathcal{H}_s = \mathbf{0} \qquad (3.3-52)$$

or equivalently, by the aid of (3.3-45) and (3.3-50),

$$\rho_0 \, \check{\mathcal{K}} \, \mathcal{L} \, \dot{\mathbf{E}} + \mathcal{H}_s = \mathbf{0}. \qquad (3.3-53)$$

The last equation can of course be obtained directly from (3.3-13) and (3.3-2), and applies also when $\check{\Phi} = 0$. From (3.3-53) and (3.2-41) it follows that the strain rate field must satisfy

$$\int_B \rho_0 \, \check{\mathcal{K}} \, \mathcal{L} \, \dot{\mathbf{E}} \, dV = \mathbf{0}, \qquad (3.3-54)$$

as a necessary condition for the stress rate to vanish at some point. In an elastic state and during unloading or loading from a plastic state, a possible loading direction corresponding to a vanishing stress rate must satisfy (3.3-52) or (3.3-53) with $\check{\mathcal{N}}$ and $\check{\mathcal{K}}$ replaced by the unity tensor $\mathcal{J}$.

We now turn our interest to the general stress rate - strain rate relation (3.3-16) involving the reponse function $\mathcal{K}$, which is rate independent only in a weak sense as discussed previously. Then, to what extent can (3.3-16) be solved for the strain rate $\dot{\mathbf{E}}$? Needless to say, we cannot expect a solution of the same type as the one given by (3.3-48), where the strain rate is a linear function of the quasi-local stress rate. However, we may proceed similarly to the derivation of (3.3-48), writing the stress rate (3.3-2) in the form

$$\dot{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}} + \frac{\pi}{\Phi} \check{\boldsymbol{\sigma}} \frac{\partial f}{\partial \mathbf{S}} \cdot \dot{\mathbf{S}} + \frac{\mathcal{H}_s}{\rho_0 \hat{g} \Phi} \frac{\partial f}{\partial \mathbf{S}} \cdot \dot{\mathbf{S}}, \qquad (3.3-55)$$

where (3.3-7), (3.3-31) and (3.2-36) have been used. Loading is presupposed and Φ must not vanish, i.e. (3.3-55) is not valid during perfectly plastic behaviour. Hence

$$\mathcal{L}\dot{\mathbf{E}} = (\mathcal{J} - \frac{1}{\Phi}(\pi\check{\dot{\boldsymbol{\sigma}}} + \frac{1}{\rho_0\hat{g}}\mathcal{H}_s) \otimes \frac{\partial f}{\partial \mathbf{S}})\dot{\mathbf{S}}, \qquad (3.3-56)$$

or

$$\dot{\mathbf{E}} = \mathcal{M}\,\mathcal{N}\,\dot{\mathbf{S}}, \qquad (3.3-57)$$

where (3.3-45) has been used and where

$$\mathcal{N} = \mathcal{J} - \frac{1}{\Phi}(\pi\check{\dot{\boldsymbol{\sigma}}} + \frac{1}{\rho_0\hat{g}}\mathcal{H}_s) \otimes \frac{\partial f}{\partial \mathbf{S}}. \qquad (3.3-58)$$

Similarly to (3.3-50) and (3.3-51) it is seen that

$$\mathcal{K}\,\mathcal{N} = \mathcal{N}\,\mathcal{K} = \mathcal{J} \qquad (3.3-59)$$

and that

$$\mathcal{N}^T\frac{\partial f}{\partial \mathbf{S}} = \frac{1}{\Phi}\frac{\partial f}{\partial \mathbf{S}}, \qquad (3.3-60)$$

$\partial f/\partial \mathbf{S}$ being an eigenvector of the transpose of $\mathcal{N}$ with eigenvalue $1/\Phi$.

It must be noted that (3.3-57) is still an integral equation for the strain rate $\dot{\mathbf{E}}$ (as is (3.3-16)), since $\mathcal{N}$ is independent of strain rates only in a weak sense due to the occurrence of the function $(1/\rho_0\hat{g})\mathcal{H}_s$ in the right-hand-side of (3.3-58), (explicit as well as implicit through the strain hardening modulus Φ).

We can arrive at expressions similar to (3.3-57) in several other ways. For example, let us use (3.3-2) and write (3.3-48) in the form

$$\dot{\mathbf{E}} = \mathcal{M}\check{\mathcal{N}}(\dot{\mathbf{S}} - \frac{1}{\rho_0}\mathcal{H}_{\mathbf{S}}). \qquad (3.3-61)$$

We recall that $\check{\Phi} \neq 0$ in (3.3-61) (and of course that $\hat{g} > 0$). If we also assume that $\Phi \neq 0$ we conclude by the aid of (3.3-31) and (3.2-36) that

$$\begin{aligned} \dot{\mathbf{E}} &= \mathcal{M}\check{\mathcal{N}}(\dot{\mathbf{S}} - \frac{1}{\rho_0\hat{g}\Phi}\frac{\partial f}{\partial \mathbf{S}} \cdot \dot{\mathbf{S}}\;\mathcal{H}_{\mathrm{s}}) \\ &= \mathcal{M}\check{\mathcal{N}}(\mathcal{J} - \frac{1}{\rho_0\hat{g}\Phi}\mathcal{H}_s \otimes \frac{\partial f}{\partial \mathbf{S}})\dot{\mathbf{S}}, \end{aligned} \qquad (3.3\text{- }62)$$

or

$$\dot{\mathbf{E}} = \mathcal{M}\bar{\mathcal{N}}\dot{\mathbf{S}}, \tag{3.3 – 63}$$

with

$$\begin{aligned}\bar{\mathcal{N}} &= \check{\mathcal{N}}(\mathcal{J} - \frac{1}{\rho_0 \hat{g}\Phi}\mathcal{H}_s \otimes \frac{\partial f}{\partial \mathbf{S}}) \\ &= \check{\mathcal{N}} - \frac{\check{\Phi}}{\rho_0 \hat{g}\Phi}\check{\mathcal{N}}\ \mathcal{H}_s \otimes \check{\mathcal{N}}^T \frac{\partial f}{\partial \mathbf{S}},\end{aligned} \tag{3.3- 64}$$

where (3.3-51) has been used to obtain the second equality.

From (3.3-57) and (3.3-63) it is seen that

$$\bar{\mathcal{N}} = \mathcal{N}_{\check{\phi}\neq 0}, \tag{3.3 – 65}$$

where $\mathcal{N}_{\check{\Phi}\neq 0}$ denotes the restriction of $\mathcal{N}$ to strain hardening behaviour with $\check{\Phi} \neq 0$. (Note that $\Phi \neq 0$ from the outset.) A direct proof (from (3.3-36) and (3.3-64)) of (3.3-65) is straightforward and can be performed in various ways. Starting from the expression for $\mathcal{N}$ found in (3.3-56), we may e.g. proceed as follows:

$$\begin{aligned}\mathcal{N}_{\check{\phi}\neq 0} &= \mathcal{J} - \frac{1}{\check{\Phi}}\frac{\check{\Phi}}{\Phi}\pi\check{\check{\boldsymbol{\sigma}}} \otimes \frac{\partial f}{\partial \mathbf{S}} - \frac{1}{\rho_0 \hat{g}\Phi}\mathcal{H}_s \otimes \frac{\partial f}{\partial \mathbf{S}} \\ &= \mathcal{J} - \frac{1}{\check{\Phi}}(1 - \frac{1}{\rho_0 \hat{g}\Phi}\mathcal{H}_s \cdot \frac{\partial f}{\partial \mathbf{S}})\pi\check{\check{\boldsymbol{\sigma}}} \otimes \frac{\partial f}{\partial \mathbf{S}} - \frac{1}{\rho_0 \hat{g}}\mathcal{H}_s \otimes \frac{\partial f}{\partial \mathbf{S}},\end{aligned} \tag{3.3- 66}$$

where $(3.3\text{-}39)_4$ has been used. From (3.3-49) it then follows that

$$\begin{aligned}\mathcal{N}_{\check{\Phi}\neq 0} &= \check{\mathcal{N}} + \frac{\mathcal{H}_s \cdot \frac{\partial f}{\partial \mathbf{S}}}{\rho_0 \hat{g}\Phi\check{\Phi}}\pi\check{\check{\boldsymbol{\sigma}}} \otimes \frac{\partial f}{\partial \mathbf{S}} - \frac{1}{\rho_0 \hat{g}\Phi}\mathcal{H}_s \otimes \frac{\partial f}{\partial \mathbf{S}} \\ &= \check{\mathcal{N}} + \frac{\check{\Phi}}{\rho_0 \hat{g}\Phi}(\frac{1}{\check{\Phi}}\pi\check{\check{\boldsymbol{\sigma}}} \otimes \frac{\partial f}{\partial \mathbf{S}}\mathcal{H}_s \cdot \check{\mathcal{N}}^T \frac{\partial f}{\partial \mathbf{S}} - \mathcal{H}_s \otimes \check{\mathcal{N}}^T \frac{\partial f}{\partial \mathbf{S}}),\end{aligned} \tag{3.3- 67}$$

where (3.3-51) has also been used. Hence[3.3-7]

[3.3-7] To obtain (3.3-68) we have taken advantage of the second equality of the identity $\mathbf{A} \otimes \mathbf{B}\ \mathbf{C} \cdot \mathbf{D} = \mathbf{A} \otimes \mathbf{C}\ \mathbf{D} \otimes \mathbf{B} = \mathbf{A} \otimes \mathbf{D}\ \mathbf{C} \otimes \mathbf{B}$ valid for second order tensors **A**, **B**, **C**, **D**.

$$\mathcal{N}_{\check{\Phi}\neq 0} = \check{\mathcal{N}} + \frac{\check{\Phi}}{\rho_0\hat{g}\Phi}(\frac{1}{\check{\Phi}}\pi\check{\check{\boldsymbol{\sigma}}} \otimes \frac{\partial f}{\partial \mathbf{S}} - \mathcal{J})\mathcal{H}_s \otimes \check{\mathcal{N}}^T\frac{\partial f}{\partial \mathbf{S}}$$

$$= \check{\mathcal{N}} - \frac{\check{\Phi}}{\rho_0\hat{g}\Phi}\check{\mathcal{N}}\ \mathcal{H}_s \otimes \check{\mathcal{N}}^T\frac{\partial f}{\partial \mathbf{S}}, \qquad (3.3\text{-}68)$$

where (3.3-49) has been used again. We note that the right-hand side of (3.3-68) equals $\bar{\mathcal{N}}$ defined by (3.3-64), which completes the proof.

Remark 3.5. The result contained in (3.3-68) may also be obtained by direct inversion of the response function $\mathcal{K}$ under the assumption that neither Φ nor $\check{\Phi}$ vanishes. From $(3.3\text{-}39)_2$ and (3.3-59) and by use of elementary algebra[3.3-8] we conclude that

$$\mathcal{N}_{\check{\Phi}\neq 0} = \mathcal{K}^{-1}_{\check{\Phi}\neq 0} = (\check{\mathcal{K}} + \frac{1}{\rho_0\hat{g}}\mathcal{H}_s \otimes \frac{\partial f}{\partial \mathbf{S}})^{-1}_{\check{\Phi}\neq 0}$$

$$= \check{\mathcal{K}}^{-1} - \frac{\det\check{\mathcal{K}}}{\det\mathcal{K}}\frac{1}{\rho_0\hat{g}}\check{\mathcal{K}}^{-1}\mathcal{H}_s \otimes (\check{\mathcal{K}}^{-1})^T\frac{\partial f}{\partial \mathbf{S}}$$

$$= \check{\mathcal{N}} - \frac{\check{\Phi}}{\rho_0\hat{g}\Phi}\check{\mathcal{N}}\ \mathcal{H}_s \otimes \check{\mathcal{N}}^T\frac{\partial f}{\partial \mathbf{S}}, \qquad (3.3\text{-}69)$$

where (3.3-28), (3.3-30) and (3.3-50) have been used in deriving the last equality.

Remark 3.6. As long as (3.3-57) is valid, keeping in mind that $\mathcal{N}$ is assumed to be bounded, it is evident that there is no direction of loading for which the stress rate vanishes. Hence, as already mentioned (in relation to (3.3-52)), a direction of loading resulting in vanishing stress rate necessarily corresponds to perfectly plastic behaviour (loading ($g = 0$, $\hat{g} > 0$) presupposed).□

3.3.5 Restriction to local theory

It may be enlightening to compare the basic equations derived so far in this chapter with analogous relations obtained in a corresponding local theory. It follows from the discussion in Section 3.2.1 that the constitutive assumptions for stress and yield in (3.2-5), (3.2-8) and (3.2-34) become local if α' satisfies the conditions

$$\frac{\partial\alpha'}{\mathcal{U}'(\mathbf{Z},t)} \equiv 0. \qquad (3.3-70)$$

[3.3-8] The inverse of $\mathcal{C} = \mathcal{J} + \mathbf{A}\otimes\mathbf{B}$ is given by $\mathcal{C}^{-1} = \mathcal{J} - \frac{1}{\det\mathcal{C}}\mathbf{A}\otimes\mathbf{B}$, $\det\mathcal{C} = 1 + \mathbf{A}\cdot\mathbf{B}$. Take $\mathcal{C} = \mathcal{D}^{-1}\mathcal{G}$ and $\mathbf{A} = \mathcal{D}^{-1}\mathbf{E}$, i.e. $\mathcal{G} = \mathcal{D} + \mathbf{E}\otimes\mathbf{B}$ and conclude that $\mathcal{G}^{-1} = \mathcal{C}^{-1}\mathcal{D}^{-1} = \mathcal{D}^{-1} - \frac{\det\mathcal{D}}{\det\mathcal{G}}\mathcal{G}^{-1}(\mathbf{E}\otimes\mathbf{B})\mathcal{D}^{-1}$.

We recall that the set $\alpha' = \{\boldsymbol{\alpha}^p, \alpha^h\}$ contains prescribed functions, which are used to define the members $< \mathbf{E}^p >$ and $< \kappa >$ of $< \mathcal{U}' >$, as explicitly shown in (3.2-6).

We choose in agreement with (3.3-70),

$$\alpha' = \frac{\mathcal{U}'(\mathbf{X}, t)}{V(B)}\ , \tag{3.3 – 71}$$

and hence, as seen from (3.2-6), we recover the basic state functions, i.e.

$$< \mathcal{U}' >= \mathcal{U}'. \tag{3.3 – 72}$$

It follows from (2.4-9) and (3.3-71) that

$$(\alpha')^* = \frac{\mathcal{U}'(\mathbf{Z}, t)}{V(B)}\ , \tag{3.3 – 73}$$

and hence

$$\frac{\partial(\alpha')^*}{\partial \mathcal{U}'} \equiv 0. \tag{3.3 – 74}$$

Substituting (3.3-70) and (3.3-74) into (3.2-21) - (3.2-24) leads to the conditions

$$\frac{\partial(\rho_0 g)^*}{\partial \mathcal{U}'} \equiv 0,\ \mathcal{H}_g \equiv 0, \tag{3.3 – 75}$$

where the first equality implies that in fact it is possible to replace $< \rho_0 g >$ with $\rho_0 g$ wherever it occurs. From (3.2-38) and (3.2-39) it follows similarly that

$$\mathcal{H}_f \equiv 0,\ \boldsymbol{\mathcal{H}}_s \equiv 0. \tag{3.3 – 76}$$

Further it is easily seen that $< \rho_0 f >$ and $< \rho_0 \tilde{\mathbf{S}} >$ should be replaced by $\rho_0 f$ and $\rho_0 \tilde{\mathbf{S}}$, respectively, and in view of (3.3-76) and $(3.3\text{-}39)_1$,

$$\check{f} = \hat{f},\ \check{\mathbf{S}} = \dot{\mathbf{S}},\ \check{\Phi} = \Phi \text{ and } \check{\mathcal{K}} = \mathcal{K}. \tag{3.3 – 77}$$

(Of course the quasi-local formulation coincides with the general formulation since $\boldsymbol{\mathcal{H}}_s = \mathbf{0}$.)

Summarizing, we have shown that results valid in local theory are obtained from the general nonlocal theory by simply discarding the $<\ \ >$ symbols and equating to zero the $\mathcal{H}$-functionals wherever they occur. Also note that the mass density ρ_0 everywhere cancels out.

As an illustration, consider the consistency condition (3.2-29). Following the discussion above we obtain immediately its local counterpart as

$$1 + \pi(\frac{\partial g}{\partial \mathbf{E}^p} \cdot \mathbf{R} + \frac{\partial g}{\partial \kappa} r) = 0. \tag{3.3 - 78}$$

Here π cannot vanish, and without loss in generality we can adopt the inequality $\pi > 0$ as a basic condition in the local theory (contrasting with nonlocal theory as discussed in Subsection (3.2.2)). Hence (3.3-78) can be used to calculate π in terms of the constitutive functions r, $\mathbf{R}$ and g.

As another example, we look at the definition of the tensor $\check{\tilde{\boldsymbol{\sigma}}}$ in (3.3-9), and note that the corresponding local expression reads

$$\tilde{\boldsymbol{\sigma}} = \frac{\partial \tilde{\mathbf{S}}}{\partial \mathbf{E}^p}\mathbf{R} + \frac{\partial \tilde{\mathbf{S}}}{\partial \kappa} r, \;\; \tilde{\boldsymbol{\sigma}} \equiv \check{\tilde{\boldsymbol{\sigma}}}, \tag{3.3 - 79}$$

where $(2.4\text{-}21)_1$ and (3.2-18) have additionally been used. It should be noted that (3.3-79) will be of interest in relation to the notion of normality conditions in local theory, which will be discussed in Chapter 4.

With regard to the discussion of the inverse relation in Subsection 3.3.4 we note that in restriction to local theory the response function $\mathcal{N}$ equals the quasi-local response function $\check{\mathcal{N}}$, and of course that the difference between $\mathcal{N}$ and $\check{\mathcal{N}}$ disappears (see (3.3-64) and (3.3-65)). Hence the local counterparts of (3.3-49), (3.3-59) and (3.3-66) all coincide, and each of them reads

$$\mathcal{N} = \mathcal{J} - \frac{1}{\Phi}\pi\tilde{\boldsymbol{\sigma}} \otimes \frac{\partial f}{\partial \mathbf{S}}. \tag{3.3 - 80}$$

Clearly $\mathcal{N}$ in (3.3-80) is rate independent. Hence, in the identical relationships (3.3-48), (3.3-57) or (3.3-63), the strain rate is expressed as a linear function of the stress rate (valid during hardening and softening but not during perfectly plastic behaviour). As a consequence, a direction of loading that produces a zero stress rate necessarily corresponds to perfectly plastic behaviour (cf. the discussion that precedes (3.3-52)).

3.4 A plastic potential function and the principle of maximum dissipation

Thermodynamic assumptions (like the Clausius-Duhem inequality) are in local, purely mechanical plasticity theory replaced by a work assumption of some kind, such as the postulate of Drucker (1952) or that of Il'iushin (1961). Drucker's postulate, concerning the nonnegativity of work in a cycle of stress, is a stability condition valid for hardening materials only, while the restrictions placed on the constitutive equations by invoking. Il'iushin's postulate, which involves a cycle of strain, remain valid for hardening as well as softening behaviour. From the postulate of either Drucker or Il'iushin, each formulated within the context of linearized theory and small deformations, conditions on normality of plastic strain rate and on convexity of yield surfaces are obtained. Another work assumption is that of Naghdi and Trapp (1975b), similar to Il'iushin's postulate but valid for finite deformations.

The work assumption of Naghdi and Trapp states that the external work done on an elastic-plastic body in any sufficiently homogeneous cycle of deformation is nonnegative, i.e.

$$\int_{t_1}^{t_2} \{ \int_{B} \mathbf{t}^0 \cdot \dot{\mathbf{x}} \, dA + \int_{B} \rho_0 \, \mathbf{f} \cdot \dot{\mathbf{x}} \, dV \} \, dt \geq 0 \qquad (3.4-1)$$

for all strain cycles beginning at time t_1 and ending at time t_2. In (3.4-1) ∂B designates the boundary surface of B, $\mathbf{f}$ the specific body force and $\mathbf{t}^0$ the traction vector measured per unit area of surface in the reference configuration.

The conditions on the motion, being a smooth homogeneous cycle of deformation, imply that it is homogeneous and that each particle of the body has the same position and the same velocity at t_1 and t_2. We recall that in a homogeneous motion the strain tensor $\mathbf{E}$ is independent of $\mathbf{X}$ and hence only a function of time. It is also assumed that mass density ρ_0, plastic strain $\mathbf{E}^p$ and strain hardening function κ do not depend on $\mathbf{X}$ at time t_1, nor do the constitutive functions for the material. Therefore, in a homogeneous motion of an elastic-plastic body which is homogeneous at time t_1, $\mathbf{E}^p$, κ and $\mathbf{S}$, for $t \geq t_1$, are functions of t only[3.4-1].

Using (3.4-1) together with the fact that the kinetic energy takes on the same value at t_1 and t_2 (due to the smoothness condition), leads to the inequality

[3.4-1] Implied e.g. by (3.2-10) and (3.2-11). Note, namely, that in any homogeneous motion of an initially homogeneous elastic-plastic body, each nonlocal equation derived in the present work takes on a purely local form. Thus (3.2-10) and (3.2-11) (or any other equation of relevance) may be applied directly.

$$\int_{t_1}^{t_2} \{ \int_B \mathbf{S} \cdot \dot{\mathbf{E}} \, dV \} dt \geq 0, \tag{3.4-2}$$

where we recall that $\mathbf{S}$ is the symmetric second Piola-Kirchhoff stress tensor. Since $\mathbf{S}$ and $\mathbf{E}$ are independent of $\mathbf{X}$, (3.4-2) can equivalently be written in the form[3.4-2]

$$\int_{t_1}^{t_2} \mathbf{S} \cdot \dot{\mathbf{E}} \, dt \geq 0. \tag{3.4-3}$$

The inequality (3.4-3) has three important consequences, namely

(i) the existence of a scalar potential function $\psi = \tilde{\psi}(\mathcal{U})$ for the stress, i.e.[3.4-3]

$$\mathbf{S} = \rho_0 \frac{\partial \tilde{\psi}}{\partial \mathbf{E}}, \tag{3.4-4}$$

by which is observed that the fourth order tensor $\mathcal{L}$ is symmetric;

(ii) convexity of the yield surface in stress space as well as strain space for a restricted class of elastic-plastic materials, including those for which the response function ψ satisfies $\psi = \bar{\psi}(\mathbf{E} - \mathbf{E}^p)$, i.e. being independent of κ and dependent on $\mathbf{E}^p$ only through the difference $\mathbf{E} - \mathbf{E}^p$;

(iii) a constitutive restriction on the function $\tilde{\boldsymbol{\sigma}}$ of the form

$$\tilde{\boldsymbol{\sigma}} = -\gamma \frac{\partial g}{\partial \mathbf{E}}, \, \gamma = \gamma(\mathcal{U}) \geq 0, \, (g = 0, \, \hat{g} \geq 0), \tag{3.4-5}$$

where the corresponding quasi-local function $\check{\tilde{\boldsymbol{\sigma}}}$ is defined by (3.3-9) and its local form recorded in (3.3-79). The result (3.4-5) is a normality condition for the tensor $\tilde{\boldsymbol{\sigma}}$, being directed parallel to the inward normal to the yield surface $\partial\mathcal{E}$ in strain space. For the class of materials considered in (ii), the condition (3.4-5) implies

$$\dot{\mathbf{E}}^p = \gamma \pi \hat{g} \frac{\partial f}{\partial \mathbf{S}}, \tag{3.4-6}$$

recognized as a classical result of (infinitesimal) plasticity.

The condition (3.4-5) was first proved by Naghdi and Trapp (1975b), while an alternative and simpler proof was provided by Casey (1984). It is important to note that each proof is carried out by invoking (3.4-3) for special choices of *homogeneous* strain cycles. Hence the normality condition (3.4-5) is proved to hold true for homogeneous

[3.4-2] Il'iushin's postulate of plasticity has essentially this form (proposed in the context of linearized theory and small deformations).

[3.4-3] Notations agree with those used previously in this work.

motions only. However, the result is in fact valid in *all* motions, since the constitutive functions appearing in (3.4-5) depend only on the local variables $\mathcal{U}$, and not on their nonlocal counterparts (or their spatial gradients). Therefore it is sufficient to consider homogeneous motions in order to obtain results valid in all motions.

So much for local theory. Evidently it would be highly ineffective to invoke the work assumption of Naghdi and Trapp for nonlocal elastic-plastic bodies, since all characteristic features of nonlocality vanish in homogeneous motions. This means that the normality condition (3.4-5) can still be derived (for homogeneous motions), but it cannot be extended to be valid in all motions, because of the constitutive dependence on the nonlocal state functions.

Looking back at the condition (3.4-1), we note that it is a global statement valid for the entire body, so it seems logical to take this inequality as a starting point for a discussion of work assumptions in nonlocal plasticity. Hence we should require (3.4-1) to hold true for general strain cycles, not only homogeneous ones, such that

$$\mathbf{E}(\mathbf{Z}, t_1) = \mathbf{E}(\mathbf{Z}, t_2) \tag{3.4-7}$$

for all particles in B. The motion should still be assumed to be a smooth cycle of deformation such that for each particle the velocity takes on the same value at t_1 and t_2. Following this concept (3.4-2) can still be derived, but not of course (3.4-3). Thus a plausible nonlocal work assumption should require (3.4-2) to be valid for all smooth deformation cycles which satisfy (3.4-7). However, it is not believed that it will be possible to derive a condition of the type (3.4-5) from such a general statement.

We leave for a moment the purely mechanical theory and look at thermodynamic plasticity. If the Clausius-Duhem inequality (2.3-4) is invoked, the nonlocal counterpart of (i) is represented by (2.4-23). It turns out that it is possible to derive restrictions on the quasi-local function $\check{\boldsymbol{\sigma}}$ which corresponds to (**iii**) by invoking a generalized form of the principle of maximum dissipation in classical plasticity, (Hill 1948)[3.4−4]. We proceed as follows. For isothermal motion, assume $\mathcal{U}'$ a given state. Define a set $\mathcal{A}_{\mathbf{E}}$ of admissible states of the Lagrangian strain $\mathbf{E}$ satisfying the yield condition,

$$\mathcal{A}_{\mathbf{E}} = \{\mathbf{E} |\; g\;(\mathbf{E},\; <\mathcal{U}'>) \leq 0\}. \tag{3.4-8}$$

The principle claims that the *actual* stress tensor $\mathbf{E}$ is the one for which the plastic dissipation D^p (per unit volume) adopts its maximum. In precise formulation, for *given* $\dot{\mathcal{U}}'$,

$$D^p(\mathbf{E},\; <\mathcal{U}'>\;;\; \dot{\mathcal{U}}'>) \geq D^p(\hat{\mathbf{E}},\; <\mathcal{U}'>\;;\; \dot{\mathcal{U}}') \tag{3.4-9}$$

[3.4−4] By Hill credited to von Mises (1928).

for any $\hat{\mathbf{E}} \in \mathcal{A}_{\mathbf{E}}$.

Define

$$\check{D}^p = -\frac{\partial < \rho_0 \check{\psi} >}{\partial \mathcal{U}'} \dot{\mathcal{U}}', \qquad (3.4-10)$$

being a measure of the specific plastic dissipation at unit volume. Upon substitution of (3.4-10) into (2.4-26) and using (2.4-12), it is observed that the total amount $\mathcal{D}^p$ of dissipation in the body is given by

$$\mathcal{D}^p = \int_B \check{D}^p dV = \int_B (\check{D}^p + \mathcal{H}_\psi) dV, \qquad (3.4-11)$$

where $\mathcal{H}_\psi$ is defined by (2.4-11). If we split $\mathcal{H}_\psi$ into parts according to

$$\mathcal{H}_\psi = \mathcal{H}_\psi^{\alpha^e} + \mathcal{H}_\psi^{\alpha'}, \qquad (3.4-12)$$

apparently $\check{D}^p + \mathcal{H}_\psi^{\alpha'}$ is another reasonable measure of specific plastic dissipation.

Being now at a point of departure, we assume as our basic postulate that

$$\check{D}^p \geq 0, \qquad (3.4-13)$$

for every $\dot{\mathcal{U}}'$ at a given state $\mathcal{U} = \mathcal{U}(\mathbf{X}, t)$, $\mathbf{X} \in B$.

Remark 4.1. It is noted that (3.4-13) implies (2.4-26) and hence is a sufficient condition for the global Clausius-Duhem inequality to hold true, while needless to say, it is not necessary.□

Remark 4.2. It is important to see that (3.4-13) does not imply any restriction whatsoever on $\mathcal{H}_\psi^{\alpha'}$, since it is *not* required that an inequality of the type $\check{D}^p + \mathcal{H}_\psi^{\alpha'} \geq 0$ should hold simultaneously.□

Now to the implications of the principle. Choose $\check{D}^p$ defined by (3.4-10) for D^p in (3.4-9) and construct a Lagrangian

$$L^p(\mathbf{E},\ \dot{\lambda}) = -\check{D}^p + \dot{\lambda} g(\mathbf{E},\ < \mathcal{U}' >), \qquad (3.4-14)$$

where $\dot{\lambda} \geq 0$ is a Lagrangian multiplier. The condition (3.4-9) is now enforced by solving the associated minimization problem implied by (3.4-14), i.e.

$$\frac{\partial L^p}{\partial \mathbf{E}} = 0, \quad \dot{\lambda} \geq 0, \quad \dot{\lambda} g(\mathbf{E},\ < \mathcal{U}' >) = 0. \qquad (3.4-15)$$

From (3.4-10), (3.4-14) and (3.4-15) it then follows that

$$\frac{\partial^2 < \rho_0 \tilde{\psi} >}{\partial \mathbf{E} \partial \mathcal{U}'} \dot{\mathcal{U}}' + \dot{\lambda} \frac{\partial g}{\partial \mathbf{E}} = 0, \qquad (3.4-16)$$

$$\dot{\lambda} \geq 0, \; g(\mathbf{E}, \; < \mathcal{U}' >) \leq 0, \quad \dot{\lambda} g(\mathbf{E}, \; < \mathcal{U}' > = 0, \qquad (3.4-17)$$

where (3.4-17) represents the loading/unloading conditions (in Kuhn-Tucker form). The restriction on $\tilde{\psi}$ implied by the condition (3.4-16) has a simple interpretation. Firstly, note from (2.4-16), (2.4-17) and (2.4-19) that

$$\frac{\partial^2 < \rho_0 \tilde{\psi}}{\partial \mathbf{E} \partial \mathcal{U}'} = \int_B \{ \rho_0 \frac{\partial^2 \tilde{\psi}}{\partial \mathbf{E} \partial < \mathcal{U}' >} \frac{\partial \alpha'}{\partial \mathcal{U}'} + \rho_0(\mathbf{Z}) \frac{\partial^2 \tilde{\psi}}{\partial \mathbf{E} \partial < \mathcal{U}' >} (\mathbf{Z}) \frac{\partial^2 (\alpha')^*}{\partial \mathcal{U}'} \} dV(\mathbf{Z}). \qquad (3.4-18)$$

Secondly, assume that

$$\frac{\partial^2 \tilde{\psi}}{\partial \mathbf{E} \partial < \mathcal{U}' >} = \frac{\partial^2 \tilde{\psi}}{\partial < \mathcal{U}' > \partial \mathbf{E}}, \qquad (3.4-19)$$

and use $(2.4\text{-}23)_2$ to conclude that

$$\frac{\partial^2 < \rho_0 \tilde{\psi} >}{\partial \mathbf{E} \partial < \mathcal{U}' >} = \frac{1}{\rho_0} \int_B \{ \rho_0 \frac{\partial \tilde{\mathbf{S}}}{\partial < \mathcal{U}' >} \frac{\partial \alpha'}{\partial \mathcal{U}'}$$

$$+ \rho_0(\mathbf{Z}) \frac{\partial \tilde{\mathbf{S}}}{\partial < \mathcal{U}' >} (\mathbf{Z}) \frac{\partial (\alpha')^*}{\partial \mathcal{U}'} \} dV(\mathbf{Z}) = \frac{1}{\rho_0} \frac{\partial < \rho_0 \tilde{\mathbf{S}} >}{\partial \mathcal{U}'}, \qquad (3.4-20)$$

where we recall that $< \rho_0 \tilde{\mathbf{S}} >$ is defined similar to $< \rho_0 \tilde{\psi} >$ (also cf. (3.2-37))[3.4-5]. Using $(3.2\text{-}19)_1$, (3.3-9) and (3.4-20) we note that (3.4-16) can be cast into the form

$$\pi \hat{g} \check{\dot{\boldsymbol{\sigma}}} + \dot{\lambda} \frac{\partial g}{\partial \mathbf{E}} = \mathbf{0}. \qquad (3.4-21)$$

Since not all components of $\partial g / \partial \mathbf{E}$ are zero, we conclude from (3.4-21) that π cannot vanish for $\dot{\lambda} > 0$. Without loss in generality, we assume that

$$\pi > 0. \qquad (3.4-22)$$

[3.4-5]To arrive at $(3.4\text{-}20)_1$ we have in fact used $\rho(\mathbf{Z}) = \rho_0$. The assumption that the body is initial homogeneous is however not essential, but is used merely to simplify notations. To write the right-hand side of $(3.4\text{-}20)_2$ as $\partial < \tilde{\mathbf{S}} > / \partial \mathcal{U}'$ would certainly lead to confusion. Anyhow we must have in mind that (3.4-20) in this form is only valid for initial homogeneous bodies.

Hence, (3.4-21) may be written as

$$\overset{\times}{\dot{\boldsymbol{\sigma}}} = -\frac{\dot{\lambda}}{\pi \hat{g}} \frac{\partial g}{\partial \mathbf{E}} = -\gamma \frac{\partial g}{\partial \mathbf{E}}, \qquad (3.4-23)$$

where $\gamma = \gamma(\mathbf{E}, < \mathcal{U}' >)$ is a positive function of the arguments indicated. Thus the restriction (3.4-23) on the quasi-local function $\overset{\times}{\dot{\boldsymbol{\sigma}}}$ has identically the same form as the normality condition (3.4-5) in local theory[3.4-6].

Remark 4.3. We recall (cf. the discussion in Section 3.2.2) that in nonlocal plasticity the consistency condition cannot be used to solve for the function π, as is the case in local theory. Neither does the consistency condition place restrictions on π of the type recorded in (3.4-22). Needless to say, the global condition (3.2-32) does not imply (3.4-22).

As noted previously the work assumption of Naghdi and Trapp in the form (3.4-3) is not admissible with regard to nonlocal bodies in nonhomogeneous motions. Only by using arguments based on thermodynamics has it been possible to derive restrictions of the type (3.4-21) or (3.4-22) on the constitutive functions.□

We are not intended to restrict the general theory by merely considering such classes of materials for which the principle of maximum dissipation applies. Hence we do not require (3.4-21) to hold true for all rate independent elastic-plastic materials. However, we will adopt (3.4-22)[3.4-7].

A nonlocal plastic potential will be assumed to exist. In Subsection 3.4.1 below a potential function is specified, by which reduced forms for the quasi-local elastic-plastic response function and strain hardening modulus, respectively, are obtained.

Special results for perfectly plastic behaviour is obtained in Subsection 3.4.2, whereas certain aspects on the relationship between loading directions and strain hardening behaviour is discussed in Subsection 3.4.3.

3.4.1 Definition of a plastic potential function

Guided by the results derived from the principle of maximum dissipation, we assume the existence of a nonlocal plastic potential function $p = p(\mathbf{E}, < \mathcal{U}' >)$, such that during loading ($g = 0$, $\hat{g} > 0$) [3.4-8]

[3.4-6] In Section 3.5.3 we will derive (3.4-6) as the restriction of (3.4-23) to the class of elastic-plastic materials addressed in **(ii)**.

[3.4-7] Since we do not accept the principle of maximum dissipation as a general postulate in plasticity, (3.4-22) is a postulate and not a proved statement.

[3.4-8] If we take $p = g$ (associated plasticity) it is seen that (3.4-24) and (3.4-23) coincide.

$$\check{\boldsymbol{\sigma}} = -\gamma\frac{\partial p}{\partial \mathbf{E}}, \quad \gamma = \gamma(\mathbf{E},\ <\mathcal{U}'>) > 0, \tag{3.4-24}$$

where $\partial p/\partial \mathbf{E}$ and γ are evaluated on the yield surface in strain space.

If we substitute (3.4-24) into (3.3-11) we obtain the reduced form of the quasi-local response function $\check{\mathcal{K}}$ as

$$\check{\mathcal{K}} = \mathcal{J} - \pi\gamma\frac{\partial p}{\partial \mathbf{E}} \otimes \frac{\partial f}{\partial \mathbf{S}}\ , \tag{3.4-25}$$

whereas the corresponding form of the quasi-local strain hardening modulus $\check{\Phi}$ reads

$$\check{\Phi} = 1 - \pi\gamma\frac{\partial p}{\partial \mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{S}}\ . \tag{3.4-26}$$

The analogue of (3.3-35) takes the form

$$\check{\mathcal{K}}\frac{\partial p}{\partial \mathbf{E}} = \check{\Phi}\frac{\partial p}{\partial \mathbf{E}}\ , \tag{3.4-27}$$

and follows from (3.4-25) and (3.4-26). Thus $\partial p/\partial \mathbf{E}$ is an eigenvector of $\check{\mathcal{K}}$ with eigenvalue $\check{\Phi}$ given by (3.4-16).

Since we have adopted (3.4-22) and since $\gamma > 0$ by assumption, we can define a function $\check{\Gamma}$ by

$$\check{\Phi} = \pi\gamma\check{\Gamma}\ . \tag{3.4-28}$$

and conclude that the function $\check{\Gamma}$ is of the same sign as the quasi-local strain hardening modulus and vanishes precisely when $\check{\Phi} = 0$. On substituting (3.4-28) into (3.4-26) we obtain

$$\pi\gamma = \frac{1}{\check{\Gamma} + \dfrac{\partial p}{\partial \mathbf{E}} \cdot \dfrac{\partial f}{\partial \mathbf{S}}} \tag{3.4-29}$$

Hence (3.4-25) can be written

$$\check{\mathcal{K}} = \mathcal{J} - \frac{\dfrac{\partial p}{\partial \mathbf{E}} \otimes \dfrac{\partial f}{\partial \mathbf{S}}}{\check{\Gamma} + \dfrac{\partial p}{\partial \mathbf{E}} \cdot \dfrac{\partial f}{\partial \mathbf{S}}}, \tag{3.4-30}$$

whereas (3.3-13) takes the explicit form

$$\check{\mathbf{S}} = \{\mathcal{L} - \frac{\frac{\partial p}{\partial \mathbf{E}} \otimes \mathcal{L}^T \frac{\partial f}{\partial \mathbf{S}}}{\check{\Gamma} + \frac{\partial p}{\partial \mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{S}}}\}\dot{\mathbf{E}}, \tag{3.4-31}$$

or equivalently, by the aid of (3.3-10) and (3.3-45),

$$\check{\mathbf{S}} = \{\mathcal{L} - \frac{\frac{\partial p}{\partial \mathbf{E}} \otimes \frac{\partial g}{\partial \mathbf{E}}}{\check{\Gamma} + \frac{\partial p}{\partial \mathbf{E}} \cdot \mathcal{M}^T \frac{\partial g}{\partial \mathbf{E}}}\}\dot{\mathbf{E}}. \tag{3.4-32}$$

The similarity of the fourth order tensor in the right-hand side of (3.4-31) or (3.4-32) with the tangential elastic-plastic stiffness tensor used in classical non-associated plasticity (with small strains) is recognized, where in particular the function $\check{\Gamma}$ plays the role of the plastic modulus. Accordingly we refer to either of these tensors in (3.4-31) or (3.4-32) as the quasi-local tangential stiffness tensor (in strain space) and to $\check{\Gamma}$ as the quasi-local plastic modulus.

In associated plasticity, p is taken equal to g. Then (3.4-27) reduces to

$$\check{\mathcal{K}} \frac{\partial g}{\partial \mathbf{E}} = \check{\Phi} \frac{\partial g}{\partial \mathbf{E}}\ , \tag{3.4-33}$$

i.e. $\partial g / \partial \mathbf{E}$ is an eigenvector of $\check{\mathcal{K}}$ with eigenvalue $\check{\Phi}$ given by (3.4-26) with $p = g$. If additionally $\mathcal{L} = \mathcal{L}^T$ we conclude from (3.4-31) or (3.4-32) that the tangential stiffness tensor is symmetric,

$$\check{\mathbf{S}} = \{\mathcal{L} - \mathcal{L} \frac{\frac{\partial f}{\partial \mathbf{S}} \otimes \mathcal{L} \frac{\partial f}{\partial \mathbf{S}}}{\check{\Gamma} + \mathcal{L} \frac{\partial f}{\partial \mathbf{S}} \cdot \frac{\partial f}{\partial \mathbf{S}}}\}\dot{\mathbf{E}}\ , \tag{3.4-34}$$

where (3.3-10) has been used to display the relationship in stress space form.

Reduced forms similar to (3.4-25), (3.4-27) and (3.4-33) in terms of the inverse quasi-local response function $\check{\mathcal{N}}$ are easily derived. From (3.4-24) and (3.4-26) it follows that (3.3-49) reduces to

$$\check{\mathcal{N}} = \mathcal{J} + \frac{\pi\gamma}{1 - \pi\gamma \frac{\partial p}{\partial \mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{S}}} \frac{\partial p}{\partial \mathbf{E}} \otimes \frac{\partial f}{\partial \mathbf{S}}\ , \tag{3.4-35}$$

and hence by (3.4-26)

$$\check{\mathcal{N}}\frac{\partial p}{\partial \mathbf{E}} = \frac{1}{\check{\Phi}}\frac{\partial p}{\partial \mathbf{E}}\,, \tag{3.4-36}$$

in associated plasticity replaced by

$$\check{\mathcal{N}}\frac{\partial g}{\partial \mathbf{E}} = \frac{1}{\check{\Phi}}\frac{\partial g}{\partial \mathbf{E}}\,, \tag{3.4-37}$$

similar to (3.4-33), i.e. $\partial g/\partial \mathbf{E}$ is an eigenvector of $\check{\mathcal{N}}$ as well, corresponding to the eigenvalue $1/\check{\Phi}$.

Remark 4.1. From the basic assumption (3.4-24) together with the restriction $\pi > 0$ used in (3.4-28), expressions for a quasi-local elastic-plastic tangential stiffness tensor have been derived, (3.4-31) and (3.4-32). No approximations whatsoever are involved in these derivations - the relationship for the quasi-local stress rate $\check{\mathbf{S}}$ in either (3.4-31) or (3.4-32) is exact. As before, however, calculation of the actual stress rate $\dot{\mathbf{S}}$ requires use of (3.3-2) in conjunction with (3.3-18).□

Remark 4.2. Even though e.g. (3.4-30) is an exact expression for the quasi-local elastic-plastic response function $\check{\mathcal{K}}$ at a point $\mathbf{X}$, there is a substantial drawback involved, to the extent that the quasi-local plastic modulus $\check{\Gamma}$ in the denominator does not afford a simple description of the strain hardening behaviour at the very point $\mathbf{X}$. E.g. hardening occurs if

$$\check{\Gamma} + \frac{1}{\pi\gamma\rho_0\hat{g}}\boldsymbol{\mathcal{H}}_s \cdot \frac{\partial f}{\partial \mathbf{S}} > 0, \tag{3.4-38}$$

as is seen from (3.3-32), $(3.3\text{-}39)_4$ and (3.4-28), $\boldsymbol{\mathcal{H}}_s$ being calculated by the aid of (3.3-18). Obviously (3.4-38) is not generally fulfilled even if the quasi-local plastic modulus satisfies the inequality $\check{\Gamma} > 0$ (cf. the discussion in Section 3.3.2). However, within the (approximate) quasi-local formulation (see Section 3.3.3) the amount of the second term in the left side of (3.4-38) is small, and if $\check{\Gamma}$ itself is not close to zero, then the sign of the quasi-local plastic modulus apparently determines whether the material is hardening or softening at the actual point (leaving perfect - or nearly perfect - plastic behaviour with some uncertainty in this approximation).□

3.4.2 Special results for perfectly plastic behaviour

In Subsection 3.3.4 we concluded that a direction of loading for which the stress rate vanishes, during plastic loading ($g = 0$, $\hat{g} > 0$), necessarily corresponded to perfectly plastic behaviour.

It was also noted that a direction of loading which results in zero quasi-local stress rate necessarily corresponded to zero quasi-local strain hardening modulus. Here we will concern ourselves with the question of existence of such loading directions. When do such directions exist and to what extent can they be specified explicitly? Our general nonlocal formulation does not provide much answer to these questions. If, however, we turn our interest to the corresponding problem with regard to zero quasi-local stress rate, it will be seen that the introduction of the plastic potential function p (defined by (3.4-24), in fact guarantees that a direction of loading for which the quasi-local stress rate $\check{\mathbf{S}}$ vanishes always exists, necessarily corresponding to vanishing quasi-local strain hardening modulus $\check{\Phi}$, and is parallel with the vector $\mathcal{M}\partial p/\partial\mathbf{E}$ in six-dimensional strain space ($\mathcal{M}$ defined by (3.2-37) and (3.3-45)).

To see this, we first observe that, for vanishing $\check{\Phi}$, (3.4-27) is reduced to

$$\check{\mathcal{K}}\frac{\partial p}{\partial \mathbf{E}} = \mathbf{0}, \quad 0 < \check{\gamma} = 1/(\frac{\partial p}{\partial \mathbf{E}} \cdot \frac{\partial f}{\partial \mathbf{S}}) \,, \tag{3.4-39}$$

where we have used (3.4-26). Thus $\partial p/\partial\mathbf{E}$ lies in the null space of $\check{\mathcal{K}}$[3.4-9]. We note that (3.4-39) by use of (3.4-25) can be written

$$\pi\gamma\frac{\partial p}{\partial \mathbf{E}} \otimes \frac{\partial f}{\partial \mathbf{S}}\frac{\partial p}{\partial \mathbf{E}} = \frac{\partial p}{\partial \mathbf{E}} \,, \tag{3.4-40}$$

that is, for $\check{\Phi} = 0$, $\partial p/\partial\mathbf{E}$ is an eigenvector of the tensor $\pi\gamma\ \partial p/\partial\mathbf{E} \otimes \partial f/\partial\mathbf{S}$ corresponding to eigenvalue unity. (Note that (3.4-28) implies $\check{\Gamma} = 0$ if $\check{\Phi} = 0$.)

To prove that there is exactly one direction of loading for which $\check{\mathbf{S}} = \mathbf{0}$, choose $\dot{\mathbf{E}}$ such that

$$\dot{\mathbf{E}} = h\ \mathcal{M}\frac{\partial p}{\partial \mathbf{E}} \,, \tag{3.4-41}$$

where $h = h(\mathbf{E},\ < \mathcal{U}' > \,,\ \dot{\mathbf{E}})$ is a positive scalar-valued function, and note by (3.2-9) and $(3.4\text{-}39)_2$ that

$$\hat{g} = \frac{h}{\check{\gamma}} > 0. \tag{3.4-42}$$

Thus (3.4-41) specifies a direction of loading to which, in view of (3.3-13), corresponds a quasi-local stress rate

$$\check{\mathbf{S}} = h\check{\mathcal{K}}\mathcal{L}\ \mathcal{M}\frac{\partial p}{\partial \mathbf{E}} \,. \tag{3.4-43}$$

From $(3.4\text{-}39)_1$ then follows that

[3.4-9] Recall from (3.3-35) that $\partial f/\partial\mathbf{S}$ belongs to the null space of $\check{\mathcal{K}}^T$ when $\check{\Phi} = 0$.

$$\check{\mathbf{S}} = \mathbf{0}, \qquad (3.4-44)$$

and hence $\mathcal{M}\partial p/\partial\mathbf{E}$ represents a loading direction for which the quasi-local stress rate vanishes for zero quasi-local strain hardening modulus.

Next we prove that in fact there is no direction of loading other than that of $\mathcal{M}\partial p/\partial\mathbf{E}$, which results in zero quasi-local stress rate for vanishing quasi-local strain hardening modulus. We set $\check{\mathbf{S}} = \mathbf{0}$ and by using (3.3-13) and (3.4-25) we get

$$(\mathcal{J} - \pi\, \gamma\frac{\partial p}{\partial \mathbf{E}} \otimes \frac{\partial f}{\partial \mathbf{S}})\mathcal{L}\dot{\mathbf{E}} = \mathbf{0},\, \gamma > 0, \qquad (3.4-45)$$

which can also be written

$$\mathcal{L}\dot{\mathbf{E}} = \pi\, \gamma\hat{g}\frac{\partial p}{\partial \mathbf{E}}\, , \qquad (3.4-46)$$

where we have used (3.3-10) and (3.2-9). Hence, in view of (3.3-45), the equation $\check{\mathbf{S}} = \mathbf{0}$ has a solution of the form

$$\dot{\mathbf{E}} = \pi\, \gamma\hat{g}\; \mathcal{M}\frac{\partial p}{\partial \mathbf{E}}\, , \qquad (3.4-47)$$

i.e. $\dot{\mathbf{E}}$ has the same direction as $\mathcal{M}\partial p/\partial\mathbf{E}$, which completes the proof.

For associated plasticity ($p = g$) and with $\mathcal{L} = \mathcal{L}^T$ we deduce from (3.4-47) that in the case of vanishing quasi-local strain hardening modulus, the only direction of loading for which the quasi-local stress rate vanishes is that of the normal to the yield surface in stress space.

Remark 4.3. Referring to Section 4.3 we recall that in the restriction to local theory the differences between quasi-local and general nonlocal quantities vanish since $\boldsymbol{\mathcal{H}}_s = \mathbf{0}$. Hence we may omit the check symbol in all entities except that of the stress rate, where $\check{\mathbf{S}}$ should be replaced by $\dot{\mathbf{S}}$. When further the independent variables are replaced by $\mathcal{U} = \{\mathbf{E},\, \mathbf{E}^p,\, \kappa\}$, all equations in this section also apply to local theory.

As a special case, taking $p = g$, we note that (3.4-24) reads

$$\tilde{\boldsymbol{\sigma}} = -\gamma\frac{\partial g}{\partial \mathbf{E}}\, , \gamma > 0, \qquad (3.4-48)$$

corresponding to (3.4-5). Thus $\tilde{\boldsymbol{\sigma}}$ is normal to the yield surface in strain space as discussed previously.

Still assuming associated plasticity and additionally $\mathcal{L} = \mathcal{L}^T$, we also note that (3.4-47), by use of (3.3-10) and (3.3-45), in local theory should be replaced by

$$\dot{\mathbf{E}} = \pi\ \gamma\ \hat{g} \frac{\partial f}{\partial \mathbf{S}}. \qquad (3.4-49)$$

Hence we conclude that for perfectly plastic behaviour, the only direction of loading for which the stress rate vanishes is that of the normal to the yield surface in stress space.□

3.4.3 On loading directions

In classical, infinitesimal plasticity it is commonly claimed that a material is of hardening type if its constitutive response satisfies $\dot{\sigma}_{ij}\ \dot{\epsilon}_{ij} > 0$ for any choice of nonvanishing loading direction $\dot{\epsilon}_{ij}$. If $\dot{\sigma}_{ij} = D_{ijkl}\ \dot{\epsilon}_{kl}$ it then follows that $D_{ijkl}\ \dot{\epsilon}_{ij}\ \dot{\epsilon}_{kl} > 0$ whenever $\dot{\epsilon}_{ij} \neq 0$, i.e. D_{ijkl} is positive definite. Conversely a material is of softening type if $\dot{\sigma}_{ij}\ \dot{\epsilon}_{ij} < 0$, and hence D_{ijkl} is negative definite for a softening material. Also recall that materials which satisfy Drucker's postulate of stability, $\dot{\sigma}_{ij}\dot{\epsilon}^{p}_{ij} > 0$, are not capable of sustaining strain softening behaviour.

In a general theory of finite plasticity (local or nonlocal), without constraints of the type of Drucker's postulate no restriction whatever is laid on the scalar $\dot{\mathbf{S}} \cdot \dot{\mathbf{E}}$, which may be of any sign (at least in principle) during loading, irrespective of the type of strain hardening response. Below we will briefly discuss restrictions on $\dot{\mathbf{S}} \cdot \dot{\mathbf{E}}$ (or $\check{\mathbf{S}} \cdot \dot{\mathbf{E}}$) for a familiar class of materials where $\mathcal{L} = \partial\tilde{\mathbf{S}}/\partial\mathbf{E}$ is symmetric and positive definite.

In fact, not much can be said about arbitrary loading directions. However, it is easy to show that the following statements hold true[3.4-10],

(i) the outward normal to the yield surface in stress space, $\partial f/\partial\mathbf{S}$, is always a possible direction of loading, and

(ii) if, in an elastic-plastic state ($g = 0$), loading is effected in the direction of the outward normal to the yield surface in stress space, then the state is one of hardening precisely if $\dot{\mathbf{S}} \cdot \dot{\mathbf{E}} > 0$, one of softening precisely if $\dot{\mathbf{S}} \cdot \dot{\mathbf{E}} < 0$ and perfectly plastic precisely if $\dot{\mathbf{S}} \cdot \dot{\mathbf{E}} = 0$.

Assume that

$$\dot{\mathbf{E}} = \xi \frac{\partial f}{\partial \mathbf{S}}, \qquad (3.4-50)$$

where ξ is an arbitrary positive function of ($\mathbf{E}$, $< \mathcal{U}' >$) and $\dot{\mathbf{E}}$. Since $\mathcal{L}$ is symmetric and positive definite, it follows from (3.2-9) and (3.3-10) that

[3.4-10] Arguments used are similar to those of Casey and Lin (1986).

$$\hat{g} = \xi \frac{\partial f}{\partial \mathbf{S}} \cdot \mathcal{L} \frac{\partial f}{\partial \mathbf{S}} > 0, \tag{3.4-51}$$

which proves (**i**).

Further, using (3.2-36), (3.3-31) and (3.4-51), it is deduced that

$$\dot{\mathbf{S}} \cdot \dot{\mathbf{E}} = \xi \dot{\mathbf{S}} \cdot \frac{\partial f}{\partial \mathbf{S}} = \xi \Phi \hat{g}, \tag{3.4-52}$$

from which the conclusions listed in (**ii**) follow immediately in view of (3.3-32).

Substitution of $(3.3\text{-}39)_1$ and $(3.3\text{-}39)_4$ into $(3.4\text{-}52)_2$ yields

$$\check{\mathbf{S}} \cdot \dot{\mathbf{E}} = \xi \check{\Phi} \hat{g}. \tag{3.4-53}$$

Hence (note that (3.4-51) still holds), if loading is effected along the outward normal to the yield surface in stress space, then $\check{\mathbf{S}} \cdot \dot{\mathbf{E}} > 0$, $\check{\mathbf{S}} \cdot \dot{\mathbf{E}} < 0$ or $\dot{\mathbf{S}} \cdot \dot{\mathbf{E}} = 0$ if and only if the quasi-local strain hardening modulus accordingly is positive, negative or zero.

Finally, we consider materials for which a potential function exists. Then, in the case of associated plasticity, we note that (3.4-25) becomes

$$\check{\mathcal{K}} = \mathcal{J} - \pi\gamma \frac{\partial g}{\partial \mathbf{E}} \otimes \frac{\partial f}{\partial \mathbf{S}}, \quad \pi > 0, \quad \gamma > 0, \tag{3.4-54}$$

during loading ($g = 0$, $\hat{g} > 0$). From (3.3-10), (3.3-13) and (3.4-54)

$$\check{\mathbf{S}} \cdot \dot{\mathbf{E}} = \dot{\mathbf{E}} \cdot \mathcal{L}\dot{\mathbf{E}} - \pi\gamma \frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} \, \frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}} \leq \dot{\mathbf{E}} \cdot \mathcal{L}\dot{\mathbf{E}}, \tag{3.4-55}$$

where we also have taken advantage of the fact that $\mathcal{L}$ is symmetric. In view of $(3.3\text{-}39)_1$ (3.2-9) and $(3.4\text{-}55)_1$ we deduce that

$$\dot{\mathbf{S}} \cdot \dot{\mathbf{E}} = \dot{\mathbf{E}} \cdot \mathcal{L}\dot{\mathbf{E}} - \pi\gamma(\hat{g})^2 + \frac{1}{\rho_0}\mathcal{H}_s \cdot \dot{\mathbf{E}} \leq \dot{\mathbf{E}} \cdot \mathcal{L}\dot{\mathbf{E}} + \frac{1}{\rho_0}\mathcal{H}_s \cdot \dot{\mathbf{E}}. \tag{3.4-56}$$

It is noted that the inequalities in (3.4-55) and (3.4-56) are the only restrictions on $\check{\mathbf{S}} \cdot \dot{\mathbf{E}}$ and $\dot{\mathbf{S}} \cdot \dot{\mathbf{E}}$, respectively, implied by the assumption of associated plasticity.

3.5 An equivalent set of constitutive variables

In this section we draw attention to the fact that it may be advantageous to express the constitutive equation (3.2-5) in terms of an equivalent set of constitutive variables in the form

$$\mathbf{S} = \bar{\mathbf{S}}(\mathbf{E} - <\mathbf{E}^p>,\ <\mathcal{U}'>) = \bar{\mathbf{S}}(\mathbf{E} - <\mathbf{E}^p>,\ <\mathbf{E}^p>, <\kappa>). \qquad (3.5-1)$$

In Subsection 3.5.1 we will discuss the consequences of (3.5-1) and derive expressions for the quasi-local response function $\check{\mathcal{K}}$ and the quasi-local strain hardening modulus $\check{\Phi}$ in terms of the new set of constitutive variables. In Subsection 3.5.2 a special case will be considered, where $\bar{\mathbf{S}}$ does not depend on its second and third argument, and in Subsection 3.5.3 some results in the restriction to local theory will be discussed.

3.5.1 Results in terms of the new set of variables

Using the chain rule of differentiation, we obtain from (3.5-1) and (3.2-5) that

$$\frac{\partial \bar{\mathbf{S}}}{\partial(\mathbf{E} - <\mathbf{E}^p>)} = \mathcal{L} \qquad (3.5-2)$$

and

$$\frac{\partial \bar{\mathbf{S}}}{\partial <\mathcal{U}'>} = \{\frac{\partial \bar{\mathbf{S}}}{\partial <\mathbf{E}^p>}, \frac{\partial \bar{\mathbf{S}}}{\partial <\kappa>}\} = \{\mathcal{L} + \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathbf{E}^p>}, \frac{\partial \tilde{\mathbf{S}}}{\partial <\kappa>}\}, \qquad (3.5-3)$$

where $\mathcal{L}$ originally is defined in relation to the derivation of (3.2-37).

Next, results corresponding to (3.5-2) and (3.5-3) will be obtained, where the differentiation involves derivatives with respect to $\mathcal{U}'$ instead *of* $<\mathcal{U}'>$. It appears that considerable simplifications can be achieved if we assume that the prescribed functions $\alpha' = \{\boldsymbol{\alpha}^p,\ \alpha^h\}$, defined in (3.2-6), satisfy restricted constitutive equations of the form

$$\left.\begin{aligned} \boldsymbol{\alpha}^p &= \boldsymbol{\alpha}^p(\mathbf{E}^p(\mathbf{X},t),\ \mathbf{E}^p(\mathbf{Z},t), \\ \alpha^h &= \alpha^h(\kappa(\mathbf{X},t),\ \kappa(\mathbf{Z},t)). \end{aligned}\right\} \qquad (3.5-4)$$

This restriction on α' is used throughout the rest of the chapter.

In view of (3.5-2) and (3.5-3) and with arguments similar to those used in the derivation of (3.2-45), we obtain (also cf. (3.2-22) and (3.2-23))

$$\frac{\partial}{\partial \mathbf{E}^p}(<\rho_0\tilde{\mathbf{S}}>) = \frac{\partial}{\partial \mathbf{E}^p}\int_B \{\rho_0 \frac{\partial \tilde{\mathbf{S}}}{\partial <\mathbf{E}^p>}\boldsymbol{\alpha}^p + \rho_0(\mathbf{Z})\frac{\partial \tilde{\mathbf{S}}}{\partial <\mathbf{E}^p>}(\mathbf{Z})(\boldsymbol{\alpha}^p)^*\}dV(\mathbf{Z})$$

$$= \frac{\partial}{\partial \mathbf{E}^p}\int_B \{\rho_0 \frac{\partial \bar{\mathbf{S}}}{\partial <\mathbf{E}^p>}\boldsymbol{\alpha}^p + \rho_0(\mathbf{Z})\frac{\partial \bar{\mathbf{S}}}{\partial <\mathbf{E}^p>}(\mathbf{Z})(\boldsymbol{\alpha}^p)^*\}dV(\mathbf{Z})$$

$$- \frac{\partial}{\partial \mathbf{E}^p}\int_B \{\rho_0 \frac{\partial \bar{\mathbf{S}}}{\partial (\mathbf{E}- <\mathbf{E}^p>)}\boldsymbol{\alpha}^p + \rho_0(\mathbf{Z})\frac{\partial \bar{\mathbf{S}}}{\partial (\mathbf{E}- <\mathbf{E}^p>)}(\mathbf{Z})(\boldsymbol{\alpha}^p)^*\}dV(\mathbf{Z}). \quad (3.5-5)$$

If we write $\partial <\rho_0\bar{\mathbf{S}}> /\partial \mathbf{E}^p$ for the first integral in $(3.5\text{-}5)_2$ and $\partial <\rho_0\bar{\mathbf{S}}> / \partial(\mathbf{E}-\mathbf{E}^p)$ for the second one we conclude that

$$\frac{\partial <\rho_0\tilde{\mathbf{S}}>}{\partial \mathbf{E}^p} = \frac{\partial <\rho_0\bar{\mathbf{S}}>}{\partial \mathbf{E}^p} - \rho_0\, \mathcal{L}^*, \quad (3.5-6)$$

where

$$\mathcal{L}^* = \frac{1}{\rho_0}\frac{\partial <\rho_0\bar{\mathbf{S}}>}{\partial(\mathbf{E}-\mathbf{E}^p)}\ . \quad (3.5-7)$$

For the derivative with respect to the strain hardening function, we correspondingly deduce that

$$\frac{\partial <\rho_0\tilde{\mathbf{S}}>}{\partial \kappa} = \frac{\partial <\rho_0\bar{\mathbf{S}}>}{\partial \kappa}\ , \quad (3.5-8)$$

where the influence of the restricted form $(3.5\text{-}4)_1$ for $\boldsymbol{\alpha}^p$ is apparent. While $<\rho_0\tilde{\mathbf{S}}>$ is unambiguously defined by an expression similar to (3.2-22), it should be noted that this is not true if $\tilde{\mathbf{S}}$ is replaced by $\bar{\mathbf{S}}$. Consequently we refer to (3.5-5) for an interpretation of (3.5-6)-(3.5-8). Note that (3.5-6)-(3.5-8) have been established in such a way, that they have exactly the same form as $(3.5\text{-}3)_2$.

We now continue as in Section 3.3.1 and define a second order tensor-valued function $\check{\boldsymbol{\sigma}}$ by

$$\check{\boldsymbol{\sigma}} = \frac{1}{\rho_0}\frac{\partial <\rho_0\bar{\mathbf{S}}>}{\partial \mathcal{U}'}\Lambda'\ , \quad (3.5-9)$$

where Λ' is defined by (3.2-18). Using (3.3-9), (3.5-6), (3.5-7) and (3.5-8) it follows that

$$\check{\boldsymbol{\sigma}} = \check{\tilde{\boldsymbol{\sigma}}} + \mathcal{L}^*\mathbf{R}, \qquad (3.5-10)$$

so that (3.3-11) becomes

$$\check{\mathcal{K}} = \mathcal{J} - \pi\mathcal{L}^*\mathbf{R} \otimes \frac{\partial f}{\partial \mathbf{S}} + \pi\check{\tilde{\boldsymbol{\sigma}}} \otimes \frac{\partial f}{\partial \mathbf{S}}. \qquad (3.5-11)$$

The quasi-local strain hardening modulus $\check{\Phi}$ defined by (3.3-27) (cf. Section 3.3.3) becomes similarly

$$\check{\Phi} = 1 - \pi\mathcal{L}^*\mathbf{R} \cdot \frac{\partial f}{\partial \mathbf{S}} + \pi\check{\tilde{\boldsymbol{\sigma}}} \cdot \frac{\partial f}{\partial \mathbf{S}}\,, \qquad (3.5-12)$$

or alternatively

$$\check{\Phi} = 1 - \pi\mathcal{M}\ \mathcal{L}^*\mathbf{R} \cdot \frac{\partial g}{\partial \mathbf{E}} + \pi\check{\tilde{\boldsymbol{\sigma}}} \cdot \frac{\partial f}{\partial \mathbf{S}}\,, \qquad (3.5-13)$$

where (3.3-10) and (3.3-45) have been used.

For future use we also record the expression of the plastic potential in terms of $\check{\tilde{\boldsymbol{\sigma}}}$,

$$\check{\tilde{\boldsymbol{\sigma}}} - \mathcal{L}^*\mathbf{R} = -\gamma\frac{\partial p}{\partial \mathbf{E}}\,,\, \gamma > 0, \qquad (3.5-14)$$

as is seen from (3.4-24) and (3.5-10).

The rate of stress can be calculated directly by differentiating the function $\mathbf{S} = \bar{\mathbf{S}}(\mathbf{E}- <\mathbf{E}^p>, <\mathcal{U}'>)$ or alternatively from the expression for $\dot{\mathbf{S}}$ in (3.2-37) by the aid of (3.5-6) and (3.5-8). We use the second possibility and conclude that

$$\rho_0\dot{\mathbf{S}} - \mathcal{H}_s = \rho_0\ \mathcal{L}\dot{\mathbf{E}} - \rho_0\ \mathcal{L}^*\dot{\mathbf{E}}^p + \frac{\partial <\rho_0\bar{\mathbf{S}}>}{\partial \mathcal{U}'}\dot{\mathcal{U}}', \qquad (3.5-15)$$

where we understand that the function $\bar{\mathbf{S}}$ is used in the calculation of $\mathcal{H}_s$. That is, in view of (3.2-39),

$$\mathcal{H}_s = \int_B \{\rho_0(-\frac{\partial\bar{\mathbf{S}}}{\partial(\mathbf{E}- <\mathbf{E}^p>)}\frac{\partial\boldsymbol{\alpha}^p}{\partial\mathcal{U}'(\mathbf{Z})} + \frac{\partial\bar{\mathbf{S}}}{\partial <\mathcal{U}'>}\frac{\partial\alpha'}{\partial\mathcal{U}'(\mathbf{Z})})\dot{\mathcal{U}}'(\mathbf{Z})$$

$$-\ \rho_0(\mathbf{Z})(-\frac{\partial\bar{\mathbf{S}}}{\partial(\mathbf{E}- <\mathbf{E}^p>)}(\mathbf{Z})\frac{\partial(\boldsymbol{\alpha}^p)^*}{\partial\mathcal{U}'} + \frac{\partial\bar{\mathbf{S}}}{\partial <\mathcal{U}'>}(\mathbf{Z})\frac{\partial(\alpha')^*}{\partial\mathcal{U}'}\dot{\mathcal{U}}'\}\ dV(\mathbf{Z}) \qquad (3.5-16)$$

where the implication of (3.5-4) should be taken into consideration.

During loading, using (3.5-9), (3.2-18) and $(3.2\text{-}19)_1$ we conclude that (3.5-15) can be written in the form

$$\dot{\mathbf{S}} - \frac{1}{\rho_0}\mathcal{H}_s = (\mathcal{L} - \pi\ \mathcal{L}^*\mathbf{R} \otimes \frac{\partial g}{\partial \mathbf{E}} + \pi\ \check{\boldsymbol{\sigma}} \otimes \frac{\partial g}{\partial \mathbf{E}})\dot{\mathbf{E}}, \qquad (3.5-17)$$

or by (3.3-2)

$$\check{\mathbf{S}} = (\mathcal{L}' + \pi\check{\boldsymbol{\sigma}} \otimes \frac{\partial g}{\partial \mathbf{E}})\dot{\mathbf{E}}\ , \qquad (3.5-18)$$

where $\mathcal{L}'$ is defined by

$$\mathcal{L}' = \mathcal{L} - \pi\ \mathcal{L}^*\mathbf{R} \otimes \frac{\partial g}{\partial \mathbf{E}}\ . \qquad (3.5-19)$$

It is immediately seen that the fourth order tensor-valued function $\mathcal{L}'$ has the same symmetry properties as those of $\mathcal{L}$ (first appearing in (3.2-37)). If (3.5-10) is used to substitute for $\check{\boldsymbol{\sigma}}$ in (3.5-17) it is noted that the result agrees with (3.3-8) as it should.

It is also noted that the inverse relation for the strain rate in terms of the quasi-local stress rate is still valid in the form (3.3-48), where $\check{\mathcal{N}}$ now is given by

$$\check{\mathcal{N}} = \mathcal{J} + \frac{\pi}{\check{\Phi}}((\mathcal{L}^*\mathbf{R} - \check{\boldsymbol{\sigma}}) \otimes \frac{\partial f}{\partial \mathbf{S}}), \qquad (3.5-20)$$

as is seen from (3.3-49) and (3.5-10). Of course (3.5-20) is valid only if the quasi-local strain hardening modulus $\check{\Phi}$, now to be calculated by (3.5-12) or (3.5-13), does not vanish.

3.5.2 A special case

In Section 3.5.1 we derived expressions for the quasi-local stress rate $\check{\mathbf{S}}$, the quasi-local response function $\check{\mathcal{K}}$ and the quasi-local strain hardening modulus $\check{\Phi}$ in terms of the response function $\bar{\mathbf{S}}$. We will here consider the special case when $\bar{\mathbf{S}}$ does not depend on its second and third argument. In view of (3.5-5) it then follows that $\partial < \rho_0\bar{\mathbf{S}} > /\partial\mathbf{E}^p$ and $\partial < \rho_0\bar{\mathbf{S}} > /\partial\kappa$ both vanish, and hence that (3.5-9) reduces to

$$\check{\boldsymbol{\sigma}} = \mathbf{0}. \qquad (3.5-21)$$

Thus (3.5-11) becomes

$$\check{\mathcal{K}} = \mathcal{J} - \pi\mathcal{L}^* \, \mathbf{R} \otimes \frac{\partial f}{\partial \mathbf{S}}, \qquad (3.5-22)$$

while (3.5-12) and (3.5-13) become

$$\check{\Phi} = 1 - \pi\mathcal{L}^* \, \mathbf{R} \cdot \frac{\partial f}{\partial \mathbf{S}}, \qquad (3.5-23)$$

$$\check{\Phi} = 1 - \pi \, \mathcal{M} \, \mathcal{L}^*\mathbf{R} \cdot \frac{\partial g}{\partial \mathbf{E}}, \qquad (3.5-24)$$

respectively. If a plastic potential function is assumed to exist, then (3.5-14) is reduced to

$$\mathcal{L}^*\mathbf{R} = \gamma\frac{\partial p}{\partial \mathbf{E}}, \quad \gamma > 0, \qquad (3.5-25)$$

and, if (3.4-29) is valid, (3.5-22) takes on the term (3.4-30). By substitution of (3.5-25) into (3.5-23), (3.4-26) is recovered as should be expected.

If $\mathcal{L}^*$ is not singular, (3.5-25) can alternatively be written in the form

$$\mathbf{R} = \gamma\mathcal{M}^*\frac{\partial p}{\partial \mathbf{E}}, \qquad (3.5-26)$$

where the fourth order tensor $\mathcal{M}^*$ satisfies

$$\mathcal{L}^*\mathcal{M}^* = \mathcal{M}^*\mathcal{L}^* = \mathcal{J}, \qquad (3.5-27)$$

and where $\mathcal{J}$ is defined by (3.3-7)[3.5-1].

By virtue of (3.5-21) we observe that the stress-strain relation (3.5-18) is reduced to

$$\check{\mathbf{S}} = \mathcal{L}' \, \dot{\mathbf{E}}, \qquad (3.5-28)$$

with $\mathcal{L}'$ defined by (3.5-19), while the inverse relation is given by

$$\dot{\mathbf{E}} = \mathcal{M}(\mathcal{J} + \frac{\pi}{\check{\Phi}} \, \mathcal{L}^*\mathbf{R} \otimes \frac{\partial f}{\partial \mathbf{S}})\check{\mathbf{S}}, \qquad (3.5-29)$$

[3.5-1] If the principle of maximum dissipation is invoked, it is easily seen that (3.5-26) should be replaced by $\mathbf{R} = \gamma\mathcal{M}^*\partial g/\partial\mathbf{E}$, or equivalently (by use of $(3.2\text{-}16)_1$ and (3.4-23)), $\dot{\mathbf{E}}^p = \dot{\lambda}\mathcal{M}^*\partial g/\partial\mathbf{E}$, being the nonlocal counterpart of (3.4-6).

where (3.3-48) and (3.5-20) have been used.

If the conditions (3.5-25) and (3.5-26) are valid we conclude from (3.2-19), (3.3-40) and (3.4-28) that

$$\dot{\mathbf{E}}^p = \frac{\check{f}}{\check{\Gamma}} \mathcal{M}^* \frac{\partial p}{\partial \mathbf{E}} \quad \dot{\kappa} = \frac{\check{f}\ r}{\check{\Gamma}\ \gamma}\ , \tag{3.5-30}$$

constituting the flow rules in the case when the quasi-local plastic modulus does not vanish. If $\check{\Gamma} = 0$, using $(3.2\text{-}16)_1$ gives

$$\dot{\mathbf{E}}^p = \hat{g} \frac{\mathcal{M}^* \dfrac{\partial p}{\partial \mathbf{E}}}{\dfrac{\partial p}{\partial \mathbf{E}} \cdot \dfrac{\partial f}{\partial \mathbf{S}}}\ , \tag{3.5-31}$$

where (3.4-29) and (3.5-26) have also been used.

3.5.3 Restriction to local theory

Below we make some comments concerning local counterparts of the nonlocal results presented previously in this section. We refer to Section 3.3.5 (also cf. *Remark 4.3*) and conclude that the quantity $< \rho_0 \bar{\mathbf{S}} >$ may be replaced by $\rho_0 \bar{\mathbf{S}}$ in a corresponding local theory. Hence deduce from (3.5-2) and (3.5-7) that

$$\mathcal{L}^* = \mathcal{L}\ . \tag{3.5-32}$$

Also note that the restricted forms of (3.5-2) and (3.5-3) are equivalent to the corresponding restricted forms of (3.5-6) and (3.5-8), respectively.

Specifically we observe that the quasi-local strain hardening modulus $\check{\Phi}$ coincides with Φ, and hence that $\check{\Gamma} = \Gamma$ represents the actual plastic modulus, describing the three different types of material reponse, i.e. hardening when $\Gamma > 0$, softening when $\Gamma < 0$ and perfectly plastic behaviour when $\Gamma = 0$.

Using (3.5-32) and (3.3-45) we conclude that the local counterpart of (3.5-13) becomes

$$\Phi = 1 - \pi\ \mathbf{R} \cdot \frac{\partial g}{\partial \mathbf{E}} + \pi\ \bar{\boldsymbol{\sigma}} \cdot \frac{\partial f}{\partial \mathbf{S}}\ , \tag{3.5-33}$$

while (3.5-15) may be written in the form

$$\dot{\mathbf{S}} = \mathcal{L}(\dot{\mathbf{E}} - \dot{\mathbf{E}}^p) + \frac{\partial \bar{\mathbf{S}}}{\partial \mathbf{E}^p}\dot{\mathbf{E}}^p + \frac{\partial \bar{\mathbf{S}}}{\partial \kappa}\dot{\kappa}. \tag{3.5-34}$$

In addition (3.5-24) by use of (3.3-45) reduces to

$$\Phi = 1 - \pi\ \mathbf{R} \cdot \frac{\partial g}{\partial \mathbf{E}} \tag{3.5-35}$$

and (3.5-25) (or equivalently (3.5-26) to

$$\mathbf{R} = \gamma\ \mathcal{M}\frac{\partial p}{\partial \mathbf{E}}\ ,\ \gamma > 0. \tag{3.5-36}$$

If $p = g$ (associated plasticity) and if in addition $\mathcal{L} = \mathcal{L}^T$, (3.5-36) by use of (3.3-10) further reduces to

$$\mathbf{R} = \gamma\frac{\partial f}{\partial \mathbf{S}},\ \gamma > 0, \tag{3.5-37}$$

i.e. $\mathbf{R}$ is parallel to the outward normal to the yield surface $\partial\mathcal{S}$ in stress space. As seen from (3.5-25) (or equivalently from (3.5-26) it is recalled that this result does not generally follow for associated nonlocal plasticity.

As a final example we conclude that the flow rule $(3.5\text{-}30)_1$ for the plastic strain rate attains the local form

$$\dot{\mathbf{E}}^p = \frac{\hat{f}}{\Gamma}\mathcal{M}\frac{\partial p}{\partial \mathbf{E}},\quad (\Gamma = \check{\Gamma}), \tag{3.5-38}$$

for hardening and softening plasticity, while for perfectly plastic behaviour we correspondingly deduce from (3.5-31), (3.5-32) and (3.5-27) that

$$\dot{\mathbf{E}}^p = \hat{g}\frac{\mathcal{M}\dfrac{\partial p}{\partial \mathbf{E}}}{\dfrac{\partial p}{\partial \mathbf{E}} \cdot \dfrac{\partial f}{\partial \mathbf{S}}}\ . \tag{3.5-39}$$

In view of $(3.2\text{-}16)_1$ and (3.5-36) it is evident that (3.5-38) and (3.5-39) may be replaced by the single condition

$$\dot{\mathbf{E}}^p = \pi\ \gamma\ \hat{g}\ \mathcal{M}\frac{\partial p}{\partial \mathbf{E}}\ , \tag{3.5-40}$$

or by

$$\dot{\mathbf{E}}^p = \pi\ \gamma\ \hat{g}\frac{\partial f}{\partial \mathbf{S}}\ ,\ \pi\ \gamma\ \hat{g} > 0, \tag{3.5 - 41}$$

if (3.5-37) is valid[3.5-2]. The last expression is identical with (3.5-41) and tells us that the plastic strain rate is directed as the outward normal to the yield surface in stress space (as in classical plasticity). Clearly (3.5-41) is not restricted to hardening plasticity, but applies to softening plasticity and perfectly plastic behaviour as well.

[3.5-2]The flow rule (3.4-6) also follows by invoking the principle of maximum dissipation (cf footnote 3.5-1).

3.6 Illustrative example

In this section we will discuss some aspects of nonlocal plasticity theory with reference to a special class of materials characterized by a stress-strain relation of the type briefly discussed in Section 3.5.2.

In Subsection 3.6.1, constitutive equations are presented for an isotropic body with linear stress-strain response and with a yield function of von Mises type.

In Subsection 3.6.2, the functions $\boldsymbol{\alpha}^p$ and α^h are constituted by the aid of time independent attenuation functions, which for fixed $\mathbf{X}$ are assumed to be rapidly decaying with the distance from $\mathbf{X}$. Further, to simplify notations, two types of averaging operators - brackets and braces - are introduced.

In Subsection 3.6.3 we derive expressions for the quasi-local and the actual stress rate. The general nonlocal consistency condition in strain space is established, decomposed into a quasi-local part and a part which vanishes in an average sense.

The classification of strain hardening behaviour is the subject of Subsection 3.6.4, while in Subsection 3.6.5 the consequences of the existence of a plastic potential are discussed.

In Subsection 3.6.6 a global form of the consistency condition is used to produce a solution for the function π appearing in the flow rules. Then, explicit - however approximate - expressions for the elastic-plastic response function and the strain hardening modulus are obtained.

Finally, in Subsection 3.6.7, the principle of maximum dissipation is applied to the class of materials defined in Subsection 3.6.1.

3.6.1 Constitutive assumptions

Consider an isotropic body B represented by a class of materials which satisfies (3.5-21) and let

$$\bar{\mathbf{S}} = \mathcal{L}(\mathbf{E} - < \mathbf{E}^p >), \qquad (3.6-1)$$

and

$$\mathcal{L} = 2\mu\mathcal{J} + (k - \frac{2}{3}\mu)\mathbf{1} \otimes \mathbf{1}, \qquad (3.6-2)$$

where $\mathbf{1}$ is the second order identity tensor, and the positive material parameters $\mu = \mu(\mathbf{X})$ and $k = k(\mathbf{X})$ are the shear modulus and bulk modulus of elasticity, respectively.

If the tensors $\mathbf{E}$, $\mathbf{E}^p$, $\mathbf{S}$ are decomposed into deviatoric parts $\boldsymbol{\gamma}$, $\boldsymbol{\gamma}^p$, $\boldsymbol{\tau}$ and spherical parts $\bar{e}\mathbf{1}$, $\bar{e}^p\mathbf{1}$, $\bar{s}\mathbf{1}$, the stress response (3.6-1) may be written as

$$\boldsymbol{\tau} = 2\mu(\boldsymbol{\gamma} - <\boldsymbol{\gamma}^p>), \quad \bar{s} = 3k(\bar{e} - <\bar{e}^p). \tag{3.6-3}$$

We choose a yield function f similar to that of von Mises type in local theory,

$$f = \boldsymbol{\tau}\cdot\boldsymbol{\tau} - <k> . \tag{3.6-4}$$

Using (3.2-33) and (3.6-3) we then have

$$g = 4\mu^2(\boldsymbol{\gamma} - <\boldsymbol{\gamma}^p>)\cdot(\boldsymbol{\gamma} - <\boldsymbol{\gamma}^p>) - <k> . \tag{3.6-5}$$

From (3.6-4) follows that

$$\frac{\partial f}{\partial \mathbf{S}} = \frac{\partial f}{\partial \boldsymbol{\tau}} = 2\boldsymbol{\tau}, \tag{3.6-6}$$

and hence by (3.3-10) and (3.6-2) (or directly from (3.6-5)) that

$$\frac{\partial g}{\partial \mathbf{E}} = \frac{\partial g}{\partial \boldsymbol{\gamma}} = 4\mu\ \boldsymbol{\tau}. \tag{3.6-7}$$

In view of the definitions (3.2-9) and (3.2-6) we obtain from (3.6-6) and (3.6-7) that

$$\hat{g} = 4\mu\ \boldsymbol{\tau}\cdot\dot{\boldsymbol{\gamma}} \tag{3.6-8}$$

and

$$\hat{f} = 2\boldsymbol{\tau}\cdot\dot{\boldsymbol{\tau}}, \tag{3.6-9}$$

where advantage has been taken of the fact that $\boldsymbol{\tau}\cdot\mathbf{1} = \boldsymbol{\gamma}\cdot\mathbf{1} = 0$, which will be used frequently below.

The relationship (3.3-2) between the actual and the quasi-local stress rate is split into a deviatoric part

$$\dot{\boldsymbol{\tau}} = \check{\boldsymbol{\tau}} + \frac{1}{\rho_0}\mathcal{H}_\tau, \tag{3.6-10}$$

and a spherical part

$$\dot{\bar{s}} = \check{\bar{s}} + \frac{1}{\rho_0}\mathcal{H}_{\bar{s}}, \tag{3.6-11}$$

where $\mathcal{H}_\tau$ is the deviatoric and $\mathcal{H}_{\bar{s}}\mathbf{1}$ the spherical part of $\mathcal{H}_s$, which in the general form is given by (3.2-38). Hence (3.3-20) takes the form

$$\hat{f} = \check{f} + \frac{1}{\rho_0}\frac{\partial f}{\partial \boldsymbol{\tau}} \cdot \mathcal{H}_\tau, \tag{3.6-12}$$

where $\check{f}$ (defined by (3.3-21)) may be written

$$\check{f} = \frac{\partial f}{\partial \boldsymbol{\tau}} \cdot \check{\boldsymbol{\tau}}, \tag{3.6-13}$$

and where $\partial f/\partial \boldsymbol{\tau}$ should be replaced by $2\boldsymbol{\tau}$ according to (3.6-6).

Flow rules are assumed in accordance with (3.2-19). Specifically, we write

$$\dot{\gamma}^p = \pi \; \hat{g} \; \boldsymbol{\rho} \quad (g = 0, \;\; \hat{g} \geq 0), \tag{3.6-14}$$

where $\boldsymbol{\rho}$ is the deviatoric part of the second order tensor $\mathbf{R}$ in (3.2-16).

3.6.2 Attenuation functions

We now turn to the matter of selecting functions $\boldsymbol{\alpha}^p$ and α^h. With the restricted form (3.5-4) as our starting point we assume

$$\boldsymbol{\alpha}^p = \frac{1}{V_p(\mathbf{X})} w^p(|\mathbf{Z} - \mathbf{X}|)\mathbf{E}^p(\mathbf{Z}), \tag{3.6-15}$$

$$\alpha^h = \frac{1}{V_h(\mathbf{X})} w^h(|\mathbf{Z} - \mathbf{X}|)\kappa(\mathbf{Z}), \tag{3.6-16}$$

where V_p and V_h are defined by

$$V_p(\mathbf{X}) = \int_B w^p(|\mathbf{Z} - \mathbf{X}|) \; dV(\mathbf{Z}), \tag{3.6-17}$$

and

$$V_h(\mathbf{X}) = \int_B w^h(|\mathbf{Z} - \mathbf{X}|) \; dV(\mathbf{Z}), \tag{3.6-18}$$

respectively. Here w^p and w^h are scalar, time independent *attenuation* or *influence functions*, by which are constructed *representative volumes* V_p andV_h, being characteristic measures for the assigned body B with volume $V(B)$. From physical considerations it is reasonable that the attenuation functions decay smoothly and rapidly with the distance from $\mathbf{X}$, as is the case when w^p and w^h are assumed to be of exponential form. It is evident that the nonlocal feature of a certain material behaviour is considerably affected by the choice of attenuation functions. For the present purpose, however, it is not necessary to specify w^p and w^h [3.6-1].

Before continuing it is convenient to introduce some new notations. Define attenuation functions $\tilde{w}(\mathbf{Z},\mathbf{X})$ and $\tilde{w}(\mathbf{X},\mathbf{Z})$ by the relationship

$$\tilde{w}(\mathbf{Z},\mathbf{X}) = \frac{1}{V(\mathbf{X})} w(|\mathbf{Z}-\mathbf{X}|) \qquad (3.6-19)$$

and

$$\tilde{w}(\mathbf{X},\mathbf{Z}) = \frac{1}{V(\mathbf{Z})} w(|\mathbf{X}-\mathbf{Z}|), \qquad (3.6-20)$$

respectively, where $w(\tilde{w})$ stands for either $w^p(\tilde{w}^p)$ or $w^h(\tilde{w}^h)$. We use *brackets* and *braces* to denote two different types of averaging operators,

$$\overline{[Q]} = \int_B \tilde{w}(\mathbf{Z},\mathbf{X}) Q(\mathbf{Z}) dV(\mathbf{Z}), \qquad (3.6-21)$$

$$\overline{\{Q\}} = \int_B \tilde{w}(\mathbf{X},\mathbf{Z}) Q(\mathbf{Z}) dV(\mathbf{Z}), \qquad (3.6-22)$$

for every scalar, vector or tensor valued function Q. Since

$$\frac{1}{V(\mathbf{X})}\int_B \tilde{w}(\mathbf{X},\mathbf{Z}) Q(\mathbf{Z}) dV(\mathbf{Z}) = \frac{1}{V(\mathbf{X})}\int_B \frac{1}{V(\mathbf{Z})} w(|\mathbf{X}-\mathbf{Z}|) Q(\mathbf{Z}) dV(\mathbf{Z})$$

$$= \int_B \tilde{w}(\mathbf{Z},\mathbf{X}) \frac{Q(\mathbf{Z})}{V(\mathbf{Z})} dV(\mathbf{Z}), \qquad (3.6-23)$$

we conclude from (3.6-21) and (3.6-22) that

$$\overline{\{Q\}}/V = \overline{[Q/V]}. \qquad (3.6-24)$$

[3.6-1]The influence of the choice of attenuation functions on localization in a strain softening solid is discussed in Chapter 4.

The function $\int_B \tilde{w}(\mathbf{X},\mathbf{Z})dV(\mathbf{Z})$ is simply denoted by β, i.e.

$$\beta = \int_B \tilde{w}(\mathbf{X},\mathbf{Z})dV(\mathbf{Z}) = \overline{\{1\}}, \qquad (3.6-25)$$

where the second equality is due to (3.6-22). Hence in view of (3.6-17), (3.6-18) and (3.6-25)

$$\overline{[1]}_p = \overline{[1]}_h = 1, \quad \overline{\{1\}}_p = \beta^p, \quad \overline{\{1\}}_h = \beta^h. \qquad (3.6-26)$$

Also note that the time derivative of the nonlocal plastic strain variable may be written

$$\begin{aligned} < \dot{\mathbf{E}}^p > &= \frac{d}{dt}\int_B \tilde{w}^p(\mathbf{Z},\mathbf{X})\mathbf{E}^p(\mathbf{Z})dV(\mathbf{Z}) \\ &= \int_B \tilde{w}^p(\mathbf{Z},\mathbf{X})\dot{\mathbf{E}}^p(\mathbf{Z})dV(\mathbf{Z}) = \overline{[\dot{\mathbf{E}}^p]}_p, \end{aligned} \qquad (3.6\text{-} 27)$$

where we have used the fact that the attenuation function $\tilde{w}^p$ does not depend on time, and where also (3.6-21) has been used to obtain the last equality. For the time derivative of the strain hardening variable we write similarly

$$< \dot{\kappa} >= \int_B \tilde{w}^h(\mathbf{Z},\mathbf{X})\dot{\kappa}(\mathbf{Z})dV(\mathbf{Z}) = \overline{[\dot{\kappa}]}_h. \qquad (3.6-28)$$

In order to evaluate the quasi-local elastic-plastic response function $\check{\mathcal{K}}$ and the corresponding quasi-local strain hardening modulus $\check{\Phi}$ we need the derivatives of the functions $\boldsymbol{\alpha}^p$ and α^h. From (3.6-15) and (3.6-16) we deduce that

$$\frac{\partial \boldsymbol{\alpha}^p}{\partial \mathbf{E}^p} = \mathcal{O}, \quad \frac{\partial \alpha^h}{\partial \kappa} = 0, \qquad (3.6-29)$$

while

$$\frac{\partial \boldsymbol{\alpha}^p}{\partial \mathbf{E}^p(\mathbf{Z})} = \frac{1}{V_p(\mathbf{X})} w^p(|\mathbf{Z}-\mathbf{X}|)\mathcal{J} = \tilde{w}^p(\mathbf{Z},\mathbf{X})\mathcal{J} \qquad (3.6-30)$$

and

$$\frac{\partial \alpha^h}{\partial \kappa(\mathbf{Z})} = \frac{1}{V_h(\mathbf{X})} w^h(|\mathbf{Z}-\mathbf{X}|) = \tilde{w}^h(\mathbf{Z},\mathbf{X}), \qquad (3.6-31)$$

where $\mathcal{J}$ is the fourth order unit tensor, defined by (3.3-12), and where (3.6-19) has been used to rewrite the right-hand side of (3.6-30) and (3.6-31). Further, in view of (2.4-9), we observe that

$$(\boldsymbol{\alpha}^p)^* = \frac{1}{V_p(\mathbf{Z})} w^p(|\mathbf{X}-\mathbf{Z}|)\mathbf{E}^p(\mathbf{X}) = \tilde{w}^p(\mathbf{X},\mathbf{Z})\mathbf{E}^p(\mathbf{X}) \qquad (3.6-32)$$

and

$$(\alpha^h)^* = \frac{1}{V_h(\mathbf{Z})} w^h(|\mathbf{X}-\mathbf{Z}|)\kappa(\mathbf{X}) = \tilde{w}^h(\mathbf{X},\mathbf{Z})\kappa(\mathbf{X}), \qquad (3.6-33)$$

where (3.6-20) has also been used. Thus

$$\frac{\partial(\boldsymbol{\alpha}^p)^*}{\partial \mathbf{E}^p(\mathbf{Z})} = \mathcal{O}, \quad \frac{\partial(\alpha^h)^*}{\partial \kappa(\mathbf{Z})} = 0, \qquad (3.6-34)$$

while

$$\frac{\partial(\boldsymbol{\alpha}^p)^*}{\partial \mathbf{E}^p} = \tilde{w}^p(\mathbf{X},\mathbf{Z})\mathcal{J}, \qquad (3.6-35)$$

and

$$\frac{\partial(\alpha^h)^*}{\partial \kappa} = \tilde{w}^h(\mathbf{X},\mathbf{Z}). \qquad (3.6-36)$$

3.6.3 Stress rates and the consistency condition

With the preliminaries worked out in the preceding section, we are ready to investigate some characteristic features of material response due to the special choice of functions $\boldsymbol{\alpha}^p$ and α^h, given by (3.6-15) and (3.6-16). Using (3.5-2), $(3.5\text{-}5)_2$, (3.5-7) and (3.6-1) we obtain

$$\rho_0\mathcal{L}^* = \int_B (\rho_0\mathcal{L}\frac{\partial\boldsymbol{\alpha}^p}{\partial\mathbf{E}^p} + \rho_0(\mathbf{Z})\mathcal{L}(\mathbf{Z})\frac{\partial(\boldsymbol{\alpha}^p)^*}{\partial\mathbf{E}^p})dV(\mathbf{Z}), \qquad (3.6-37)$$

which in view of $(3.6\text{-}29)_1$, (3.6-35) and (3.6-22) reduces to

$$\rho_0\mathcal{L}^* = \int_B \rho_0(\mathbf{Z})\mathcal{L}(\mathbf{Z})\tilde{w}^p(\mathbf{X},\mathbf{Z})dV(\mathbf{Z}) = \overline{\{\rho_0\mathcal{L}\}}_p. \qquad (3.6-38)$$

If we construct quantities

$$\rho_0\mu^* = \int_B \rho_0(\mathbf{Z})\mu(\mathbf{Z})\tilde{w}^p(\mathbf{X},\mathbf{Z})dV(\mathbf{Z}) = \overline{\{\rho_0\mu\}}_p \qquad (3.6-39)$$

and

$$\rho_0 k^* = \int_B \rho_0(\mathbf{Z})k(\mathbf{Z})\tilde{w}^p(\mathbf{X},\mathbf{Z})dV(\mathbf{Z}) = \overline{\{\rho_0 k\}}_p, \qquad (3.6-40)$$

then (3.6-38) in combination with (3.6-2) takes the form

$$\mathcal{L}^* = 2\mu^* \mathcal{J} + (k^* - \frac{2}{3}\mu^*)\mathbf{1} \otimes \mathbf{1}. \tag{3.6-41}$$

Apparently μ^* and k^* are nonlocal measures of the elastic properties of the material. However, due to the dependence on the representative volume V_p, μ^* and k^* cannot be replaced by μ and k, even if the body is initially homogeneous and the elastic moduli are constants. In that case, i.e. when μ and k do not depend on position (3.6-38), (3.6-39) and (3.6-40) reduce to

$$\left.\begin{aligned} \mathcal{L}^* &= \mathcal{L}\overline{\{1\}}_p = \beta^p \mathcal{L}, \\ \mu^* &= \mu\overline{\{1\}}_p = \beta^p \mu, \\ k^* &= k\overline{\{1\}}_p = \beta^p k, \end{aligned}\right\} \tag{3.6-42}$$

where (3.6-25) has also been used. Nevertheless we will refer to μ^* and k^* as the nonlocal shear modulus and the nonlocal bulk modulus, respectively.

From (3.5-14) we conclude that the quasi-local stress rate (originally defined by (3.3-2)) becomes

$$\check{\mathbf{S}} = \mathcal{L}\,\dot{\mathbf{E}} - \mathcal{L}^*\,\dot{\mathbf{E}}^p, \tag{3.6-43}$$

and hence in view of (3.6-41), allowing the decompositions

$$\check{\boldsymbol{\tau}} = 2\mu\,\dot{\boldsymbol{\gamma}} - 2\mu^*\,\dot{\boldsymbol{\gamma}}^p, \tag{3.6-44}$$

and

$$\check{s} = 3k\dot{e} - 3k^*\dot{e}^p, \tag{3.6-45}$$

into deviatoric and spherical parts, respectively.

Remark 6.1 If (3.6-42) is valid (3.6-43) reduces to

$$\check{\mathbf{S}} = \mathcal{L}(\dot{\mathbf{E}} - \beta^p \dot{\mathbf{E}}^p) \tag{3.6-46}$$

(with similar expressions for $\check{\boldsymbol{\tau}}$ and $\check{s}$). The almost entirely local form of the quasi-local stress rate is due to the simple constitutive assumption (3.6-1). The appearance of β^p in (3.6-46) may be looked upon as the result of boundary effects. If w^p is a rapidly decaying attenuation function, then at a distance from the boundary of the body, β^p is close to unity, as is seen from its definition (3.6-25). However, a second look at (3.5-5) and (3.5-7) shows that for a general, nonlinear response, $\check{\mathbf{S}}$ will depend on the nonlocal inelastic variables (but of course not on their corresponding rates).□

From (3.6-1) and (3.6-29)–(3.6-36) we deduce that (3.5-16) becomes

$$\mathcal{H}_s = \int_B \{-\rho_0 \mathcal{L}\tilde{w}^p(\mathbf{Z},\mathbf{X})\dot{\mathbf{E}}^p(\mathbf{Z}) + \rho_0(\mathbf{Z})\mathcal{L}(\mathbf{Z})\tilde{w}^p(\mathbf{X},\mathbf{Z})\dot{\mathbf{E}}^p\}\ dV(\mathbf{Z}), \qquad (3.6-47)$$

and hence by (3.6-27) and (3.6-38)

$$\frac{1}{\rho_0}\mathcal{H}_s = -\mathcal{L} < \dot{\mathbf{E}}^p > +\mathcal{L}^*\ \dot{\mathbf{E}}^p. \qquad (3.6-48)$$

Then (3.3-2) in combination with (3.6-43) and (3.6-48) gives

$$\dot{\mathbf{S}} = \mathcal{L}(\dot{\mathbf{E}}^p - < \dot{\mathbf{E}}^p >), \qquad (3.6-49)$$

which of course follows directly from (3.6-1) by differentiation with respect to time.

Remark 6.2. It should come as no surprise that the term $\mathcal{L}^*\dot{\mathbf{E}}^p$ present in (3.6-43) and (3.6-48) cancels out in (3.6-49). If, namely, we decompose the stress rate into two parts according to

$$\dot{\mathbf{S}} = (\mathcal{L}\dot{\mathbf{E}}^p - \mathcal{L}^*\dot{\mathbf{E}}^p) + (\mathcal{L}^*\dot{\mathbf{E}}^p - \mathcal{L} < \dot{\mathbf{E}}^p >), \qquad (3.6-50)$$

it is easily seen by use of (3.6-27) and (3.6-38) that

$$\int_B \rho_0(-\mathcal{L} < \dot{\mathbf{E}}^p > +\mathcal{L}^*\dot{\mathbf{E}}^p)dV = \mathbf{0}. \qquad (3.6-51)$$

Hence we may identify the second part of (3.6-50) with $\mathcal{H}_s/\rho_0$ and the first part with $\check{\mathbf{S}}$, recovering the results contained in (3.6-43) and (3.6-48). (Recall that $\mathcal{H}_s$ according to (3.2-41) is defined such that $\int_B \mathcal{H}_s\ dV = \mathbf{0}$.) Needless to say, $\mathcal{H}_s$ cannot in general be obtained so easily, but must be evaluated from (3.2-39). The simple argument used here to decompose the stress rate relies heavily on the fact that $\bar{\mathbf{S}}$ is linear in $(\mathbf{E} - < \mathbf{E}^p >)$, and $\boldsymbol{\alpha}^p$ is linear in $\mathbf{E}^p$.)□

It is also enlightening to evaluate the decomposition (3.2-21) of $\dot{g}$ and (3.3-35) of $\dot{f}$, respectively. From (3.2-22) and (3.2-23) in combination with (3.6-29) and (3.6-35) we conclude that

$$\frac{\partial < \rho_0 g >}{\partial \mathbf{E}^p} = \int_B \rho_0(\mathbf{Z})\frac{\partial g}{\partial < \mathbf{E}^p >}(\mathbf{Z})\tilde{w}^p(\mathbf{X},\mathbf{Z})dV(\mathbf{Z}), \qquad (3.6-52)$$

and hence by (3.6-3), (3.6-5) and (3.6-22) that

$$\frac{\partial < \rho_0 g >}{\partial \mathbf{E}^p} = -4\overline{\{\rho_0 \mu \boldsymbol{\tau}\}}_p. \qquad (3.6-53)$$

Similarly

$$\frac{\partial < \rho_0 g >}{\partial k} = \int_B \rho_0(\mathbf{Z}) \frac{\partial g}{\partial < k >}(\mathbf{Z}) \tilde{w}^h(\mathbf{X}, \mathbf{Z}) dV(\mathbf{Z}) = -\overline{\{\rho_0\}}_h. \qquad (3.6-54)$$

Further, it follows from (3.2-24), (3.6-30), (3.6-31), (3.6-35) and (3.6-36) that

$$\begin{aligned} \mathcal{H}_g &= \int_B \{\rho_0 (\frac{\partial g}{\partial < \mathbf{E}^p >} \tilde{w}^p(\mathbf{Z}, \mathbf{X}) \dot{\mathbf{E}}^p(\mathbf{Z}) + \frac{\partial g}{\partial < k >} \tilde{w}^h(\mathbf{Z}, \mathbf{X}) \dot{\kappa}(\mathbf{Z})) \\ &- \rho_0(\mathbf{Z}) (\frac{\partial g}{\partial < \mathbf{E}^p >}(\mathbf{Z}) \tilde{w}^p(\mathbf{X}, \mathbf{Z}) \dot{\mathbf{E}}^p \\ &+ \frac{\partial g}{\partial < k >}(\mathbf{Z}) \tilde{w}^h(\mathbf{X}, \mathbf{Z}) \dot{\kappa}(\mathbf{X}))\} dV(\mathbf{Z}). \end{aligned} \qquad (3.6\text{- }55)$$

Hence, again using (3.6-3) and (3.6-5), and additionally (3.6-27) and (3.6-28),

$$\mathcal{H}_g = -4\rho_0 \mu \boldsymbol{\tau} \cdot < \dot{\mathbf{E}}^p > - \rho_0 < \dot{\kappa} > + 4\overline{\{\rho_0 \mu \boldsymbol{\tau}\}}_p \cdot \dot{\mathbf{E}}^p + \overline{\{\rho_0\}}_h \dot{\kappa}. \qquad (3.6\text{- }56)$$

A simple calculation shows that (3.6-56) satisfies the condition (3.2-41). If we now insert (3.6-8), (3.6-53), (3.6-54) and (3.6-56) into (3.2-21) we obtain the result expected,

$$\dot{g} = 4\mu \boldsymbol{\tau} \cdot \dot{\boldsymbol{\gamma}} - 4\mu \boldsymbol{\tau} \cdot < \dot{\boldsymbol{\gamma}}^p > - < \dot{\kappa} >, \qquad (3.6-57)$$

in agreement with (3.6-5). (We have also made use of the fact that $< \dot{\mathbf{E}}^p >$ may be replaced by $< \dot{\boldsymbol{\gamma}} >$ in the second term of (3.6-57).)

For the yield function in stress space, expressions corresponding to (3.6-53), (3.6-54) and (3.6-56) read

$$\frac{\partial < \rho_0 f >}{\partial \mathbf{E}^p} = \mathbf{0}, \quad \frac{\partial < \rho_0 f >}{\partial \kappa} = -\overline{\{\rho_0\}}_h, \qquad (3.6-58)$$

and

$$\mathcal{H}_f = -\rho_0 < \dot{\kappa} > + \overline{\{\rho_0\}}_h \dot{\kappa}. \qquad (3.6-59)$$

Applying (3.2-35) then verifies that

$$\dot{f} = 2\boldsymbol{\tau} \cdot \dot{\boldsymbol{\tau}} - < \dot{\kappa} > . \qquad (3.6-60)$$

From (3.2-19), (3.6-14), (3.6-53) and (3.6-54) it follows that the consistency condition (3.2-31) becomes

$$1-\frac{\pi}{\rho_0}(4\overline{\{\rho_0\mu\boldsymbol{\tau}\}}_p\cdot\boldsymbol{\rho}+\overline{\{\rho_0\}}_h r)+\frac{1}{\rho_0\hat{g}}\mathcal{H}_g=0, \qquad (3.6-61)$$

where $\mathcal{H}_g$, recorded in (3.6-56), should be reduced by means of the flow rules (cf. (3.2-30)). The corresponding global statement (3.2-32) becomes

$$\int_{B_L}(1-\frac{\pi}{\rho_0}\left(4\overline{\{\rho_0\mu\boldsymbol{\tau}\}}_p\cdot\boldsymbol{\rho}+\overline{\{\rho_0\}}_h r)\rho_0\hat{g}\right)dV=0, \qquad (3.6-62)$$

a condition which the function π necessarily must satisfy.

Substitution of (3.6-56) into (3.6-61) (or directly from (3.6-57)) gives the consistency condition in strain space in the explicit form

$$1-4\mu\boldsymbol{\tau}\cdot\frac{1}{\hat{g}}\overline{[\pi\hat{g}\boldsymbol{\rho}]}_p-\frac{1}{\hat{g}}\overline{[\pi\hat{g}r]}_h=0, \quad (g=0,\ \hat{g}>0), \qquad (3.6-63)$$

where (3.6-21) has also been used and where $\hat{g}$ is given by (3.6-8), while the corresponding expression in stress space reads (cf. (3.6-60))

$$2\boldsymbol{\tau}\cdot\dot{\boldsymbol{\tau}}-\overline{[\pi\hat{g}r]}_h=0, \quad (f=0,\ \hat{g}>0). \qquad (3.6-64)$$

As a final illustration before leaving this subsection we verify (3.2-46) concerning the relationship between the strain space formulation and the corresponding stress space formulation. In view of (3.5-6) and (3.5-8), in combination with (3.6-8), (3.6-9) and (3.6-6), it follows that the left-hand side of (3.2-46) can be written

$$\begin{aligned}&\rho_0\boldsymbol{\tau}\cdot(4\mu\ \dot{\boldsymbol{\gamma}}-2\dot{\boldsymbol{\tau}})+2\boldsymbol{\tau}(-\rho_0\mathcal{L}^*\dot{\mathbf{E}}^p\\ &-\rho_0\mathcal{L}<\dot{\mathbf{E}}^p>+\rho_0\mathcal{L}^*\dot{\mathbf{E}}^p)=-2\rho_0\boldsymbol{\tau}\cdot(\dot{\boldsymbol{\tau}}-2\mu(\dot{\boldsymbol{\gamma}}-<\dot{\boldsymbol{\gamma}}^p>)).\end{aligned} \qquad (3.6-65)$$

By substituting $(3.6\text{-}3)_1$ into $(3.6\text{-}65)_2$ we conclude that (3.2-46) is satisfied.

3.6.4 Classification of strain hardening behaviour

The quasi-local elastic-plastic response function related to the quasi-local stress rate (3.6-43) is recorded in (3.5-22) for the special case when $\breve{\boldsymbol{\sigma}}=\mathbf{0}$. Hence by (3.6-6)

$$\breve{\mathcal{K}}=\mathcal{J}-2\pi\mathcal{L}^*\mathbf{R}\otimes\boldsymbol{\tau}, \qquad (3.6-66)$$

where the explicit form of $\mathcal{L}^*$ is given by (3.6-38). In view of (3.5-23) the corresponding quasi-local strain hardening modulus becomes

$$\check{\Phi} \;=\; 1 - 2\pi\mathcal{L}^*\mathbf{R}\cdot\boldsymbol{\tau} = 1 - 4\pi\mu^*\boldsymbol{\rho}\cdot\boldsymbol{\tau}, \tag{3.6- 67}$$

where the second equality is due to (3.6-41) and (3.6-14). Alternatively $(3.6\text{-}67)_2$ may be obtained by substitution of (3.6-8), (3.6-13), (3.6-14) and (3.6-44) into (3.3-40). Of course (3.6-67) also follows from (3.3-28) i. e. by the use of the fact that $\check{\Phi}$ equals the determinant of $\check{\mathcal{K}}$.

The relationship between the general nonlocal elastic-plastic response function $\mathcal{K}$ and $\check{\mathcal{K}}$ is given by (3.3-17). Hence by use of (3.6-48), (3.2-16), (3.6-6), (3.6-21) and (3.6-66)

$$\mathcal{K} = \mathcal{J} - \frac{1}{\hat{g}}2\mathcal{L}\,\overline{[\pi\hat{g}\mathbf{R}]}_p \otimes \boldsymbol{\tau}, \tag{3.6 – 68}$$

with $\hat{g}$ given by (3.6-8). We recall that (3.6-66) and (3.6-68) are derived, presupposing loading ($g = 0$, $\hat{g} > 0$). However it is evident that (3.6-66) is valid also at unloading conditions ($\pi = 0$), while $\mathcal{K}$ in general is not.

The fact that $\mathcal{K}$ is rate independent only in a weak sense is here manifested by the appearance of $\hat{g} = \hat{g}(\mathbf{Z})$ inside the averaging bracket in the right-hand side of (3.6-68). Cf. the discussion in Section 3.3.1.

The general nonlocal strain hardening modulus Φ is defined by (3.3-29). Arguing as in the derivation of (3.6-68), we use (3.6-67) to conclude that

$$\Phi \;=\; 1 - \frac{1}{\hat{g}}2\mathcal{L}\,\overline{[\pi\hat{g}\mathbf{R}]}_p\cdot\boldsymbol{\tau} = 1 - \frac{1}{\hat{g}}4\mu\,\overline{[\pi\hat{g}\boldsymbol{\rho}]}_p\cdot\boldsymbol{\tau}, \tag{3.6- 69}$$

which of course also follows directly from (3.6-68) and (3.3-30). As mentioned in Section 3.3.2 the strain hardening modulus Φ may be expressed in various ways. By use of e.g. (3.3-33) in combination with (3.6-58), (3.6-59) and the flow rules (3.2-19) we deduce that

$$\Phi = -(-\frac{1}{\rho_0}\,\overline{\{\rho_0\}}_h\pi\, r + \frac{1}{\rho_0\hat{g}}(-\rho_0\overline{[\pi\hat{g}r]}_h + \overline{\{\rho_0\}}_h\pi\hat{g}r)) = \frac{1}{\hat{g}}\,\overline{[\pi\hat{g}r]}_h. \tag{3.6- 70}$$

Notice that this result also follows from (3.3-31) together with the stress space consistency condition (3.6-64), apparently providing the easiest way to obtain (3.6-70).

Since we have adopted (3.4-22), i. e. $\pi(\mathbf{Z}) > 0$ during loading at point $\mathbf{Z}$, it follows from (3.6-70) and (3.3-32) that

$$\left.\begin{array}{llll} r(\mathbf{Z}) > 0 & \Rightarrow & \Phi(\mathbf{X}) > 0, & \text{hardening behaviour at } \mathbf{X}, \\ r(\mathbf{Z}) < 0 & \Rightarrow & \Phi(\mathbf{X}) < 0, & \text{softening behaviour at } \mathbf{X}, \\ r(\mathbf{Z}) = 0 & \Rightarrow & \Phi(\mathbf{X}) = 0, & \text{perfectly plastic behaviour at } \mathbf{X}, \end{array}\right\} \qquad (3.6-71)$$

where e.g. $r(\mathbf{Z}) > 0$ means $r(\mathbf{Z}) > 0$ for all points $\mathbf{Z}$ at loading. It is noted that the sign of r at $\mathbf{X}$ does not necessarily determine the state of strain hardening at the very point $\mathbf{X}$ (unless r is constant). For instance $r(\mathbf{X}) = 0$ does not imply that Φ vanishes at $\mathbf{X}$, and conversely, perfectly plastic behaviour at $\mathbf{X}$ ($\Phi(\mathbf{X}) = 0$) does not imply that $r(\mathbf{X}) = 0$.

Later, in Subsection 3.6.6, an approximate formulation is outlined of the exact theory with respect to the special class of materials involved here. We postpone until then a discussion about the relationship between the quasi-local strain hardening modulus $\check{\Phi}$ and the different types of strain hardening behaviour.

Finally we note that (3.6-70) is reduced to

$$\Phi = \pi r \qquad (3.6-72)$$

in the restriction to local theory. Hence, since $\pi > 0$, the sign of r uniquely determines the type of strain hardening behaviour.

3.6.5 Associated plasticity

We now presuppose the existence of a plastic potential. If we also assume associated plasticity ($p = g$) it follows from (3.5-25), (3.6-2) and (3.3-10) that

$$\mathcal{L}^* \mathbf{R} = 4\gamma\mu\boldsymbol{\tau}, \qquad (3.6-73)$$

or in view of (3.6-4) that

$$\boldsymbol{\rho} = 2\gamma\frac{\mu}{\mu^*}\boldsymbol{\tau} \ (= \mathbf{R}). \qquad (3.6-74)$$

We note that (3.6-73) implies that $\mathbf{R}$ in fact is deviatoric. Hence (3.6-14), represents evolution of the total plastic strain rate.

Substitution of (3.6-73) into (3.6-66) and (3.6-67) yields

$$\check{\mathcal{K}} = \mathcal{J} - 8\gamma\mu\pi\boldsymbol{\tau} \otimes \boldsymbol{\tau}, \qquad (3.6-75)$$

and

$$\check{\Phi} = 1 - 8\gamma\mu\pi\boldsymbol{\tau}\cdot\boldsymbol{\tau}, \tag{3.6 – 76}$$

respectively. Due to (3.4-13), we note that (3.4-15) can be used to express (3.6-75) in the form

$$\check{\mathcal{K}} = \mathcal{J} - \frac{4\mu\boldsymbol{\tau}\otimes\boldsymbol{\tau}}{\check{\Gamma} + 4\mu\boldsymbol{\tau}\cdot\boldsymbol{\tau}}, \tag{3.6 – 77}$$

where

$$\check{\Gamma} = \frac{1}{\pi\gamma} - 8\mu\boldsymbol{\tau}\cdot\boldsymbol{\tau} \tag{3.6 – 78}$$

is the quasi-local plastic modulus.

Upon substituting (3.6-74) into (3.6-68) and by use of (3.6-2) we note that we can write the general nonlocal elastic-plastic function in the form

$$\mathcal{K} = \mathcal{J} - \frac{1}{\hat{g}}8\mu\overline{[\pi\hat{g}\gamma(\mu/\mu^*)\boldsymbol{\tau}]}_p\otimes\boldsymbol{\tau}, \tag{3.6 – 79}$$

while correspondingly (3.6-69) yields

$$\Phi = 1 - \frac{1}{\hat{g}}8\mu\overline{[\pi\hat{g}\gamma(\mu/\mu^*)\boldsymbol{\tau}]}_p\cdot\boldsymbol{\tau}, \tag{3.6 – 80}$$

for the nonlocal strain hardening modulus.

Remark 6.3. For homogeneous materials the quotient μ/μ^* appearing in (3.6-74), (3.6-79) and (3.6-80) should be replaced by $1/\beta^p$ in accordance with $(3.6\text{-}42)_2$. If we choose γ such that

$$\gamma(\mu/\mu^*) = 1, \tag{3.6 – 81}$$

(3.6-74) becomes

$$\boldsymbol{\rho} = 2\boldsymbol{\tau} = \frac{\partial f}{\partial\boldsymbol{\tau}} \tag{3.6 – 82}$$

– the second equality being due to (3.6-6) – hence rendering (3.6-14) the well-known form

$$\dot{\gamma}^p = \pi \hat{g} \frac{\partial f}{\partial \boldsymbol{\tau}}, \tag{3.6-83}$$

where $\pi \hat{g}$ may be identified as a (nonlocal) plastic multiplier. More generally, if we assume

$$\mathbf{R} = \frac{\partial f}{\partial \mathbf{S}}, \tag{3.6-84}$$

it is necessary (cf. (3.5-25)) that

$$\mathcal{L}^* \frac{\partial f}{\partial \mathbf{S}} = \gamma \mathcal{L}^T \frac{\partial f}{\partial \mathbf{S}}, \tag{3.6-85}$$

i.e. $\partial f / \partial \mathbf{S}$ must be an eigenvector of the tensor $(\mathcal{L}^{-1})^T \mathcal{L}^*$ with eigenvalue γ. Particularly if $\mathcal{L}$ is symmetric and $\mathcal{L}^* = \beta \mathcal{L}$, then $\gamma = \beta$.□

3.6.6 An approximate formulation

In local theory the consistency condition is represented by an algebraic equation, by which the function π may be solved for, as discussed in Section 3.2.2 previously. Here the nonlocal consistency condition is recorded explicitly in (3.6-63). This relationship constitutes an integral equation for π which must be solved by numerical methods, an issue that will be dealt with in Chapter 4. For the moment we will proceed by solving (3.6-63) (or equivalently (3.6-61) only in an average sense - precisely by providing a solution that satisfies (3.6-62). Trivially [3.6-2]

$$\frac{1}{\pi} = \frac{1}{\rho_0} (4 \overline{\{\rho_0 \mu \boldsymbol{\tau}\}}_p \cdot \boldsymbol{\rho} + \overline{\{\rho_0\}}_h r) \tag{3.6-86}$$

is such a solution. Though (3.6-86) in view of (3.6-61) corresponds to $\mathcal{H}_g = 0$, note that no such restriction is imposed, i. e. (3.6-86) is not in general an exact solution of (3.6-61). Since $\mathcal{H}_g$ (or correspondingly $\mathcal{H}_f$) may be considered as a measure of the nonlocality of the yield function in strain space (stress space), it is clear that the less the yield function deviates from its local counterpart, the more accurate is (3.6-86).

To make it easier to recognize typical features of nonlocal material behaviour and to facilitate comparison with local theory, we restrict the class of materials defined by (3.6-1) and (3.6-2) by the assumption that μ and k do not depend on position and that the body in consideration is initially homogeneous. Further, we assume that the

[3.6-2] We recall that π cannot vanish at loading points (cf. (3.4-22)).

constitutive function r is of the form

$$r = \frac{1}{\beta^h} 4\mu' \overline{\{\boldsymbol{\tau}\}_p} \cdot \boldsymbol{\rho} = \frac{1}{\beta^h \beta^p} 8\gamma\mu' \overline{\{\boldsymbol{\tau}\}_p} \cdot \boldsymbol{\tau}, \qquad (3.6\text{-}87)$$

where the second equality is due to (3.6-74) and $(3.6\text{-}42)_2$, and where μ' is a material constant subject to the condition

$$-\mu' < \mu. \qquad (3.6-88)$$

Then from (3.6-86), $(3.6\text{-}87)_1$ and (3.6-26) it follows that

$$\frac{1}{\pi} = 4(\mu + \mu')\overline{\{\boldsymbol{\tau}\}_p} \cdot \boldsymbol{\rho}, \qquad (3.6-89)$$

or by repeated use of (3.6-74) and $(3.6\text{-}42)_2$,

$$\frac{1}{\pi} = 8\gamma(\mu + \mu') \frac{1}{\beta^p} \overline{\{\boldsymbol{\tau}\}_p} \cdot \boldsymbol{\tau}. \qquad (3.6-90)$$

We note that the solution (3.6-90) requires that

$$\overline{\{\boldsymbol{\tau}\}_p} \cdot \boldsymbol{\tau} > 0 \qquad (3.6-91)$$

at all loading points. Since $\overline{\{\boldsymbol{\tau}\}_p} = \overline{\{\boldsymbol{\tau}\}_p}(\mathbf{X})$ represents an averaged measure of the stress $\boldsymbol{\tau}$ at $\mathbf{X}$ due to nonlocal interaction, and since we tacitly assume rapidly decaying attenuation functions, it is reasonable to suppose that (3.6-91) always holds true.

The quasi-local response function $\check{\mathcal{K}}$ and the corresponding quasi-local strain hardening modulus $\check{\Phi}$ are now obtained by substituting (3.6-90) into (3.6-75) and (3.6-76), respectively. Similar expressions for $\mathcal{K}$ and Φ follow from (3.6-79) and (3.6-79). As for the quasi-local strain hardening modulus we conclude that

$$\check{\Phi} = 1 - \frac{\mu}{\mu + \mu'} \frac{\beta^p \boldsymbol{\tau} \cdot \boldsymbol{\tau}}{\overline{\{\boldsymbol{\tau}\}_p} \cdot \boldsymbol{\tau}}, \qquad (3.6-92)$$

while the general nonlocal strain hardening modulus becomes

$$\Phi = 1 - \frac{\mu}{\mu + \mu'} \frac{1}{\hat{g}} \overline{[\hat{g}\boldsymbol{\tau}/(\overline{\{\boldsymbol{\tau}\}_p} \cdot \boldsymbol{\tau})]_p} \cdot \boldsymbol{\tau}. \qquad (3.6-93)$$

Recall that (3.9-92) and (3.6-93) are approximations. Another approximate expression for Φ may be obtained by substituting (3.6-87) and (3.6-90) into (3.6-70). Note, however, that the result is *not* equivalent to (3.6.93), since the consistency condition is not exactly fulfilled.

We observe that each of the above expressions for $\check{\Phi}$ and Φ reduces to

$$\Phi = \check{\Phi} = \frac{\mu'}{\mu + \mu'} \qquad (3.6-94)$$

in local theory, i.e. hardening, softening and perfectly plastic behaviour then occur in accordance with whether

$$\mu' > 0, \ \mu' < 0 \quad \text{or } \mu' = 0, \qquad (3.6-95)$$

respectively.

In the nonlocal case the sign of μ' affords a corresponding simple classification of the state of strain hardening - though not quite as immediately as in local theory. (Cf. the discussion in relation to (3.6-71)). Substituting (3.6-87) into (3.6-70) yields

$$\Phi = 8\mu' \frac{1}{\hat{g}} \overline{[(\gamma/(\beta^h \ \beta^p))\pi\hat{g}\overline{\{\boldsymbol{\tau}\}}_p \cdot \boldsymbol{\tau}]}_h, \qquad (3.6-96)$$

an expression in which no approximations are involved. Hence, adopting (3.6-91), we conclude that μ' classifies the strain hardening behaviour in the same manner as in the corresponding local formulation. Does the same conclusion apply with regard to the approximate formulation? A look at (3.6-93) reveals that this is not the case. If we consider the case $\mu' = 0$, we note that Φ apparently may be of any sign, in contradiction to the fact that the material is behaving perfectly plastic.

Before further discussing this deviation of Φ from zero in the case of perfectly plastic behaviour, we recall that (3.6-86) is justified as an approximate solution of the general nonlocal consistency condition (3.6-61) if

$$\frac{1}{\rho_0 \hat{g}} |\mathcal{H}_g| << 1. \qquad (3.6-97)$$

If nothing else is stated we adopt (3.6-97)[3.6-3]. If we want a unified structure of approximations we should also require that[3.6-4]

$$\frac{1}{\rho_0 \hat{g}} |\mathcal{H}_f| << 1 \qquad (3.6-98)$$

[3.6-3] Previously we did not; only consistency in an average sense was required.

[3.6-4] In view of (3.2-45) it is evident that the conditions (3.6-97)-(3.6-99) are not completely independent.

and that

$$\frac{1}{\rho_0 \hat{g}} |\boldsymbol{\mathcal{H}}_s \cdot \frac{\partial f}{\partial \mathbf{S}}| << 1. \qquad (3.6-99)$$

The scalars $\mathcal{H}_g$ and $\mathcal{H}_f$ and the tensor $\boldsymbol{\mathcal{H}}_s$ are recorded in general forms in Section 3.6.3. Here, however, it will not be necessary to evaluate these inequalities explicitly.

Due to (3.6-99) it follows from (3.3-29) that

$$|\Phi - \check{\Phi}| << 1, \qquad (3.6-100)$$

i.e. the quasi-local strain hardening modulus approximately equals the corresponding general nonlocal modulus. Continuing the discussion with regard to perfectly plastic behaviour, we note that (3.6.92) and (3.6-100) give rise to the inequality

$$\left| \overline{\{\boldsymbol{\tau}\}}_p \cdot \boldsymbol{\tau} - \beta^p \boldsymbol{\tau} \cdot \boldsymbol{\tau} \right| << 1, \qquad (3.6-101)$$

substantially a condition of a form which should be expected. By virtue of (3.6-25) we observe that the presence of the function β^p forces the left-hand side of (3.6-101) to vanish for a homogeneous stress state.

In accordance with the discussion above, we may conclude that $\check{\Phi}$ uniquely determines the state of strain hardening - unless $\check{\Phi}$ is close to zero, i.e. in the case of nearly perfectly plastic behaviour. Evidently the same argument applies to the quasi-local plastic modulus $\check{\Gamma}$, as discussed in general terms in Section 3.4.1.

Remark 6.4. Approximations of different significance have been involved here. We recall that it is (3.6-97) alone that legitimates the approximate evaluation of π from the general nonlocal consistency condition. The *ad hoc* condition (3.6-99) facilitates the classification of strain hardening considerably and constitutes the base for the quasi-local formulation discussed in Section 3.3.3.□

For the sake of completeness we also record the expressions for the response functions $\check{\mathcal{K}}$ and $\mathcal{K}$. Substitution of (3.6-90) into (3.6-75) and (3.6-79) yields

$$\check{\mathcal{K}} = \mathcal{J} - \frac{\mu}{\mu + \mu'} \frac{\beta^p \boldsymbol{\tau} \otimes \boldsymbol{\tau}}{\overline{\{\boldsymbol{\tau}\}}_p \cdot \boldsymbol{\tau}}, \qquad (3.6-102)$$

and

$$\mathcal{K} = \mathcal{J} - \frac{\mu}{\mu + \mu'} \frac{1}{\hat{g}} \overline{[\hat{g}\boldsymbol{\tau}/(\overline{\{\boldsymbol{\tau}\}}_p \cdot \boldsymbol{\tau})]}_p \otimes \boldsymbol{\tau}, \qquad (3.6-103)$$

respectively. The corresponding stress rates (3.3-13) and (3.3-15) are easily derived by the aid of (3.6-2) and (3.6-8) and become

$$\check{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}} - \frac{2\mu^2\beta^p\boldsymbol{\tau}\cdot\dot{\boldsymbol{\gamma}}}{(\mu+\mu')\overline{\{\boldsymbol{\tau}\}}_p\cdot\boldsymbol{\tau}}\boldsymbol{\tau} \tag{3.6 – 104}$$

and

$$\dot{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}} - \frac{2\mu^2}{\mu+\mu'}\overline{[\boldsymbol{\tau}\cdot\dot{\boldsymbol{\gamma}}\boldsymbol{\tau}/(\overline{\{\boldsymbol{\tau}\}}_p\cdot\boldsymbol{\tau})]}_p, \tag{3.6 – 105}$$

respectively. Of course these results may alternatively be derived using (3.6-46) and (3.6-49) as the starting point.

In local theory, (cf. (3.6-94)) we note that

$$\check{\mathcal{K}} = \mathcal{K} = \mathcal{J} - \frac{\mu}{\mu+\mu'}\frac{\boldsymbol{\tau}\otimes\boldsymbol{\tau}}{\boldsymbol{\tau}\cdot\boldsymbol{\tau}}, \tag{3.6 – 106}$$

while

$$\check{\mathbf{S}} = \dot{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}} - \frac{2\mu^2\boldsymbol{\tau}\cdot\dot{\boldsymbol{\gamma}}}{(\mu+\mu')\boldsymbol{\tau}\cdot\boldsymbol{\tau}}\boldsymbol{\tau}. \tag{3.6 – 107}$$

Remark 6.5. The explicit nonlocal character of the quasi-local stress rate does not justify that the actual stress rate be replaced by its quasi-local counterpart. The case $\mathcal{H}_s \equiv \mathbf{0}$ represents restricted nonlocality of quite another kind than that induced by (3.6-97)-(3.6-99), as discussed in Section 3.3.3. Most significant in the comparison between the expressions above for $\check{\mathbf{S}}$ and $\dot{\mathbf{S}}$ is of course the fact that the strain rate $\dot{\boldsymbol{\gamma}}$ appears locally in the former, while the latter depends on $\dot{\boldsymbol{\gamma}}$ at every point subjected to plastic loading throughout the body.□

Remark 6.6. Note that (3.6-103) as well as (3.6-105) are valid only during loading. Obviously (3.6-103) cannot be reduced to incorporate elastic behaviour - contrasting with (3.6-102) which reduces to $\check{\mathcal{K}} = \mathcal{J}$ - while (3.6-105) can be written in the form

$$\left.\begin{aligned} &\dot{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}} - 2\mu^2\,\overline{[8(\gamma/\beta^p)\pi\boldsymbol{\tau}\cdot\dot{\boldsymbol{\gamma}}\,\boldsymbol{\tau}]}_p, \\ &\pi = \begin{cases} 1/(8(\gamma/\beta^p)(\mu+\mu')\overline{\{\boldsymbol{\tau}\}}_p\cdot\boldsymbol{\tau}) & \text{if } g=0,\ \hat{g}>0, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}\right\}\square \tag{3.6 – 108}$$

3.6.7 Maximum plastic work

In this subsection the principle of maximum dissipation will be applied to the class of materials defined by (3.6-1). A potential which complies with the constitutive assumptions in Section 3.5.2 may be written in the form

$$\rho_0\tilde{\psi} = \bar{\psi}(\mathbf{E}- < \mathbf{E}^p >) + \hat{\psi}(< \kappa >). \tag{3.6 - 109}$$

In particular

$$\bar{\psi} = \frac{1}{2}(\mathbf{E}- < \mathbf{E}^p >)\mathcal{L}(\mathbf{E}- < \mathbf{E}^p >), \tag{3.6 - 110}$$

where $\mathcal{L}$ is independent of $\mathbf{E}$ and $< \mathbf{E}^p >$, is a potential for the stress appearing in (3.6-1), as is easily seen from $(2.4\text{-}23)_2$. We employ (3.5-21) and deduce from (3.5-10), (3.4-23), and $(3.2\text{-}16)_1$ that[3.6-5]

$$\mathcal{L}^*\dot{\mathbf{E}} = \dot{\lambda}\frac{\partial g}{\partial \mathbf{E}}, \tag{3.6 - 111}$$

where $\mathcal{L}^*$ is defined by (3.5-7).

We recall that the normality condition (3.6-111) together with the associated condition of convexity of the yield surface, are the only restrictions placed on the constitutive equations by the principle of maximum dissipation in its classical form. However, by employing the principle in a more restrictive sense (than is implied by (3.4-8) and (3.4-9)) further restrictions will be laid on the constitutive functions. To see how, proceed as follows. Replace (3.4-8) with

$$\mathcal{A}_{\mathbf{E}} = \{(\mathbf{E}, < \mathcal{U}' >) | \ g(\mathbf{E}, < \mathcal{U}' >) \leq 0\} \tag{3.6 - 112}$$

and claim that

$$\check{D}^p(\mathbf{E}, < \mathcal{U}' >; \dot{\mathcal{U}}') \geq \check{D}^p(\hat{\mathbf{E}}, < \hat{\mathcal{U}}' > ; \dot{\mathcal{U}}') \tag{3.6 - 113}$$

for any state $(\hat{\mathbf{E}}, < \hat{\mathcal{U}}' >) \in \mathcal{A}_{\mathbf{E}}$. Hence define a Lagrangian,

$$L^p(\mathbf{E}, < \mathcal{U}' >, \dot{\lambda}) = -\check{D}^p(\mathbf{E}, < \mathcal{U}' > ; \dot{\mathcal{U}}') + \dot{\lambda} g(\mathbf{E}, < \mathcal{U}' >), \tag{3.6 - 114}$$

and conclude that the associated constrained minimization problem is solved by employing (3.4-15), when additionally

[3.6-5] Cf. (3.5-26) and footnote 3.5-2.

$$\frac{\partial L^p}{\partial < \mathbf{E}^p >} = \mathbf{0}, \qquad \frac{\partial L^p}{\partial < \kappa >} = 0. \tag{3.6 - 115}$$

We will not carry out the differentiation implied by (3.6-115) for general functions $\boldsymbol{\alpha}^p$ and α^h, but restrict ourselves to those defined by (3.6-15) and (3.6-16).

In view of (2.4-16)-(2.4-19), (3.6-15)-(3.6-20), (3.6-29)-(3.6-36) and (3.6-109) it is then seen that

$$\frac{\partial < \rho_0 \tilde{\psi} >}{\partial \mathcal{U}'} = \int_B \{\rho_0(\mathbf{Z}) \frac{\partial \tilde{\psi}}{\partial < \mathcal{U}' >}(\mathbf{Z}) \tilde{w}'(\mathbf{X}, \mathbf{Z}) dV(\mathbf{Z}), \tag{3.6 - 116}$$

and hence (3.6-114) becomes

$$L^p(\mathbf{E}, < \mathbf{E}^p >, < \kappa >, \dot{\lambda}) = \int_B \frac{\partial \bar{\psi}}{\partial < \mathbf{E}^p >}(\mathbf{Z}) \tilde{w}^p(\mathbf{X}, \mathbf{Z}) dV(\mathbf{Z}) \cdot \dot{\mathbf{E}}^p$$

$$+ \int_B \frac{\partial \hat{\psi}}{\partial < \kappa >} \tilde{w}^h(\mathbf{X}, \mathbf{Z}) dV(\mathbf{Z}) \, \dot{\kappa} + \dot{\lambda} g(\mathbf{E}, < \mathbf{E}^p >, < \kappa >). \tag{3.6 - 117}$$

Employing $(3.4\text{-}15)_1$ for the case when $\bar{\psi}$ satisfies (3.6-110) yields

$$- \int_B \mathcal{L}(\mathbf{Z}) \tilde{w}^p(\mathbf{X}, \mathbf{Z}) dV(\mathbf{Z}) \, \dot{\mathbf{E}}^p + \dot{\lambda} \frac{\partial g}{\partial \mathbf{E}} = \mathbf{0}, \tag{3.6 - 118}$$

which, by the use of (3.6-38) reduces to (3.6-111) for initial homogeneous bodies (as expected).

In addition, substitution of (3.6-117) into $(3.6\text{-}115)_1$ and $(3.6\text{-}115)_2$, respectively, and repeated use of (3.6-109) and (3.6-110) will leave us with two additional restrictions on the constitutive functions,

$$\mathcal{L}^* \dot{\mathbf{E}}^p + \dot{\lambda} \frac{\partial g}{\partial < \mathbf{E}^p >} = \mathbf{0}, \tag{3.6 - 119}$$

and

$$\int_B \frac{\partial^2 \hat{\psi}}{\partial < \kappa >^2}(\mathbf{Z}) \, \tilde{w}^h(\mathbf{X}, \mathbf{Z}) dV(\mathbf{Z}) \, \dot{\kappa} + \dot{\lambda} \frac{\partial g}{\partial < \kappa >} = 0, \tag{3.6 - 120}$$

respectively. From (3.6-111) and (3.6-119) we note that g must satisfy

$$\frac{\partial g}{\partial \mathbf{E}} = - \frac{\partial g}{\partial < \mathbf{E}^p >}, \tag{3.6 - 121}$$

a restriction which in view of (3.6-1) is trivially fulfilled for yield functions of von Mises type. Finally we observe that (3.6-120), in view of (3.6-5), can be written

$$\dot{\kappa} = \dot{\lambda} r, \quad r \neq 0, \tag{3.6 - 122}$$

where $r = r(< \kappa >)$ satisfies

$$\frac{1}{r} = \overline{\left\{ \frac{\partial^2 \hat{\psi}}{\partial < \kappa >^2} \right\}}. \tag{3.6 - 123}$$

In summary, we conclude that a generalization of the classical principle of maximum dissipation may be used to derive the flow rules of associated nonlocal plasticity. In addition, a restriction on the yield function in strain space is obtained, which is consistent with the constitutive equations for materials of von Mises type.

Chapter 4

Numerical aspects

4.1 Introduction

Analysis of the elastic-plastic response of real structures cannot successfully be performed without reliable numerical techniques. In statically indeterminate problems the constitutive equations of plasticity must be solved simultaneously with the equilibrium equations.[4.1-1] The final problem is constituted by a system of nonlinear partial differential equations, to be solved numerically. In engineering mechanics this is widely accomplished by means of the finite element method.

It should be noted that some basic knowledge of nonlinear finite element analysis is a prerequisite[4.1-2]. Accordingly we will concentrate on such features of the numerical implementation as are primarily due to the nonlocal character of the constitutive equations. Following this intention the finite element analysis will be performed in the context of small deformations. Though it is rather straightforward to provide for large deformations, the corresponding added complexity does not seem to be of essential merit at this stage - no new characteristic features of nonlocality will emerge.

In Section 4.2. the basic features of the finite element formulation are outlined, while in Section 4.3 a numerical algorithm of Newton-Raphson type for solving nonlinear equations is presented.

In Section 4.4 we address the issue of formulating an elastic-plastic tangential stiffness matrix to be used in the numerical algorithm, which with regard to nonlocal plasticity turns out to be a far from trivial problem.

In Section 4.5 simplified Newton-Raphsson schemes are discussed, whereas Section 4.6 deals with the problem of integrating the rate equations.

In Section 4.7 localization in a strain softening bar is numerically analyzed for two types of yield functions, one linear and one nonlinear, the linear one chosen such, that

[4.1-1] In the context of the analysis provided here, only static problems will be recognized, inertia effects being completely ignored.

[4.1-2] Consult e.g. Chen and Han (1988) or Criesfield (1991).

it is also provides for an analytical solution. Finally, in Section 4.8, some concluding remarks are made.

4.2 Finite element formulation

In a theory of small deformations, to leading order, the Piola-Kirchhoff stress reduces to the Cauchy stress, and the Lagrangian strain tensor accordingly reduces to the small strain tensor

$$\mathbf{E} = \frac{1}{2}(\mathbf{F} + \mathbf{F}^{\mathrm{T}} - \mathbf{1}), \qquad (4.2-1)$$

where we have retained the same notation for the strain tensor as in the large deformation theory. Likewise stress and plastic strain are still denoted $\mathbf{S}$ and $\mathbf{E}^p$, respectively. This should not lead to confusion, since small deformations are assumed throughout the chapter.

In finite element analysis it is convenient to use matrix notation - e.g. the symmetric stress tensor is then represented by a column matrix,

$$\mathbf{S}^{\mathrm{T}} = \{S_{11},\ S_{22},\ S_{33},\ S_{12},\ S_{13},\ S_{23}\}. \qquad (4.2-2)$$

As can be seen, we use the same symbol for the stress tensor as for the stress in matrix notation, commonly referred to as the stress vector in six-dimensional space. The practice of not distinguishing between tensor and matrix notation will currently be used. We leave to the reader to work out the details for the transcription of current tensor expressions into matrix notation.

Based on a weak formulation of the equilibrium equation and a Galerkin approximation, the total equilibrium condition in standard notations becomes

$$\int_B \mathbf{B}^{\mathrm{T}}\mathbf{S}\ dV = \mathbf{F}, \qquad (4.2-3)$$

where

$$\mathbf{F} = \int_{\partial B} \mathbf{N}^{\mathrm{T}}\ \mathbf{t}\ dA + \int_B \mathbf{N}^{\mathrm{T}}\ \mathbf{f}\ dV \qquad (4.2-4)$$

represents the external forces. Here $\mathbf{N}$ is the shape function matrix, by which

$$\mathbf{u} = \mathbf{N}\ \mathbf{U} \qquad (4.2-5)$$

where $\mathbf{u}$ is displacement and $\mathbf{U}$ the displacement vector of nodal points. Further, in a small deformation analysis, strains are related to displacements through

$$\mathbf{E} = \mathbf{L}\ \mathbf{u} = \mathbf{B}\ \mathbf{U}, \tag{4.2-6}$$

where $\mathbf{L}$ is the differential operator matrix, and

$$\mathbf{B} = \mathbf{L}\ \mathbf{N} \tag{4.2-7}$$

is the strain-displacement matrix present in (4.2-3).

Now assume that the state at load step n is completely known. The problem is recognized as one of determining the state at load step $(n+1)$ corresponding to external forces ${}_{n+1}\mathbf{F}$. Hence we must solve the equation

$$\int_B \mathbf{B}^{\mathrm{T}}{}_{n+1}\mathbf{S}\ dV = {}_{n+1}\mathbf{F} \tag{4.2-8}$$

for the unknown stresses ${}_{n+1}\mathbf{S}$, corresponding to the yet unknown displacements ${}_{n+1}\mathbf{U}$.

Two separate types of numerical approaches are in fact involved in solving (4.2-8) for the displacement increments

$$\Delta\mathbf{U} = {}_{n+1}\mathbf{U} - {}_{n}\mathbf{U} \tag{4.2-9}$$

and the stress increments

$$\Delta\mathbf{S} = {}_{n+1}\mathbf{S} - {}_{n}\mathbf{S}. \tag{4.2-10}$$

The one requires an algorithm for solving systems of nonlinear equations, and the other an algorithm for updating the stresses by integration of the constitutive equations. The first issue will be dealt with briefly in the next three sections, while the second one will be discussed in Section 4.6.

4.3 The Newton-Raphson method

A variety of different approaches exists for solving nonlinear equations. It is common in finite element analysis to use an algorithm of the Newton-Raphson type. Below we outline the iterative scheme of a full Newton-Raphson algorithm for the case when the stress-strain relation is generally nonlocal.

Let

$$ {}^{1}_{n+1}\mathbf{U} = {}_{n}\mathbf{U} + {}^{1}(\Delta\mathbf{U}) \qquad (4.3-1) $$

represent a first estimate of the unknown displacement ${}_{n+1}\mathbf{U}$. Integration of the constitutive equations provides the corresponding stresses

$$ {}^{1}_{n+1}\mathbf{S} = {}_{n}\mathbf{S} + {}^{1}(\Delta\mathbf{S}), \qquad (4.3-2) $$

which henceforth are treated as known quantities. The corresponding external forces are then given by

$$ {}^{1}_{n+1}\mathbf{F} = \int_B \mathbf{B}^{\mathrm{T}}\, {}^{1}_{n+1}\mathbf{S}\; dV. \qquad (4.3-3) $$

By a second estimate

$$ {}_{n+1}\mathbf{S} \approx {}^{2}_{n+1}\mathbf{S} = {}^{1}_{n+1}\mathbf{S} + {}^{2}(\Delta\mathbf{S}), \qquad (4.3-4) $$

we conclude from (4.2-8), (4.3-3) and (4.3-4) that

$$ \int_B \mathbf{B}^{\mathrm{T}}\, {}^{2}(\Delta\mathbf{S})\; dV = {}_{n+1}\mathbf{F} - {}^{1}_{n+1}\mathbf{F}. \qquad (4.3-5) $$

In view of (3.3-14) we may write[4.3-1]

$$ \Delta\mathbf{S} = \check{\mathcal{K}}\mathcal{L}\Delta\mathbf{E} + \frac{1}{\rho_0}\mathcal{H}_s, \qquad (4.3-6) $$

where $\mathcal{H}_s$ by virtue of (3.2-9) is now understood to have the form (cf. (3.3-18))

$$ \begin{aligned} \mathcal{H}_s = \int_B \{ \rho_0 \frac{\partial\tilde{\mathbf{S}}}{\partial < \mathcal{U}' >} \frac{\partial\alpha'}{\partial\mathcal{U}'(\mathbf{Z})} \pi(\mathbf{Z}) \frac{\partial g}{\partial\mathbf{E}}(\mathbf{Z}) \cdot \Delta\mathbf{E}(\mathbf{Z})\Lambda'(\mathbf{Z}) \\ -\rho_0(\mathbf{Z}) \frac{\partial\tilde{\mathbf{S}}}{\partial < \mathcal{U}' >}(\mathbf{Z}) \frac{\partial(\alpha')^*}{\partial\mathcal{U}'} \pi \frac{\partial g}{\partial\mathbf{E}} \cdot \Delta\mathbf{E}\Lambda' \} dV(\mathbf{Z}). \end{aligned} \qquad (4.3-7) $$

As usual it is assumed that π vanishes at non-loading points.

If the rate dependent functional in the right-hand side of (4.3-6) is approximated in accordance with

[4.3-1] Notice that we retain the symbol $\mathcal{H}_s$ also when strain rates are replaced by strain increments in the arguments of the functional.

$$\mathcal{H}_s \approx {}^1\mathcal{H}_s, \tag{4.3-8}$$

the estimate of the stress increment may be written

$${}^2(\Delta\mathbf{S}) = {}_{n+1}^{\;\;1}(\check{\mathcal{K}}\mathcal{L})\ {}^2(\Delta\mathbf{E}) + \frac{1}{\rho_0}\ {}_{n+1}^{\;\;1}\mathcal{H}_s. \tag{4.3-9}$$

Hence we conclude by substituting (4.3-9) into (4.3-5) and by use of (4.2-6) that

$$\int_B \mathbf{B}^{\mathrm{T}}\ {}_{n+1}^{\;\;1}(\check{\mathcal{K}}\mathcal{L})\mathbf{B}\ dV)\ {}^2(\Delta\mathbf{U}) = {}_{n+1}\mathbf{F} - {}_{n+1}^{\;\;1}\mathbf{F} - \int_B \mathbf{B}^{\mathrm{T}}\ {}_{n+1}^{\;\;1}\mathcal{H}_s/\rho_0\ dV. \tag{4.3-10}$$

Due to its dependence on the quasi-local elastic-plastic response function $\check{\mathcal{K}}$ we refer to

$$\mathbf{K} = \int_B \mathbf{B}^{\mathrm{T}}\ \check{\mathcal{K}}\mathcal{L}\mathbf{B}\ dV \tag{4.3-11}$$

as the *structural quasi-local stiffness matrix.* Using (4.3-3), (4.3-10) and (4.3-11) allows us to write a general iteration scheme of the Newton-Raphson algorithm in the form

$${}_{n+1}^{i-1}\mathbf{K}\ {}^i(\Delta\mathbf{U}) = {}_{n+1}\mathbf{F} - {}_{n+1}^{i-1}\tilde{\mathbf{F}}, \quad (i = 1, 2, \ldots), \tag{4.3-12}$$

where

$${}_{n+1}^{i-1}\tilde{\mathbf{F}} = \int_B \mathbf{B}^{\mathrm{T}}\left({}_{n+1}^{i-1}\mathbf{S} + {}_{n+1}^{i-1}\mathcal{H}_s/\rho_0\right)\ dV \tag{4.3-13}$$

and

$${}_{n+1}^{\;\;0}\mathbf{K} = {}_n\mathbf{K}\ , \quad {}_{n+1}^{\;\;0}\tilde{\mathbf{F}} = {}_n\mathbf{F} + \int_B \mathbf{B}^{\mathrm{T}}\ {}_n\mathcal{H}_s/\rho_0\ dV. \tag{4.3-14}$$

Of course the integral $\int_B \mathbf{B}^{\mathrm{T}}\ {}_{n+1}^{i-1}\mathcal{H}_s/\rho_0\ dV$ does not represent an actual contribution to the external forces, but due to the way it appears in (4.3-13) we will refer to $\int_B \mathbf{B}^T\mathcal{H}_s/\rho_0\ dV$ as (incremental) *external pseudo forces.*

The iterations continue until some accurate convergence criterion is fulfilled. For instance, until the *residual forces* approximately vanish,

$$\|\ {}_{n+1}\mathbf{F} - {}_{n+1}^{\;\;i}\mathbf{F}\| \leq \epsilon_F\|\ {}_{n+1}\mathbf{F} - {}_n\mathbf{F}\|, \tag{4.3-15}$$

where $\epsilon_F > 0$ is a prescribed number and $\|\ \|$ denotes some proper vector norm. Obviously the equilibrium equation (4.2-8) is satisfactorily fulfilled if only (4.3-15) is satisfied for a sufficiently small number ϵ_F. A convergence criterion involving the iterative strain increments and the iteration increments of the functional $\mathcal{H}_s$ may be expressed as (cf. (4.3-1)

$$\left.\begin{aligned} &\| \,^i(\Delta\mathbf{U})\| \leq \epsilon_D \| \,_{n+1}^{\;i}\mathbf{U} - \,_n\mathbf{U}\|, \\ &\| \,_{n+1}^{\;i}\tilde{\mathbf{F}} - \,_{n+1}^{\;i}\mathbf{F}\| \leq \epsilon_H \| \,_{n+1}\mathbf{F} - \mathbf{F}\|, \end{aligned}\right\} \qquad (4.3-16)$$

where the inequalities are required to be satisfied simultaneously. It is noted that the left-hand side of $(4.3\text{-}16)_2$ in view of (4.3-13) equals $\| \int_B \mathbf{B}^{\mathrm{T}}\,_{n+1}^{\;i}\mathcal{H}_s/\rho_0 \, dV\|$, and hence that the external pseudo forces must vanish if the condition of equilibrium is to be met.

Remark 3.1. Since, by virtue of (4.3-4) and (4.3-6),

$$\frac{1}{\rho_o}\,_{n+1}^{i-1}\mathcal{H}_s = \,_{n+1}^{i-2}\mathbf{S} - \,_{n+1}^{i-1}\mathbf{S} - \,_{n+1}^{i-2}(\check{\mathcal{K}}\mathcal{L})\,^{i-1}(\Delta\mathbf{E}), \qquad (4.3-17)$$

we note in view of (4.2-6) and (4.2-8) that the iterative scheme (4.3-12)-(4.3-14) may be replaced by the procedure:

$$\,_{n+1}^{\;i}\mathbf{K}\,^{i+1}(\Delta\mathbf{U}) = \,_{n+1}\mathbf{F} - 2\,_{n+1}^{\;i}\mathbf{F} + \,_{n+1}^{i-1}\mathbf{F} + \,_{n+1}^{i-1}\mathbf{K}\,^{i}(\Delta\mathbf{U}), \quad (i = 1, 2, \ldots), \; (4.3-18)$$

$$\,_n\mathbf{K}\,^1(\Delta\mathbf{U}) = \,_{n+1}\mathbf{F} - \,_n\mathbf{F} - \int_B \mathbf{B}^{\mathrm{T}}\,_n\mathcal{H}_s/\rho_0 \, dV. \square \qquad (4.3-19)$$

Remark 3.2. The numerical procedure outlined in this section may apparently be simplified by altogether disregarding the influence of the pseudo forces (recall that $\int_B \mathcal{H}_s \, dV = \mathbf{0}$). Then

$$\,_{n+1}^{\;i}\tilde{\mathbf{F}} = \,_{n+1}^{\;i}\mathbf{F} \qquad (4.3-20)$$

and (4.3-12) becomes

$$\,_{n+1}^{i-1}\mathbf{K}\,^{i}(\Delta\mathbf{U}) = \,_{n+1}\mathbf{F} - \,_{n+1}^{i-1}\mathbf{F}, \quad (i = 1, 2, \ldots), \qquad (4.3-21)$$

recognized as the usual form of the Newton-Raphson algorithm. The convergence properties will surely change - how and to what extent is impossible to say at this level of generality. Notice, however, that the computation of the pseudo forces is easy to perform, since all quantities at the actual iteration step are known in advance. Anyhow, if nothing else is asserted, we refer to (4.3-12)-(4.3-14) as the general iteration scheme for systems with nonlocal interactions involved. □

4.4 The elastic-plastic stiffness matrix

In view of (3.3-8) the quasi-local tangential stiffness tensor can be written in the form

$$\check{\mathcal{K}}\mathcal{L} = \mathcal{L} + \pi \check{\boldsymbol{\sigma}} \otimes \frac{\partial g}{\partial \mathbf{E}}, \qquad (4.4-1)$$

where the second order tensor-valued function $\check{\boldsymbol{\sigma}}$ is defined by (3.3-9). As discussed in Section 3.2.2 and again in Section 3.6.6 the consistency condition cannot be used to solve analytically for the function π, which hence must be determined by a numerical procedure - unless consistency is required only in an average sense, as was the case in Section 3.6.6. The latter possibility will be considered in Subsection 4.4.2, while the first one is discussed below.

Remark 4.1. Attention has not been paid to the elastic-plastic response function $\mathcal{K}$, since it is in general rate independent only in a weak sense, and hence it is inappropriate to employ $\mathcal{K}\mathcal{L}$ as the tangential stiffness matrix in a Newton-Raphson scheme. If, however, $\mathcal{K}$ actually appears to be independent of rates, then of course $\mathcal{K}\mathcal{L}$ may be used for the tangential stiffness matrix in the equilibrium iterations. Nonlocal elastic-plastic response functions of such type correspond, of course, to very special constitutive assumptions. A special case, attributed to a class of familiar materials with linear stress-strain relation and yield function of von Mises type, is discussed in Section 4.7.2.□

4.4.1 The incremental tangential stiffness matrix

It turns out to be convenient to introduce the *plastic multiplier* $\dot{\lambda}$ by the definition

$$\dot{\lambda} = \pi \hat{g} = \pi \frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}}, \quad \hat{g} > 0, \ \pi \geq 0, \qquad (4.4-2)$$

where advantage has been taken to (3.4-22) and where (3.2-9) has also been used. As usual $\pi(\mathbf{X}) = 0$ corresponds to unloading at $\mathbf{X}$. The flow rules (3.2-19) are then replaced by

$$\dot{\mathcal{U}}' = \left\{ \begin{array}{ll} \dot{\lambda}\Lambda' & (g = 0, \ \dot{\lambda} \geq 0), \\ 0 & (\text{otherwise}), \end{array} \right\} \qquad (4.4-3)$$

where the first equation explicitly reads

$$\dot{\mathbf{E}}^p = \dot{\lambda}\, \mathbf{R}, \ \dot{\kappa} = \dot{\lambda}\, r, \quad (g = 0, \ \dot{\lambda} \geq 0). \qquad (4.4-4)$$

By use of (4.4-2) in incremental form, i.e.

$$\Delta\lambda = \pi \frac{\partial g}{\partial \mathbf{E}} \cdot \Delta \mathbf{E}, \quad \frac{\partial g}{\partial \mathbf{E}} \cdot \Delta \mathbf{E} > 0, \ \pi \geq 0, \tag{4.4-5}$$

the quasi-local tangential stiffness matrix corresponding to the displacement ${}^{i-1}_{n+1}\mathbf{U}$ can be written as

$${}^{i-1}_{n+1}(\check{\mathcal{K}}\mathcal{L}) = {}^{i-1}_{n+1}(\mathcal{L} + \Delta\lambda \frac{\check{\check{\boldsymbol{\sigma}}} \otimes \dfrac{\partial g}{\partial \mathbf{E}}}{\dfrac{\partial g}{\partial \mathbf{E}} \cdot \Delta \mathbf{E}}), \tag{4.4-6}$$

where ${}^{i-1}(\Delta \mathbf{E})$ results from the preceding Newton-Raphson iteration, and ${}^{i-1}(\Delta\lambda)$ correspondingly from the integration of the constitutive equations. To distinguish the tangential stiffness tensor appearing in (4.4-1) from the matrix recorded in (4.4-6), we will refer to the latter as the *quasi-local incremental tangential stiffness matrix.*

4.4.2 The continuous tangential stiffness matrix

In Section 3.6.6 we used the consistency condition to solve for π in an average sense. In the special case discussed in Section 3.6 with linear stress-strain response and yield condition of von Mises type, the result is recorded in (3.6-86). Using the same argument as in the derivation of (3.6-86) we conclude that

$$\begin{aligned} \frac{1}{\pi} &= -\frac{1}{\rho_0} \frac{\partial < \rho_0 g >}{\partial \mathcal{U}'} \Lambda' \\ &= -\frac{1}{\rho_0} (\frac{\partial < \rho_0 g >}{\partial \mathbf{E}^p} \cdot \mathbf{R} + \frac{\partial < \rho_0 g >}{\partial \kappa} r) \end{aligned} \tag{4.4-7}$$

is a solution of (3.2-32)[4.4-1]. Substitution of $(4.4\text{-}7)_2$ into (4.4-1) then yields

$$\check{\mathcal{K}}\mathcal{L} = \mathcal{L} - \frac{\rho_0 \check{\check{\boldsymbol{\sigma}}} \otimes \dfrac{\partial g}{\partial \mathbf{E}}}{\dfrac{\partial < \rho_0 g >}{\partial \mathbf{E}^p} \cdot \mathbf{R} + \dfrac{\partial < \rho_0 g >}{\partial \kappa} r}, \tag{4.4-8}$$

where again $\check{\check{\boldsymbol{\sigma}}}$ is given by (3.3-9). When (4.4-8) is used for the elastic-plastic stiffness matrix in the Newton-Raphson algorithm, we refer to $\check{\mathcal{K}}\mathcal{L}$ as the *quasi-local continuous tangential stiffness matrix.* The reason is of course due to its resemblance with local theory. However, it is emphasized that the right-hand side of (4.4-8) - in contrast to

[4.4-1] Recall that (3.4-22) is valid at loading points.

local theory - is an approximate expression for the quasi-local stiffness tensor $\check{\mathcal{K}}\mathcal{L}$, the tacit assumption being that consistency is only required to be satisfied in an average sense. If the consistency condition is not enforced otherwise, this means that the yield condition in general will not be satisfied, but the final strains (after equilibrium iterations) may lie outside the yield surface in strain space.

For further details concerning consistency in an average sense we refer to Section 3.6. With regard to the class of familiar materials presented there, it is noted that the corresponding quasi-local continuous stiffness tensor is obtained from (3.6-102). It should also be mentioned that a simplified integration procedure for the determination of $\Delta\lambda$, in which (4.4-7) is used as a predictor, is presented in Section 4.6.5.

4.5 Simplified Newton-Raphson schemes

Various modified Newton-Raphson approaches may be utilized in order to avoid evaluation and factorization of the structural tangential stiffness matrix ${}^{i-1}_{n+1}\mathbf{K}$ at each iteration step. Difficulties in the iteration process due to singularities or ill-conditioned behaviour may also call for a modification of the numerical scheme. In this respect the differences between local and nonlocal appearance are not significant. As noted introductorily our main purpose is to describe features of the numerical implementation, not present in local theory, but characteristic for nonlocal theory. Hence, the various existing matrix update methods commonly used to reduce the computational effort within each iteration in the Newton-Raphson algorithm will not be treated here. The special case, however, when the tangential elastic-plastic stiffness tensor $\check{\mathcal{K}}\mathcal{L}$ is replaced by merely $\mathcal{L}$, has certain important implications from a nonlocal point of view and will be discussed in some detail below.

A straighforward and simple modification of the numerical scheme given by (4.3-12)-(4.3-14) is of course to replace ${}^{i-1}_{n+1}\mathbf{K}$ by the tangential stiffness matrix of the structure at an earlier load step, ${}_m\mathbf{K}$, $m \leq n$. Especially if the quasi-local response function $\check{\mathcal{K}}$ is replaced by unity in (4.3-11), i.e.

$$\mathbf{K} = \int_B \mathbf{B}^{\mathrm{T}}\mathcal{L}\mathbf{B}\ dV, \qquad (4.5-1)$$

we obtain a modified Newton-Raphson method, which - as in local theory - may be called the *initial stress method*, in which no updating at all of $\mathbf{K}$ is required if $\mathcal{L}$ is constant (or kept constant) [4.5-1].

It would be improper to employ (4.5-1) with (4.3-12) otherwise unchanged. Recall, namely, that the general Newton-Raphson scheme (4.3-12)-(4.3-14) (through (4.3-6)) is

[4.5-1] Recall that $\mathcal{L} = \mathcal{L}(\mathbf{E}, <\mathbf{E}^p>, <\kappa>)$, in general is not constant (cf. the discussion in Section 3.3.1).

based on the decomposition (3.3-2) of the stress rate into a quasi-local part (the quasi-local stress rate $\check{\mathbf{S}}$) and a general nonlocal part (the functional stress rate $\mathcal{H}_s/\rho_0$). However, the stress rate may also be written (cf. (2.4-6))

$$\dot{\mathbf{S}} = \mathcal{L}\dot{\mathbf{E}} \quad + \quad \int_B \{ \frac{\partial \tilde{\mathbf{S}}}{\partial < \mathcal{U}' >} \frac{\partial \alpha'}{\partial \mathcal{U}'} \pi \Lambda' \frac{\partial g}{\partial \mathbf{E}} \cdot \dot{\mathbf{E}}$$

$$+ \frac{\partial \tilde{\mathbf{S}}}{\partial < \mathcal{U}' >} \frac{\partial \alpha'}{\partial \mathcal{U}'(\mathbf{Z})} \pi(\mathbf{Z}) \Lambda'(\mathbf{Z}) \frac{\partial g}{\partial \mathbf{E}}(\mathbf{Z}) \cdot \dot{\mathbf{E}}(\mathbf{Z}) \} \, dV(\mathbf{Z}), \qquad (4.5\text{-} 2)$$

without introducing $\mathcal{H}_s$, which evidently is of no benefit in a Newton-Raphson algorithm, in which $\mathcal{L}$ is employed as tangential stiffness matrix. In the initial stress method, $\mathcal{H}_s$ in (4.3-13) should therefore be replaced by the incremental form of the integral in the right-hand side of (4.5-2) - or should be disregarded altogether.

Notice that (4.3-6) and the incremental form of (4-5-2) are identical expressions for the incremental stress rate, but used in a scheme of the type (4.3-5) they will give rise to two entirely disparate numerical procedures probably with quite different convergence properties. On the whole it is impossible, at least at this stage of generality, to make accurate predictions about the success of the iterations with respect to the one numerical scheme or the other.

4.6 Integration of rate equations

Accurate calculation of the iterative external forces ${}_{n+1}^{\ i}\mathbf{F}$ defined by (4.3-3) (for $i = 1$), is of decisive importance for a successful result of the Newton-Raphson procedure. Prior to the evaluation of the integral in (4.3-3) (by some numerical integration technique), the stresses ${}_{n+1}^{\ i}\mathbf{S}$ must be calculated, which requires integration of the constitutive equations for the rates of the plastic strain and the strain hardening function. Various numerical procedures may be used for the updating of the stresses in agreement with the consistency condition. The literature on this subject, with regard to local plasticity, is comprehensive; for a full list of references, consult e.g. Criesfield (1991) or Chen and Han (1988), previously cited.

Due to the long-range interaction between the particles in a nonlocal body, the integration of the rate equations in nonlocal plasticity appears in general to be a far more complex process than in a corresponding local theory. One difficulty, as discussed in Section 3.2.2 and again in Section 3.6.6, is due to the fact that the consistency condition in nonlocal plasticity is expressed by an integral equation expanded over all points at loading throughout the body under consideration. As a consequence, integration of the rate equations subject to the constraint implied by the consistency condition cannot be done pointwise at one point at the time, but must be performed simultaneously for all Gauss points in a spatially discretized model of the body. Another difficulty, closely related to the one already mentioned, is linked to the characteristic property of the yield surface of not being necessarily stationary at a point **X**, even if the rates of the plastic strain and the hardening function vanish at this very point **X** (since the yield surface may change due to plastic deformation in regions outside **X**).

One strategy for the stress updating in nonlocal plasticity is easily identified: enforce the consistency condition by solving the associated integral equation for the incremental plastic multiplier $\Delta\lambda$, then use the flow rules (the incremental form of (4.4-4)) for updating the inelastic variables, calculate the corresponding nonlocal quantities and finally use the constitutive assumption (3.2-5) to determine the new stresses. An entirely explicit method, simple to grasp and simple to implement (at least in principle, using some efficient numerical technique for solving the integral equation). Unfortunately this method will fail unless the strain and stress increments are really small, so small that the error made by replacing $\dot{\lambda}$ by $\Delta\lambda$ in the updating scheme is of no significance whatsoever. This will, however, never be the case for real problems; strains will not be infinitesimal, errors will accumulate and the yield condition will be violated in the end of the integration step. To overcome the problem, measures should be taken to adjust the yield surface in strain space to comply with the updated strains - or else an essentially different approach should be used.

To simplify terminology, nonlocal integration strategies, which do not take explicit advantage of solving the integral equation of consistency, will henceforth be called implicit[4.6−1]. Explicit strategies are not treated in this work, except for the special case when the consistency condition is enforced only in an average sense. Although general explicit strategies cannot be ruled out, it is believed that they are inferior to the special type of implicit strategy described in Subsection 4.6.1 below, referred to as a *nonlocal generalized Euler procedure*, by which the rate equations are integrated by a generalized Euler method, while consistency is enforced by equating to zero a truncated Taylor series of the yield function in strain space. Due to the nonlocal interaction, this must be done at all loading points simultaneously - instead of solving one equation at a time, say $g(x) = 0$ (as would be the case in local theory), we must solve a system of coupled equations, $g_\alpha(x_\beta) = 0$, α and β representing Gauss points at plastic loading. Finally, the inelastic variables at the end of the integration step are updated such that the yield condition is satisfied at all Gauss points of the body.

In Subsections 4.6.2 and 4.6.3 two special cases of the nonlocal generalized Euler procedure are derived - the forward and backward Euler schemes, while illustrative examples are provided in Subsection 4.6.3 with regard to a simple choice of yield function in strain space.

[4.6−1] The notions 'explicit' and 'implicit' are frequently used in computational plasticity without being precisely defined. As can be seen, we are not anxious to stray too far from that tradition.

Finally, in Subsection 4.6.5, we present the simplified (explicit) method mentioned above. It is recognized as a *quasi-local integration technique* being in particular characterized by the fact that it entirely avoids the laborious process of enforcing the consistency condition by solving a system of coupled equations as was the case for the nonlocal generalized Euler procedure.

4.6.1 Nonlocal generalized Euler procedure

The rate equations to be integrated are given by (4.4-4). To simplify we introduce the notations[4.6-2]

$$\left.\begin{aligned} \mathbf{P} &= [\mathbf{E}^p,\ \kappa]^T, \\ \boldsymbol{\rho} &= [\mathbf{R},\ r]^T, \end{aligned}\right\} \tag{4.6-1}$$

through which the flow rules are re-expressed as

$$\dot{\mathbf{P}} = \dot{\lambda}\boldsymbol{\rho}, \tag{4.6-2}$$

subject to the condition[4.6-3]

$$\dot{\lambda} \geq 0, \quad g \leq 0, \quad g\dot{\lambda} = 0. \tag{4.6-3}$$

We recall that

$$\boldsymbol{\rho} = \boldsymbol{\rho}(\mathbf{E}, <\mathbf{E}^p>, <\kappa>) = \boldsymbol{\rho}(\mathbf{E},\ <\mathbf{P}>) = \boldsymbol{\rho}(\lambda), \tag{4.6-4}$$

where $(4.6\text{-}4)_2$ is a consequence of (4.6-1) and $(4.6\text{-}4)_3$ is a definition. The differential equation (4.6-2) may then be recast into the form

$$\frac{d\mathbf{P}}{d\lambda} = \boldsymbol{\rho}(\lambda). \tag{4.6-5}$$

Some methods frequently used to integrate (4.6-5) are special cases of the generalized Euler method, according to which, approximately,

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[4.6-2] Notice that $\mathbf{P}$ corresponds to $\mathcal{U}'$ defined by (2.4-21) and $\boldsymbol{\rho}$ to Λ' defined by (3.2-18). Further it must be noted that $\mathbf{P}$ and $\boldsymbol{\rho}$ in a strict sense are *not* elements in a seven-dimensional vector space. If, e.g., we write $F = F(\mathbf{P})$ for a scalar function F, we do not assert that F is merely a function of the invariants of the 'vector' $\mathbf{P}$, in fact being a general function of the invariants of $\mathbf{E}^p$ and κ.

[4.6-3] It is tacitly understood that (4.4-2) is satisfied.

$$\Delta \mathbf{P} = \Delta\lambda[(1-\eta)\boldsymbol{\rho}(\lambda) + \eta\boldsymbol{\rho}(\lambda+\Delta\lambda)], \tag{4.6-6}$$

where

$$\begin{aligned}\boldsymbol{\rho}(\lambda+\Delta\lambda) &= \boldsymbol{\rho}(\mathbf{E}+\Delta\mathbf{E},\ <\mathbf{E}^p> +\Delta<\mathbf{E}^p>,\ <\kappa> +\Delta<\kappa>)\\ &= \boldsymbol{\rho}(\mathbf{E}+\Delta\mathbf{E},\ <\mathbf{P}> +\Delta<\mathbf{P}>),\end{aligned} \tag{4.6-7}$$

and η is a parameter between 0 and 1. If we choose $\eta = 0$, (4.6-6) turns into the forward Euler method[4.6-4] while $\eta = 1$ represents the backward Euler method and $\eta = 1/2$ the Crank-Nicholson method.

Assume now that the state corresponding to λ is completely known for all Gauss points related to the finite element discretization. Hence we know $\mathbf{E}(\lambda)$, $\mathbf{P}(\lambda)$ and $\boldsymbol{\rho}(\lambda)$ as well as the corresponding value of the yield function in strain space consistent with the yield condition, i. e.

$$g(\lambda) \leq 0, \tag{4.6-8}$$

where

$$g(\lambda) = g(\mathbf{E},\ <\mathbf{E}^p>,\ <\kappa>) = g(\mathbf{E},\ <\mathbf{P}>). \tag{4.6-9}$$

Assume further that $\Delta\mathbf{E} = {}^i(\Delta\mathbf{E})$ has been calculated by the aid of the equilibrium iteration scheme (4.3-12)-(4.3-14), and hence is known at each Gauss point throughout the body.

The problem to be solved may now be formulated as the one of calculating $\Delta\lambda$ for all Gauss points, such that (at least approximately)

$$\Delta\lambda \geq 0, \quad g(\lambda+\Delta\lambda) \leq 0, \quad g(\lambda+\Delta\lambda)\Delta\lambda = 0. \tag{4.6-10}$$

Let $\mathbf{d}$ represent the difference between the current state $\mathbf{P}$ and the state $\mathbf{P}(\lambda)+\Delta\mathbf{P}$, i.e., in view of (4.6-6),

$$\mathbf{d} = \mathbf{P} - (\mathbf{P}(\lambda) + \Delta\lambda[(1-\eta)\boldsymbol{\rho}(\lambda) + \eta\boldsymbol{\rho}(\lambda+\Delta\lambda)]). \tag{4.6-11}$$

Keeping in mind that $\mathbf{P}(\lambda)$ and $\boldsymbol{\rho}(\lambda)$ represent a known (fixed) state, the expression for $\mathbf{d}$ may be expanded by a truncated Taylor series,

[4.6-4] Since then $\Delta\mathbf{P}$ disappears from the right-hand side of (4.6-6), the integration becomes 'explicit'. According to our terminology, however, the integration strategy as a whole (with due concern to the consistency condition) is still implicit.

$$\mathbf{d}_\alpha \approx \mathbf{d}_{0,\alpha} + \delta\mathbf{P}_\alpha - (\delta\lambda)_\alpha[(1-\eta)\boldsymbol{\rho}_\alpha(\lambda) + \eta\boldsymbol{\rho}_\alpha(\lambda+\Delta\lambda)]$$
$$-\eta(\Delta\lambda)_\alpha \left.\frac{\partial\boldsymbol{\rho}_\alpha}{\partial <\mathbf{P}>_\alpha}\right|_{(\lambda+\Delta\lambda)} \delta <\mathbf{P}>_\alpha, \qquad (4.6-12)$$

where $\delta\lambda$ and $\delta\mathbf{P}$ are iterative changes of $\Delta\lambda$ and $\mathbf{P}$, respectively[4.6-5], while $\mathbf{d}_0$ represents a starting estimate for $\mathbf{d}$ (to be evaluated from (4.6-11) given a starting value of $\Delta\lambda$). Subscript α emphasizes that (4.6-12) should be established for all Gauss points (at loading).

Assume that the nonlocal inelastic variables may be calculated by

$$<\mathbf{P}>(\mathbf{X}) = \int_B \tilde{w}(\mathbf{Z},\mathbf{X})\mathbf{P}(\mathbf{Z})\; dV(\mathbf{Z}), \qquad (4.6-13)$$

where the function $\tilde{w}$ is defined by (3.6-19) [4.6-6]. A numerical approximation of (4.6-13) is written as [4.6-7]

$$<\mathbf{P}>_\alpha \approx \tilde{w}_{\alpha\beta}\mathbf{P}_\beta \quad , \quad \alpha,\beta = 1,2,\ldots,N, \qquad (4.6-14)$$

where N is the total number of Gauss points, and where summation with regard to β is implied. Since $\tilde{w}$ is independent of the state variables it then follows that

$$\delta <\mathbf{P}>_\alpha = \tilde{w}_{\alpha\beta}\; \delta\mathbf{P}_\beta. \qquad (4.6-15)$$

Substitution of (4.6-15) into (4.6-12) yields

$$\begin{aligned}\mathbf{d}_\alpha &\approx \mathbf{d}_{0,\alpha} + \delta\mathbf{P}_\alpha - \eta(\Delta\lambda)_\alpha \left.\frac{\partial\boldsymbol{\rho}_\alpha}{\partial <\mathbf{P}>_\alpha}\right|_{(\lambda+\Delta\lambda)} \tilde{w}_{\alpha\beta}\delta\mathbf{P}_\beta \\ &\quad -(\delta\lambda)_\alpha[(1-\eta)\boldsymbol{\rho}_\alpha(\lambda) + \eta\boldsymbol{\rho}_\alpha(\lambda+\Delta\lambda)] \qquad (4.6-16) \\ &= \mathbf{d}_{0,\alpha} - (\delta\lambda)_\alpha[(1-\eta)\boldsymbol{\rho}_\alpha(\lambda) + \eta\boldsymbol{\rho}_\alpha(\lambda+\Delta\lambda)] + \mathbf{Q}_{\alpha\beta}\delta\mathbf{P}_\beta,\end{aligned}$$

where

[4.6-5] Hence, if $\delta = \frac{d(\Delta\lambda)}{dt}\delta t$, then $\delta\mathbf{P} = \frac{d\mathbf{P}}{dt}\delta t = \mathbf{P}(\lambda+\delta\lambda) - \mathbf{P}(\lambda)$.

[4.6-6] Note that $\tilde{w}\mathbf{P} = [\tilde{w}^p\mathbf{E}^p,\ \tilde{w}^p\kappa]^T = \begin{bmatrix} \tilde{w}^p & 0 \\ 0 & \tilde{w}^h \end{bmatrix} \begin{bmatrix} \mathbf{E}^p \\ \kappa \end{bmatrix}$.

[4.6-7] Alternatively, a standard finite element discretization may be applied to (4.6-13) to produce a result similar to (4.6-14).

$$\mathbf{Q}_{\alpha\beta} = \delta_{\alpha\beta}\mathbf{1} - \eta(\Delta\lambda)_\alpha \left.\frac{\partial \boldsymbol{\rho}_\alpha}{\partial < \mathbf{P} >_\alpha}\right|_{(\lambda+\Delta\lambda)} \tilde{w}_{\alpha\beta}, \qquad (4.6-17)$$

$\delta_{\alpha\beta}$ being the Kronecker symbol and $\mathbf{1}$ a unit matrix in seven-dimensional space. By setting $\mathbf{d}_\alpha = \mathbf{0}$ we obtain

$$\mathbf{Q}_{\alpha\beta}\delta\mathbf{P}_\beta = -\mathbf{d}_{0,\alpha} + (\delta\lambda)_\alpha[(1-\eta)\boldsymbol{\rho}_\alpha(\lambda) + \eta\boldsymbol{\rho}_\alpha(\lambda+\Delta\lambda)]. \qquad (4.6-18)$$

Next calculate the value of the yield function g at point $(\mathbf{E}+\Delta\mathbf{E}, < \mathbf{P} > (\lambda+\Delta\lambda))$,

$$g_{0,\alpha} = g_\alpha(\mathbf{E}_\alpha + \Delta\mathbf{E}_\alpha,\ < \mathbf{P} >_\alpha (\lambda+\Delta\lambda)). \qquad (4.6-19)$$

For the sake of clarity, assume simply that[4.6-8]

$$g_{0,\alpha} > 0 \quad \text{for } \alpha = 1, \dots m,\ m \le N, \qquad (4.6-20)$$

N being the number of Gauss points, and perform a truncated Taylor expansion for these values of α,

$$g_\alpha \approx g_{0,\alpha} + \left.\frac{\partial g_\alpha}{\partial < \mathbf{P} >_\alpha}\right|_{(\lambda+\Delta\lambda)} \cdot \tilde{w}_{\alpha\beta}\delta\mathbf{P}_\beta, \quad (\text{sum over } \beta = 1, \dots, N), \qquad (4.6-21)$$

where advantage has been taken of (4.6-15). Consistency then requires that

$$g_{0,\alpha} + \mathbf{G}_{\alpha\beta} \cdot \delta\mathbf{P}_\beta = 0 \quad \alpha = 1, \dots m, \qquad (4.6-22)$$

where we have defined

$$\mathbf{G}_{\alpha\beta} = \left.\frac{\partial g_\alpha}{\partial < \mathbf{P} >_\alpha}\right|_{(\lambda+\Delta\lambda)} \tilde{w}_{\alpha\beta}. \qquad (4.6-23)$$

Since, in agreement with (4.6-2) and (4.6-3),

$$\delta\mathbf{P}_\beta = \mathbf{0} \quad \text{for } \beta = 1+1, \dots, N, \qquad (4.6-24)$$

[4.6-8] Of course it is convenient to collect all the $(\tilde{w}_{\alpha\beta})s$ in an ordered set (say in a matrix $\bar{W}$ of dimension $N \times N$) once and for all, and correspondingly all the $(g_{0,\alpha})s$ in a vector of dimension N to comply with $\bar{W}$.

the set of equations (4.6-18) and (4.6-22) provides as with an iterative scheme for the calculation of $\delta\lambda$ and $\delta\mathbf{P}$.

Alternatively, explicit use of (4.6-18) leaves us with

$$\delta\mathbf{P}_\beta = -[\mathbf{Q}_{\beta\gamma}]^{-1}\mathbf{d}_{0,\alpha} + [\mathbf{Q}_{\beta\gamma}]^{-1}(\delta\lambda[(1-\eta)\boldsymbol{\rho}(\lambda) + \eta\boldsymbol{\rho}(\lambda+\Delta\lambda)])_\gamma, \qquad (4.6-25)$$

$[\mathbf{Q}]_{\beta\gamma}$ denoting a $m \times m$ matrix where the elements $\mathbf{Q}$ are quadratic matrices of dimension seven. (Summation in (4.6-25) with respect to γ is implied.) If we introduce the notation

$$\mathbf{A}_{\beta\gamma} = [\mathbf{Q}_{\beta\gamma}]^{-1}\boldsymbol{\rho}_\gamma \quad \text{(no sum)}, \qquad (4.6-26)$$

we can write (4.6-25) in the form

$$\delta\mathbf{P}_\beta = -[\mathbf{Q}_{\beta\gamma}]^{-1}\mathbf{d}_{0,\gamma} + (\delta\lambda)_\gamma[(1-\eta)\mathbf{A}_{\beta\gamma}(\lambda) + \eta\mathbf{A}_{\beta\gamma}(\lambda+\Delta\lambda)], \qquad (4.6\text{- }27)$$

and hence (4.6-22) ends up with a system of m equations for the m unknown $(\delta\lambda)s$:

$$\begin{aligned} g_{0,\alpha} \quad &- \quad \mathbf{G}_{\alpha\beta}\cdot[\mathbf{Q}_{\beta\gamma}]^{-1}\mathbf{d}_{0,\gamma} \\ &+ \quad \mathbf{G}_{\alpha\beta}\cdot[(1-\eta)\mathbf{A}_{\beta\gamma}(\lambda) \\ &\quad + \eta\mathbf{A}_{\beta\gamma}(\lambda+\Delta\lambda)](\delta\lambda)_\gamma = 0 \quad \text{(sum over } \beta,\gamma). \end{aligned} \qquad (4.6\text{- }28)$$

In matrix notation (4.6-28) becomes[4.6−9]

$$\bar{g}_0 - \bar{\mathbf{G}}\cdot\bar{\mathbf{Q}}^{-1}\bar{\mathbf{d}}_0 + \bar{\mathbf{G}}\cdot[(1-\eta)\bar{\mathbf{A}}(\lambda) + \eta\bar{\mathbf{A}}(\lambda+\Delta\lambda)]((\overline{\delta\lambda}) = 0, \qquad (4.6-29)$$

where an overbar indicates matrices corresponding to m-dimensional space.

When (4.6-29) has been solved for $\delta\lambda$, a new $\Delta\lambda$ is known, whereupon the updated inelastic state is calculated by use of (4.6-6). Corresponding nonlocal quantities are evaluated from (4.6-14), the constitutive function $\boldsymbol{\rho}$ is updated, a new value of the yield function g_0 is calculated at all Gauss points, and the iterative scheme is repeated. As a result of the nonlocal interaction it is emphasized that the number of points at loading may have been changed, i.e. the new and the old m do not in general coincide.

The iteration procedure continues until the condition (4.6-10) is approximately satisfied, say

$$|g| < \epsilon_g \quad \text{if} \quad \Delta\lambda > 0, \qquad (4.6-30)$$

[4.6−9] To avoid confusion between $\overline{\mathbf{G}^{\mathrm{T}}}$ and $\overline{\mathbf{G}}^{\mathrm{T}}$ we use dot notation instead of transpose of a matrix to indicate the scalar product appearing in (4.6-29) and elsewhere.

where ϵ_g is a prescribed tolerance. Alternatively, a criterion on the iteration vector **d** of the type

$$\|\mathbf{d}\| < \epsilon_d, \tag{4.6-31}$$

may be used.

Remark 6.1. The restriction of the generalized Euler procedures to local theory is easy to perform. Choose $\tilde{w}_{\alpha\beta} = \delta_{\alpha\beta}$ in (4.6-14) and in subsequent equations, and deduce that (4.6-17), (4.6-23) and (4.6-26) should be replaced by

$$\left.\begin{aligned} \mathbf{Q} &= \mathbf{1} - \Delta\lambda \left.\frac{\partial \boldsymbol{\rho}}{\partial < \mathbf{P} >}\right|_{(\lambda+\Delta\lambda)}, \\ \mathbf{G} &= \left.\frac{\partial g}{\partial < \mathbf{P} >}\right|_{(\lambda+\Delta\lambda)}, \\ \mathbf{A} &= \mathbf{Q}^{-1}\boldsymbol{\rho}, \end{aligned}\right\} \tag{4.6-32}$$

while (4.6-28) may be written in the form (using matrix notation)

$$\delta\lambda = \frac{\mathbf{G}^{\mathrm{T}}\mathbf{Q}^{-1}\mathbf{d}_0 - g_0}{(1-\eta)\mathbf{G}^{\mathrm{T}}\mathbf{A}(\lambda) + \eta\mathbf{G}^{\mathrm{T}}\mathbf{A}(\lambda+\Delta\lambda)}. \tag{4.6-33}$$

As expected, the final result is represented by one single equation for the unknown $\delta\lambda$.□

4.6.2 Nonlocal forward Euler procedure

The choice $\eta = 0$ in (4.6-6) represents the forward Euler method. The iteration vector **d** now vanishes identically. Hence (4.6-6) and (4.6-11) coincide, and leave us with the single relationship

$$\Delta\mathbf{P} = \mathbf{P} - \mathbf{P}(\lambda) = \Delta\lambda\boldsymbol{\rho}(\lambda) \tag{4.6-34}$$

for the incremental inelastic variable. All equations subsequent to (4.6-11) are easily seen to comply with the forward Euler method, if we merely set $\Delta\lambda = 0$ and replace $\delta\lambda$ and $\delta\mathbf{P}$ by $\Delta\lambda$ and $\Delta\mathbf{P}$, respectively. Especially, we note that also (4.6-18) (or equivalently (4.6-25)) becomes identical with (4.6-34), since the second term in the right-hand side of (4.6-17) now vanishes.

As for the final result, we conclude that (4.6-28) by use of (4.6-6) becomes

$$g_{0,\alpha} + \mathbf{G}_{\alpha\beta} \cdot (\boldsymbol{\rho}\Delta\lambda)_\beta = 0, \tag{4.6-35}$$

or

$$g_{0,\alpha} + U_{\alpha\beta}(\Delta\lambda)_\beta = 0, \tag{4.6-36}$$

where

$$U_{\alpha\beta} = \mathbf{G}_{\alpha\beta} \cdot \boldsymbol{\rho}_\beta, \quad \text{(no sum)}. \tag{4.6-37}$$

In matrix form

$$\bar{g}_0 + \bar{U}(\overline{\Delta\lambda}) = \bar{0}, \tag{4.6-38}$$

which of course also follows directly from (4.6-29) by using (4.6-26) and (4.6-37).

Based on the solution of (4.6-38) we may construct a N-dimensional vector, consistent with the loading conditions (4.6-10), of the form

$$(\overline{\Delta\lambda})^+ = \{\Delta\lambda)_1^+, (\Delta\lambda)_2^+, \ldots, (\Delta\lambda)_m^+,\ 0, \ldots, 0\}^T, \tag{4.6-39}$$

where

$$(\Delta\lambda)^+ = \left\{ \begin{array}{lll} \Delta\lambda & \text{if} & \Delta\lambda > 0, \\ 0 & \text{if} & \Delta\lambda < 0. \end{array} \right\}. \tag{4.6-40}$$

From (4.6-34) we obtain the increments of the inelastic variables,

$$(\Delta\mathbf{P})_\alpha = ((\Delta\lambda)^+ \boldsymbol{\rho}_\alpha, \quad \alpha = 1, \ldots N, \tag{4.6-41}$$

while the corresponding nonlocal quantities are calculated by use of (4.6-15).

The forward Euler integration is then completed. However, the updated state $(g, < \mathbf{P} >)$ does not necessarily satisfy the conditions (4.6-10) and (4.6-30) for all Gauss points. Not even for points being elastic at the beginning of the increment (i.e. at state λ). If e.g. $g_\alpha(\lambda) < 0$ at $\mathbf{X}$ (Gauss point α), we cannot ignore the possibility that $g_\alpha(\lambda + \Delta\lambda) > 0$ at the same point $\mathbf{X}$, due to plastic deformation outside $\mathbf{X}$. The forward Euler procedure may therefore be extended to include iterations in order to fulfill the loading conditions[4.6-10]. Thus the above integration scheme is repeated, now recognizing $(g,\ \mathbf{E},\ \mathbf{P})_{\lambda+\Delta\lambda}$ as the known state. Calculate new values ${}^1g_0$ of the yield functions at all Gauss points. Then for all ${}^1g_{0,\alpha} > 0$, (4.6-34) (or (4.6-38)) is employed again. Calculate

$$ {}^2(\Delta\lambda)_\alpha = {}^1(\Delta\lambda)^+_\alpha + (\delta\lambda)_\alpha \ ,\ \alpha = 1,\dots N, \tag{4.6 – 42}$$

where the iterative changes $\delta\lambda$ are the new solution of (4.6-36), and then construct a new vector ${}^2(\overline{\Delta\lambda})^+$ with elements ${}^2(\Delta\lambda)^+$ satisfying (4.6-40)[4.6-11].

The procedure may be repeated until agreement with (4.6-10) and (4.6-30) is obtained.

Remark 6.2. For restriction to local theory, proceed as in *Remark 6.1* and conclude from (4.6-35) or directly from (4.6-33) (with $\eta = 0$) that

$$ \Delta\lambda = -\frac{g_0}{\boldsymbol{\rho}^{\mathrm{T}}(\lambda)\left.\dfrac{\partial g}{\partial < \mathbf{P} >}\right|_\lambda}.\Box \tag{4.6 – 43}$$

4.6.3 Nonlocal backward Euler procedure

With the choice $\eta = 1$ in (4.6-6) we obtain the backward Euler method for the integration of first-order ordinary differential equations. We need starting values of $\Delta\lambda$ and the iteration vector $\mathbf{d}_0$. These may conveniently be established by use of the forward Euler procedure, explicitly by employing (4.6-38)-(4.6-41) in the preceding subsection. Then, knowing ${}^1(\Delta\lambda)^+$ and ${}^1(\Delta\mathbf{P})$, we use (4.6-11) to obtain the starting value ${}^1\mathbf{d}_0$ of the iteration vector[4.6-12]

$$ {}^1\mathbf{d}_0 = {}^1\mathbf{P} - {}^0\mathbf{P} - {}^1(\Delta\lambda){}^1\boldsymbol{\rho} = {}^1(\Delta\lambda)({}^0\boldsymbol{\rho} - {}^1\boldsymbol{\rho}), \tag{4.6 – 44}$$

where

$$ \left. \begin{aligned} {}^0\mathbf{P} &= \mathbf{P}(\lambda), \\ {}^1\mathbf{P} &= {}^0\mathbf{P} + {}^1(\Delta\mathbf{P}), \end{aligned} \right\} \tag{4.6 – 45}$$

and

[4.6-10] In fact justifying the notation 'implicit' strategy.

[4.6-11] Note that $\delta\lambda$ in (4.6-42) may be negative. Also note that the iterative procedure outlined above corresponds to a Newton-Raphson solution of the nonlinear equation $g(x) = 0$.

[4.6-12] For simplicity we write ${}^1(\Delta\lambda)$ instead of ${}^1(\Delta\lambda)^+$.

$$\left.\begin{aligned} {}^{0}\boldsymbol{\rho} &= \boldsymbol{\rho}(\lambda) = \boldsymbol{\rho}(\mathbf{E}, <\mathbf{P}>), \\ {}^{1}\boldsymbol{\rho} &= \boldsymbol{\rho}(\lambda + {}^{1}(\Delta\lambda)) = \boldsymbol{\rho}(\mathbf{E} + \Delta\mathbf{E}, {}^{1}(<\mathbf{P}>) \end{aligned}\right\}. \tag{4.6-46}$$

Hence, from (4.6-17) we obtain

$${}^{1}\mathbf{Q}_{\alpha\beta} = \delta_{\alpha\beta}\mathbf{1} - {}^{1}(\Delta\lambda)_{\alpha}\, {}^{1}(\frac{\partial \boldsymbol{\rho}_{\alpha}}{\partial < \mathbf{P} >_{\alpha}})\tilde{w}_{\alpha\beta}, \tag{4.6-47}$$

and from (4.6-26)

$${}^{1}\mathbf{A}_{\beta\gamma} = [{}^{1}\mathbf{Q}_{\beta\gamma}]^{-1}\, {}^{1}\boldsymbol{\rho}_{\gamma}, \quad \text{(no sum)}. \tag{4.6-48}$$

Finally, (4.6-28) provides us with a system of equations for the unknown $(\delta\lambda)s$,

$${}^{1}g_{0,\alpha} - {}^{1}\mathbf{G}_{\alpha\beta} \cdot [{}^{1}\mathbf{Q}_{\beta\gamma}]^{-1}\, {}^{1}\mathbf{d}_{0,\alpha} + {}^{1}\mathbf{G}_{\alpha\beta} \cdot {}^{1}\mathbf{A}_{\beta\gamma}(\delta\lambda)_{\gamma} = 0, \tag{4.6-49}$$

where we understand that ${}^{1}\mathbf{G}_{\alpha\beta}$ has been calculated by use of (4.6-23). In matrix notation

$${}^{1}\bar{\mathbf{D}}(\overline{\delta\lambda}) = {}^{1}(\bar{\mathbf{G}} \cdot \bar{\mathbf{Q}}^{-1}\bar{\mathbf{d}}_{0}) - {}^{1}\bar{g}_{0}, \tag{4.6-50}$$

where

$$\bar{\mathbf{D}} = \bar{\mathbf{G}} \cdot \bar{\mathbf{A}}. \tag{4.6-51}$$

Remark 6.3. We recall that the number of equations to be solved equals the number of points at plastic loading, i.e. m according to (4.6-20). It is seen that (4.6-49) can be solved by means of the proposed technique, only if the matrix $\bar{\mathbf{Q}}$ has been inverted in advance. This may be a critical drawback if the number m is large. In many applications, however, yielding is confirmed to a region which is small compared to the entire body in consideration, and hence $m << N$.□

Solving (4.6-50) for $(\overline{\delta\lambda})$ and using (4.6-42) provides us with a new set of increments ${}^{2}(\Delta\lambda)_{\alpha}$ of the plastic multiplier, from which we construct the vector ${}^{2}(\overline{\Delta\lambda})^{+}$ with elements defined by (4.6-40). Then proceed, following the scheme of the nonlocal generalized Euler procedure in Subsection 4.6.1 (cf. the discussion subsequent to (4.6-29)).

Remark 6.4. The local form of the backward Euler procedure is obtained by choosing $\eta = 1$ in (4.6-33), i.e.

$$\delta\lambda = \frac{\mathbf{G}^{\mathrm{T}}\mathbf{Q}^{-1}\mathbf{d}_0 - g_0}{\mathbf{G}^{\mathrm{T}}\mathbf{A}(\lambda + \Delta\lambda)}. \qquad (4.6-52)$$

The result (4.6-52) is derived within a strain space formulation. A result of similar form also applies with regard to stress space, and is then recognized as a *backward Euler return*, cf. e.g. Criesfield (1991).□

4.6.4 Yield functions of von Mises type

The integration technique derived in the preceding subsections will be demonstrated with regard to the yield function in the illustrative example of Section 3.6, though, due to the apparent complexities, we will confine ourselves to the corresponding, degenerate problem in one dimension. Prior to this, however, we look at an even simpler problem, also in one dimension, which in familiar notations is stated as

$$\left.\begin{aligned} &\sigma = E(\epsilon - <\epsilon^p>), \quad \sigma > 0 \\ &f = \sigma - \sigma_y - H <\kappa>, \; E + H > 0 \\ &\dot{\epsilon}^p = \dot{\lambda}\frac{\partial f}{\partial \sigma} = \dot{\lambda} = \dot{\kappa}. \end{aligned}\right\} \qquad (4.6-53)$$

From (4.6-1) and (4.6-53) is deduced that

$$\mathbf{P} = [\epsilon^p \; \kappa]^T \;, \; \boldsymbol{\rho} = [1 \; 1]^T, \qquad (4.6-54)$$

and

$$g = E\epsilon - \sigma_y - \mathbf{F}\cdot <\mathbf{P}>, \qquad (4.6-55)$$

where[4.6-13]

$$\mathbf{F} = [E \; H]^T. \qquad (4.6-56)$$

By virtue of $(4.6\text{-}54)_2$ we note that (4.6-6) becomes

$$\Delta\mathbf{P} = \Delta\lambda\boldsymbol{\rho}, \qquad (4.6-57)$$

[4.6-13] Since g depends linearly on $<\epsilon>^p$ and $<\kappa>$, it is obvious that $\mathbf{P}$ here may be treated as a proper element in a two-dimensional vector space. Cf. footnote 4.6-2.

and hence that the nonlocal generalized Euler procedure in fact turns into a corresponding forward Euler procedure.

By use of (4.6-55) and (4.6-56) we note that (4.6-23) becomes

$$\mathbf{G}_{\alpha\beta} = -\tilde{w}_{\alpha\beta}\mathbf{F}_{\alpha} = -\begin{bmatrix} \tilde{w}^p_{\alpha\beta} & 0 \\ 0 & \tilde{w}^h_{\alpha\beta} \end{bmatrix}\begin{bmatrix} E \\ H \end{bmatrix} = -\begin{bmatrix} E & \tilde{w}^p_{\alpha\beta} \\ H & \tilde{w}^h_{\alpha\beta} \end{bmatrix} \qquad (4.6-58)$$

and hence it follows from (4.6-37) and (4.6-56) that

$$U_{\alpha\beta} = -(E\tilde{w}^p_{\alpha\beta} + H\tilde{w}^p_{\alpha\beta}). \qquad (4.6-59)$$

In view of (4.6-36), the final system of equations to be solved now becomes[4.6-14]

$$g_{0,\alpha} - (E\tilde{w}^p_{\alpha\beta} + H\tilde{w}^h_{\alpha\beta})(\Delta\lambda)_\beta = 0, \qquad (4.6-60)$$

where

$$g_0 = E(\epsilon + \Delta\epsilon) - \sigma_y - \mathbf{F}\cdot\ {}^0(<\mathbf{P}>), \qquad (4.6-61)$$

${}^0(<\mathbf{P}>)$ being defined by

$${}^0(<\mathbf{P}>) =<\mathbf{P}>(\lambda). \qquad (4.6-62)$$

Then, proceed in accordance with the general scheme outlined in Subsection 4.6.2.

Remark 6.5. The choice $\tilde{w}^h_{\alpha\beta} = \delta_{\alpha\beta}$ converts (4.6-60) into the equation

$$g_{0,\alpha} - E(\Delta\lambda)_\alpha - H\tilde{w}^h_{\alpha\beta}(\Delta\lambda)_\beta = 0, \qquad (4.6-63)$$

representing the special case of restricted nonlocality, discussed in *Remark 3.4* in Section 3.3.3.□

We now return to the problem discussed initially. In the one-dimensional case 'deviatoric' stress and 'deviatoric' strain become actual stress and strain (say, σ and ϵ). Hence $(3.6\text{-}3)_1$, (3.6-4) and (3.6-5) should be replaced by the following set of equations,

[4.6-14] It should be noted that the left-hand side of (4.6-60) is an *exact* expression for the Taylor expansion of g at the state $(\epsilon + \Delta\epsilon, <\mathbf{P}>(\lambda))$, since g is linear in $<\mathbf{P}>$.

$$\left.\begin{aligned} \sigma &= E(\epsilon - <\epsilon^p>),\ E = 2\mu, \\ f &= \sigma^2 - <\kappa>, \\ g &= E^2(\epsilon - <\epsilon^p>)^2 - <\kappa>. \end{aligned}\right\} \qquad (4.6-64)$$

If we assume that (3.6-84) is valid, we note that (4.6-1) becomes

$$\left.\begin{aligned} \mathbf{P} &= [\epsilon^p\ \kappa]^T, \\ \boldsymbol{\rho} &= [R\ r]^T, \end{aligned}\right\} \qquad (4.6-65)$$

where

$$R = \frac{\partial f}{\partial \sigma} = 2E(\epsilon - <\epsilon^P>) \qquad (4.6-66)$$

and where it is assumed that r is a constant (independent of $<\mathbf{P}>$ and $\mathbf{X}$)[4.6−15].

Continuing, we perform the nonlocal backward Euler procedure, outlined in Subsection 4.6.3, to the problem stated by $(4.6\text{-}64)_3$, (4.6-65) and (4.6-66).

To begin with we introduce notations in agreement with those in Subsection 4.6.3,

$$^0\mathbf{P} = \mathbf{P}(\lambda) = [^0\epsilon^p\ \ ^0\kappa], \qquad (4.6-67)$$

$$^0R = R(\lambda) = 2E(^0\epsilon -\ ^0(<\epsilon^p>)), \qquad (4.6-68)$$

$$^0\boldsymbol{\rho} = \rho(\lambda) = [^0R\ r]^T, \qquad (4.6-69)$$

and

$$^0g = g(\lambda) = E^2(^0\epsilon -\ ^0(<\epsilon^p>))^2 -\ ^0(<\kappa>) \qquad (4.6-70)$$

The first step is then to use the nonlocal forward Euler method derived in Subsection 4.6.2 to produce a start solution. From $(4.6\text{-}64)_3$ we obtain

$$\frac{\partial g_\alpha}{\partial <\mathbf{P}>_\alpha} = -[ER\ 1]^T, \qquad (4.6-71)$$

[4.6−15] Hence the sign of r determines uniquely the state of strain hardening. Cf. the discussion in Section 3.6.4.

and hence (4.6-23) becomes (set $\Delta\lambda = 0$),

$$\mathbf{G}_{\alpha\beta} = -\begin{bmatrix} \tilde{w}^p_{\alpha\beta} & 0 \\ 0 & \tilde{w}^h_{\alpha\beta} \end{bmatrix} \begin{bmatrix} E\,{}^0R_\alpha \\ 1 \end{bmatrix} = -\begin{bmatrix} E^0R_\alpha \tilde{w}^p_{\alpha\beta} \\ \tilde{w}^h_{\alpha\beta} \end{bmatrix}, \quad \text{(no sum)}. \qquad (4.6-72)$$

Then, by use of (4.6-69) and (4.6-72) we conclude that (4.6-37) becomes

$$U_{\alpha\beta} = -({}^0R_\alpha\,{}^0R_\beta\,\tilde{w}^p_{\alpha\beta} + r\,\tilde{w}^h_{\alpha\beta}), \text{ (no sum)}, \qquad (4.6-73)$$

or, in matrix notation,

$$\bar{U} = -\begin{bmatrix} {}^0R_1\,{}^0R_1\,\tilde{w}^p_{11} + r\,\tilde{w}^h_{11} & \cdots & {}^0R_1\,{}^0R_m\,\tilde{w}^p_{1m} + r\,\tilde{w}^h_{1m} \\ \vdots & & \vdots \\ {}^0R_m\,{}^0R_1\,\tilde{w}^p_{m1} + r\,\tilde{w}^h_{m1} & \cdots & {}^0R_m\,{}^0R_m\,\tilde{w}^p_{mm} + r\,\tilde{w}^h_{mm} \end{bmatrix}. \qquad (4.6-74)$$

Calculate the vector $\bar{g}_0$ from (4.6-70), use the result together with (4.6-74) and solve (4.6-38) for the unknown vector $(\overline{\Delta\lambda})$. Construct the corresponding vector $(\overline{\Delta\lambda})^+$ defined by (4.6-39) and (4.6-40) and use (4.6-41) to determine the increments of the plastic strain ϵ^p and the hardening variable κ.

Now we know ${}^1(\Delta\lambda)^+$ and ${}^1(\Delta\mathbf{P})$ and may proceed as in Subsection 4.6.3, calculating ${}^1\mathbf{d}_0$ from (4.6-44), with ${}^1\boldsymbol{\rho}$ given explicitly by

$${}^1\boldsymbol{\rho} = \boldsymbol{\rho}(\lambda + {}^1(\Delta\lambda)) = [{}^1R \;\; r]^T = [2E({}^0\epsilon + \Delta\epsilon - {}^1(<\epsilon^p>) \;\; r]^T, \qquad (4.6-75)$$

where $\Delta\epsilon = {}^i(\Delta\epsilon)$ is iterative strain, consistent with the equilibrium iteration scheme (4.3-12)-(4.3-14).

In the next step, when we now know ${}^1\mathbf{d}_0$ and ${}^1(\Delta\lambda)$, - as in (4.6-44) we write ${}^1(\Delta\lambda)$ instead of ${}^1(\Delta\lambda)^+$ - we calculate the matrix $\mathbf{Q}_{\alpha\beta}$, defined originally by (4.6-17) and again recorded in (4.6-47) for the initial integration step of the backward Euler procedure.

From (4.6-65) and (4.6-66) we observe that

$$\frac{\partial\boldsymbol{\rho}}{\partial\mathbf{P}} = \begin{bmatrix} -2E & 0 \\ 0 & 0 \end{bmatrix}, \qquad (4.6-76)$$

and hence

$$\mathbf{Q}_{\alpha\beta} = \begin{bmatrix} \delta_{\alpha\beta} & 0 \\ 0 & \delta_{\alpha\beta} \end{bmatrix} - (\Delta\lambda)_\alpha \begin{bmatrix} \tilde{w}^p_{\alpha\beta} & 0 \\ 0 & \tilde{w}^h_{\alpha\beta} \end{bmatrix} \begin{bmatrix} -2E & 0 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \delta_{\alpha\beta} + 2E\tilde{w}^p_{\alpha\beta}(\Delta\lambda)_\alpha & 0 \\ 0 & \delta_{\alpha\beta} \end{bmatrix}. \qquad (4.6\text{- }77)$$

In matrix notation

$$\bar{\mathbf{Q}} = \begin{bmatrix} 1 + 2E(\Delta\lambda)_1\tilde{w}^p_{11} & 0 & \cdots & 2E(\Delta\lambda)_1\tilde{w}^p_{1m} & 0 \\ 0 & 1 & & 0 & 0 \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ \cdot & & & \cdot & \\ 2E(\Delta\lambda)_m\tilde{w}^p_{m1} & 0 & \cdots & 1 + 2E(\Delta\lambda)_m\tilde{w}^p_{mm} & 0 \\ 0 & 0 & & 0 & 1 \end{bmatrix} \qquad (4.6-78)$$

The inverse of $\bar{\mathbf{Q}}$ is a $2m \times 2m$ matrix, which may written in the form

$$\bar{\mathbf{Q}}^{-1} = \begin{bmatrix} \mathbf{q}^{11} & \cdots & \mathbf{q}^{1m} \\ \vdots & & \vdots \\ \mathbf{q}^{m1} & \cdots & \mathbf{q}^{mm} \end{bmatrix}, \qquad (4.6-79)$$

where $\mathbf{q}^{\beta\gamma}$ is the 2×2 matrix,

$$\mathbf{q}^{\beta\gamma} = \begin{bmatrix} q^{\beta\gamma}_{11} & q^{\beta\gamma}_{12} \\ q^{\beta\gamma}_{21} & q^{\beta\gamma}_{22} \end{bmatrix}. \qquad (4.6-80)$$

Substitution of (4.6-79) and (4.6-80) into (4.6-26) then yields (no sum)

$$\mathbf{A}_{\beta\gamma} = \mathbf{q}^{\beta\gamma}\boldsymbol{\rho}^\gamma = \begin{bmatrix} q^{\beta\gamma}_{11}R^\gamma & + & q^{\beta\gamma}_{12}r \\ q^{\beta\gamma}_{21}R^\gamma & + & q^{\beta\gamma}_{22}r \end{bmatrix} = \begin{bmatrix} A^{\beta\gamma}_1 \\ A^{\beta\gamma}_2 \end{bmatrix}, \qquad (4.6-81)$$

where we also have used $(4.6\text{-}65)_2$. In matrix form (4.6-8) reads

$$\bar{\mathbf{A}} = \begin{bmatrix} A_1^{11} & \cdots & A_1^{1m} \\ A_2^{11} & & A_2^{1m} \\ \vdots & & \vdots \\ A_1^{m1} & \cdots & A_1^{mm} \\ A_2^{m1} & & A_2^{mm} \end{bmatrix}, \qquad (4.6-82)$$

where $A_1^{\beta\gamma}$ and $A_2^{\beta\gamma}$ are defined by $(4.6\text{-}81)_3$.

We also need the matrix $\bar{\mathbf{G}}$, which in view of (4.6-72) has the form

$$\bar{\mathbf{G}} = -\begin{bmatrix} ER^1\tilde{w}_{11}^p & \cdots & ER^1\tilde{w}_{1m}^p \\ \tilde{w}_{11}^h & & \tilde{w}_{1m}^h \\ \vdots & & \vdots \\ ER^m\tilde{w}_{m1}^p & \cdots & ER^m\tilde{w}_{mm}^p \\ \tilde{w}_{m1}^h & & \tilde{w}_{mm}^h \end{bmatrix}. \qquad (4.6-83)$$

Now we can calculate ${}^1\bar{\mathbf{Q}}^{-1}$ (by (4.7-78)-(4.6-80)), ${}^1\bar{\mathbf{A}}$ and ${}^1\bar{\mathbf{G}}$. Further, the vector ${}^1\bar{g}_0$ is easy to perform, keeping in mind that (cf. (4.6-19) and (4.6-70))

$${}^1g_0 = g_0(\lambda + \Delta\lambda) = E^2({}^0\epsilon + \Delta\epsilon - \ {}^1 < \epsilon^p >)^2 - \ {}^1(< \kappa >)), \qquad (4.6-84)$$

and hence we are ready to employ (4.6-50) and solve for $(\overline{\delta\lambda})$. Then proceed as described in Subsection 4.6.1 with regard to the nonlocal generalized Euler procedure.

Remark 6.6. The local counterpart of $\delta\lambda$ is obtained by emplyoing (4.6-52). Using (4.6-78)-(4.6-83) it is straightforward to deduce that

$$\mathbf{Q} = \begin{bmatrix} 1 + 2E(\delta\lambda) & 0 \\ 0 & 1 \end{bmatrix}, \qquad (4.6-85)$$

$$\mathbf{Q}^{-1} = \begin{bmatrix} (1 + 2E(\delta\lambda))^{-1} & 0 \\ 0 & 1 \end{bmatrix}, \qquad (4.6-86)$$

$$\mathbf{A} = [R(1 + 2E(\Delta\lambda))^{-1} \quad r]^T, \qquad (4.6-87)$$

and

$$\mathbf{G} = -[ER \quad 1]^T. \tag{4.6-88}$$

Further, it follows from (4.6-44) and (4.6-65) that

$${}^1\mathbf{d}_0 = {}^1(\Delta\lambda)[{}^1d_0^p \quad 0]^T \tag{4.6-89}$$

where

$${}^1d_0^p = {}^1(\Delta\lambda)({}^0R - {}^1R). \tag{4.6-90}$$

Substitution of (4.6-86)-(4.6-90) into (4.6-52) then gives

$$\delta\lambda = \frac{ER(1+2E(\Delta\lambda))^{-1}\ d_0^p + g_0}{ER^2(1+2E(\Delta\lambda))^{-1} + r}. \tag{4.6-91}$$

The forward Euler solution is obtained from (4.6-91) by setting $d_0^p = \Delta\lambda = 0$ and replacing $\delta\lambda$ by $\Delta\lambda$, i.e.

$$\Delta\lambda = {}^1(\Delta\lambda) = \frac{g_0}{r+R^2} \quad , \quad g_0 = {}^0g_0 \quad , \quad R = {}^0R. \tag{4.6-92}$$

With $\Delta\lambda$ known, ${}^1d_0^p$ is determined by (4.6-90).□

4.6.5 Quasi-local integration technique

In this subsection a simplified method for the updating of the plastic multiplier will be derived. It corresponds to a certain extent to the approximate formulation discussed in Section 3.6.6, and is closely related to the type of Newton-Raphson algorithm that uses (4.4-8) as elastic-plastic stiffness matrix (see Section 4.4.2). We recall that the quasi-local continuous tangential stiffness matrix, defined by (4.4-8), is derived by use of (4.4-7), which represents a trivial solution of the 'global' consistency condition (3.2-32). The method is iterative, but simple to implement numerically, since the calculations at each iteration step do not involve a system of equations to be solved simultaneously, as was the case for the integration technique of the preceding subsections.

The basic idea of the quasi-local integration technique to be derived below is easy to grasp: use the function π approximately determined by (4.4-7) as a predictor in an iterative process to update the incremental plastic multiplier ${}^i(\Delta\lambda)$ in order to finally obtain[4.6-16]

[4.6-16]The loading/unloading conditions (4.6-93) are recorded in (4.6-10) and (4.6-30) in a similar form.

$$
\left.\begin{array}{ll}
{}^{i}g \le 0 & \text{if } \ {}^{i}(\Delta\lambda) = 0, \\
|{}^{i}g| < \epsilon_g & \text{if } \ {}^{i}(\Delta\lambda) > 0,
\end{array}\right\} \qquad (4.6-93)
$$

for all Gauss points. In (4.6-93) ϵ_g is a prescribed tolerance with respect to the yield function g in strain space.

We continue using $\mathbf{P}$ and $\boldsymbol{\rho}$, defined by (4.6-1), for the inelastic variables and for the constitutive functions $\mathbf{R}$ and r, respectively. Hence, (4.4-7) becomes[4.6−17]

$$
\frac{1}{\pi} = -\frac{1}{\rho_0}\frac{\partial < \rho_0 g >}{\partial \mathbf{P}} \cdot \boldsymbol{\rho}, \quad \pi > 0, \qquad (4.6-94)
$$

where $< \rho_0 g >$ is defined by (3.2-22)[4.6−18]. An explicit form of (4.6-94) is given by (3.6-86), in which the yield function is of von Mises type, defined by (3.6-5).

We use notations similar to those employed in Subsection 4.6.1, but for the sake of clarity they are repeated below. Thus, for $i = 1, 2 \ldots$, we define[4.6−19]

$$
\mathbf{E}(\lambda) = {}^{0}\mathbf{E}, \qquad (4.6-95)
$$

$$
{}^{i-1}\mathbf{P} = \mathbf{P}(\lambda + {}^{i-1}(\Delta\lambda)), \quad {}^{0}(\Delta\lambda) = 0, \qquad (4.6-96)
$$

$$
\left.\begin{array}{rcl}
{}^{i-1}\boldsymbol{\rho} & = & \boldsymbol{\rho}(\lambda + {}^{i-1}(\Delta\lambda)) = \boldsymbol{\rho}({}^{0}\mathbf{E} + \Delta\mathbf{E}, \ {}^{i-1}(< \mathbf{P} >)), \\
{}^{0}\boldsymbol{\rho} & = & \boldsymbol{\rho}(\lambda) = \boldsymbol{\rho}({}^{0}\mathbf{E}, \ {}^{0}(< \mathbf{P} >)),
\end{array}\right\} \qquad (4.6-97)
$$

$$
{}^{i-1}g_0 = g({}^{0}\mathbf{E} + \Delta\mathbf{E}, \ {}^{i-1} < \mathbf{P} >), \qquad (4.6-98)
$$

$$
{}^{i-1}\hat{g} = {}^{i-1}\Big(\frac{\partial g}{\partial \mathbf{E}}\Big) \cdot \Delta\mathbf{E}, \quad {}^{i-1}\Big(\frac{\partial g}{\partial \mathbf{E}}\Big) = \frac{\partial g}{\partial \mathbf{E}}\bigg|_{(\lambda + {}^{i-1}(\Delta\lambda))} \qquad (4.6-99)
$$

and

$$
\frac{1}{{}^{i-1}\pi} = -\frac{1}{\rho_0}\, {}^{i-1}\Big(\frac{\partial < \rho_0 g >}{\partial \mathbf{P}}\Big) \cdot \ {}^{i-1}\boldsymbol{\rho}. \qquad (4.6-100)
$$

[4.6−17] Again it is noted that (4.6-94) reflects the consistency condition only in an approximate sense.

[4.6−18] Though similar in form it must be remembered that e.g. $< \kappa >$ and $< \rho_0 g >$ are defined quite differently.

[4.6−19] We recall that λ corresponds to a known state and that $\Delta\mathbf{E}$ has been calculated by equilibrium iterations, using (4.4-8) for the elastic-plastic stiffness matrix in the Newton-Raphson scheme.

Remark 6.7. The notations are in accordance with those of Subsection 4.6.1. It is noticable, however, that the presence of g_0 in (4.6-21) indicates that a Taylor series of g has been used, whereas such an expansion is not necessary to perform here.□

After these preliminaries we are ready to present the iterative scheme. Assume that $\mathbf{P}$, $< \mathbf{P} >$ and $\boldsymbol{\rho}$ are known for all Gauss points at iteration step $i-1$. Then calculate the incremental plastic multiplier in accordance with

$$ {}^{i}(\Delta\lambda) = \begin{cases} {}^{i-1}\pi \; {}^{i-1}\hat{g}, & ({}^{i-1}g_0 > 0, \; {}^{i-1}\hat{g} > 0, \; {}^{i-1}\pi > 0), \\ 0 & (\text{otherwise}). \end{cases} \qquad (4.6-101) $$

That is, calculate ${}^{i-1}g_0$ for all Gauss points and for those with ${}^{i-1}g_0 > 0$ (corresponding to plastic loading) calculate ${}^{i}(\Delta\lambda)$ in agreement with (4.6-101).

Now when ${}^{i}(\Delta\lambda)$ is known at all points, the increments of the inelastic variables are calculated by a forward Euler technique,

$$ {}^{i}(\Delta\mathbf{P}) = {}^{i}(\Delta\lambda)^{i-1}\boldsymbol{\rho}. \qquad (4.6-102) $$

Updated values of $\mathbf{P}$ are calculated,

$$ {}^{i}\mathbf{P} = {}^{0}\mathbf{P} + {}^{i}\Delta\mathbf{P}, \qquad (4.6-103) $$

as well as the corresponding nonlocal quantities ${}^{i} < \mathbf{P} >$. The state is now known for all Gauss points at iteration step i, and a new iteration may be performed to produce a new set of $(\Delta\lambda)s$. Repeat until (4.6-93), if possible, is satisfied.

Remark 6.8. In local theory the counterpart of (4.6-94) is an exact solution of the equation $\dot{g} = 0$ (the continuous strain space form of the consistency condition). Hence, for infinitesimally small increments of strain, a local procedure corresponding to the proposed quasi-local integration technique certainly will provide a correct solution. However, increments are not infinitesimal and the theory is not local! Since the consistency condition is not properly enforced (by currently solving the equation ${}^{i}g = 0$), it cannot be asserted that (4.6-93) can ever be satisfied simultaneously for all points[4.6-20].□

If the proposed quasi-local integration method tends to violate the yield condition significantly, use of sub-incrementation technique will probably reduce the errors. Improved accuracy and convergence properties may also be obtained by employing the strategy outlined below.

[4.6-20] This argument also applies to local theory.

Replace the initial step ($i = 1$) of the integration scheme above by the following procedure. Define

$$^{j-1}g = g(^0\mathbf{E} + \ ^{j-1}r\ \Delta\mathbf{E}, \quad ^0(<\mathbf{P}>)), \qquad (4.6-104)$$

where ^jr is a scalar parameter which satisfies[4.6-21]

$$0 \leq \ ^jr \leq 1 \quad , \quad ^0r = 0. \qquad (4.6-105)$$

We recall that ^0g represents an admissible state and hence $^0g \leq 0$. Calculate $^0g_0$ defined by (4.6-98) and assume that $^0g_0 > 0$. For *fixed* $\mathbf{P} = \ ^0\mathbf{P}$ at *all* Gauss points, solve the equation

$$g(^0\mathbf{E} + r\Delta\mathbf{E}), ^0(<\mathbf{P}>)) = 0, \qquad (4.6-106)$$

for the unknown parameter r. In general (4.6-106) is a nonlinear equation which must be solved iteratively. Expand $g = g(r)$ into a truncated Taylor series

$$^jg \approx \ ^{j-1}g + \ ^{j-1}(\frac{\partial g}{\partial \mathbf{E}}) \cdot \Delta\mathbf{E}\ ^j(\delta r), \qquad (4.6-107)$$

^jg defined by (4.6-104) and $^{j-1}(\partial g/\partial\mathbf{E})$ by

$$^{j-1}(\frac{\partial g}{\partial \mathbf{E}}) = \left.\frac{\partial g}{\partial \mathbf{E}}\right|_{(\mathbf{E}+\ ^{i-1}r\ \Delta\mathbf{E})}, \qquad (4.6-108)$$

and solve the equation $^jg = 0$ for $^j(\delta r)$. The solution is

$$\left.\begin{aligned} ^j(\delta r) &= -\frac{^{j-1}g}{^{j-1}(\frac{\partial g}{\partial \mathbf{E}}) \cdot \Delta\mathbf{E}}, \\ ^jr &= \ ^{j-1}r + \ ^j(\delta r), \end{aligned}\right\} \qquad (4.6-109)$$

where in particular, in view of $(4.6\text{-}105)_2$, we note that

$$^1r = \ ^1(\delta\lambda) = -\frac{^0g}{^0(\frac{\partial g}{\partial \mathbf{E}}) \cdot \Delta\mathbf{E}}. \qquad (4.6-110)$$

[4.6-21] For $j = 1$ we note that $^0g = g(^0\mathbf{E},\ ^0<\mathbf{P}>) = g(\lambda)$ in agreement with notations used in Section 4.6.3.

If r denotes the final solution of (4.6-109), then ${}^1(\Delta\lambda)$ may be calculated from $(4.6\text{-}101)_1$, if we just replace $\Delta\mathbf{E}$ by $(1-r)\Delta\mathbf{E}$, i.e.

$$ {}^1(\Delta r) = {}^0\pi\ {}^0(\frac{\partial g}{\partial \mathbf{E}}) \cdot \Delta\mathbf{E}\ (1-r), \qquad (4.6-111) $$

whereas the updated value of $\mathbf{P}$ is calculated by using (4.6-102) and 4.6-103).

In particular, if the initial estimate of r is used, it is noted that from (4.6-110) and (4.6-111) that

$$ {}^1(\Delta\lambda) = {}^0\pi({}^0g + {}^0(\frac{\partial g}{\partial \mathbf{E}}) \cdot \Delta\mathbf{E}). \qquad (4.6-112) $$

Performing this procedure for each Gauss point at loading produces a set of parameters r, by which ${}^1\mathbf{P}$ and ${}^1(<\mathbf{P}>)$ can be calculated at all points. The state is now known for all Gauss points at iteration step $i=1$, and hence a new iteration may be performed to obtain a new set of $(\Delta\lambda)$s in accordance with the general scheme (4.6-101)-(4.6-103).

Remark 6.9. On condition that the parameter r is accurately calculated, the scheme proposed above corresponds to a reliable (and commonly used) technique in *local* plasticity. Clearly, in local theory, incremental strain $r\Delta\mathbf{E}$ corresponds to purely elastic deformation ($\Delta\mathbf{P}=\mathbf{0}$), whereas plastic deformation takes place during the increment $(1-r)\Delta\mathbf{E}$, $(\Delta\mathbf{P} \neq 0)$.

Estimating the location of the intersection of the strain increment with the yield surface in strain space produces parameters r which differ from point to point throughout the discretized model of the body. Assume, e.g. $\Delta\mathbf{P}_\alpha = \mathbf{0}$ during increment $r_\alpha\Delta\mathbf{E}$ and $\Delta\mathbf{P}_\beta = \mathbf{0}$ during $r_\beta\Delta\mathbf{E}$. Then, if $r_\beta < r_\alpha$, plastic deformation along $(1-r_\beta)\Delta\mathbf{E}$ contributes to the increment of the nonlocal inelastic variable $\mathbf{P}_\alpha$, i.e. $\Delta <\mathbf{P}>_\alpha$ does not vanish in general[4.6−22]. Hence $<\mathbf{P}> = {}^0(<\mathbf{P}>)$ in (4.6-104) corresponds to a *virtual* motion of the yield surface during the strain increment $r\Delta\mathbf{E}$, which coincides with the *actual* one for homogeneous motions.

Nevertheless, if the quasi-local integration technique is augmented by the procedure of locating the intersection of the strain increment with the yield surface, it is believed that it becomes more efficient than if it is not.□

Remark 6.10. In general, not much can be said about accuracy and convergence properties with regard to the quasi-local integration technique. If convergence is not obtained the idea behind the simplified integration technique must be abandoned for the general technique derived in Section 4.6.1.

[4.6−22] The general consequences of this behaviour have been examined briefly in Chapter 3. Also cf. the introductory discussion in Section 4.6.

Obviously the strength of nonlocality is highly affected by the choice of attenuation functions, say $w(\mathbf{X}, \mathbf{Z})$. Hence - independent of the type of integration procedure - it may be conjectured that accurate convergence properties are obtained if w decays smoothly and rapidly with the distance from $\mathbf{X}$ (on condition that solutions to the corresponding *local* problem converge).□

4.7 Numerical examples and analytical solutions

In Section 4.6.4 the general nonlocal integration technique was applied to two different yield functions of von Mises type, one linear and one nonlinear, with regard to stress.

In this section localization in a strain softening bar will be analyzed for these two types of yield functions. The bar has length $L = 0.1$ m, Young's modulus $E = 20000$ MPa (for the entire bar) and is loaded in uniaxial tension. The initial yield stress is $\sigma_y = 2$ MPa, except for a small region in the middle of the bar, where the corresponding yield stress is reduced with a certain amount.

The linear problem is easy to treat analytically which will be demonstrated in Subsection 4.7.1 below, whereas finite element solutions for both problems are presented in Subsection 4.7.2.

4.7.1 Analytical solutions

The linear problem, defined by (4.6-53) will here be solved by analytical methods.

To start with we discuss briefly the choice of attenuation functions; the necessity of their being smooth and rapidly decaying has been emphasized already. Functions of the type $w(x) = exp(-k^2x^2/\ell^2)$ obviously satisfy these demands. Also $w(x) = exp(-k|x|/\ell)$, though not smooth, will certainly prove to be efficient.

Consider first the Gaussian function

$$w = w(|z - x|) = e^{-\frac{k^2(z-x)^2}{\ell^2}} . \qquad (4.7-1)$$

The corresponding representative volume (length) is defined by (3.6-17) or (3.6-18) and becomes

$$V(x) = \int_{-L/2}^{L/2} e^{-\frac{k^2(z-x)^2}{\ell^2}} dz. \qquad (4.7-2)$$

If we require the parameter ℓ to satisfy

$$\lim_{L \to \infty} V(0) = \ell, \qquad (4.7-3)$$

then

$$k = \sqrt{\pi}, \qquad (4.7-4)$$

as is seen from (4.7-2).

Hence, if we choose k in agreement with (4.7-4) the representative volume recorded in (4-7-2) has the characteristic property of attaining the value ℓ at the centre of a bar of infinite length.

The function $\tilde{w}(z, x)$ defined by (3.6-19) is graphically shown in Figure 4.7.1 for $\ell = 0.0157$ m. Clearly, in view of (4.7-3),

$$V(0) \approx \ell \qquad (4.7-5)$$

if

$$\frac{\ell}{L} << 1. \qquad (4.7-6)$$

In Figure 4.7.2 is shown the dependency of $V(0)$ on ℓ/L. For $\ell/L < 0.4$ it is seen that (4.7-5) affords a good approximation to $V(0)$. We also recall that the function $\tilde{w}$ by definition satisfies $\int \tilde{w}(z, x)dz = 1$, i.e. the area under each graph always has unit value.

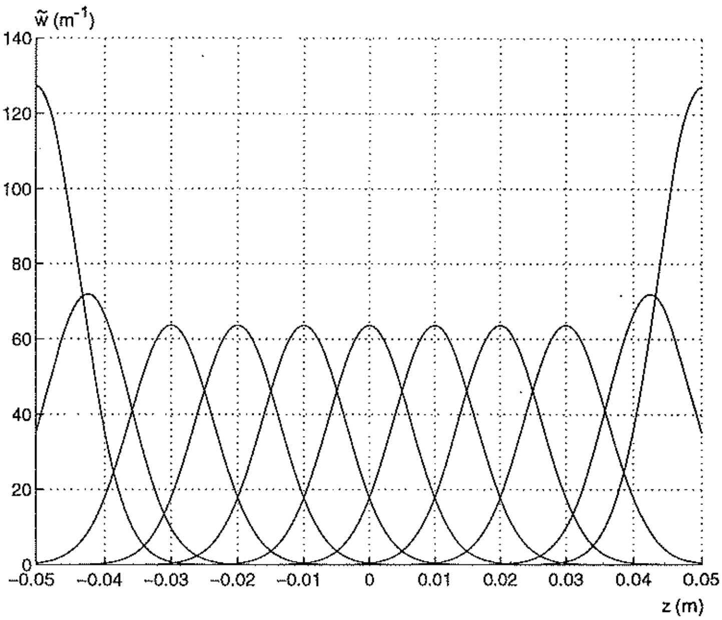

Figure 4.7.1. *The function* $\tilde{w}(z, x) = 1/V(x)exp(-\pi(z-x)^2/\ell^2)$.

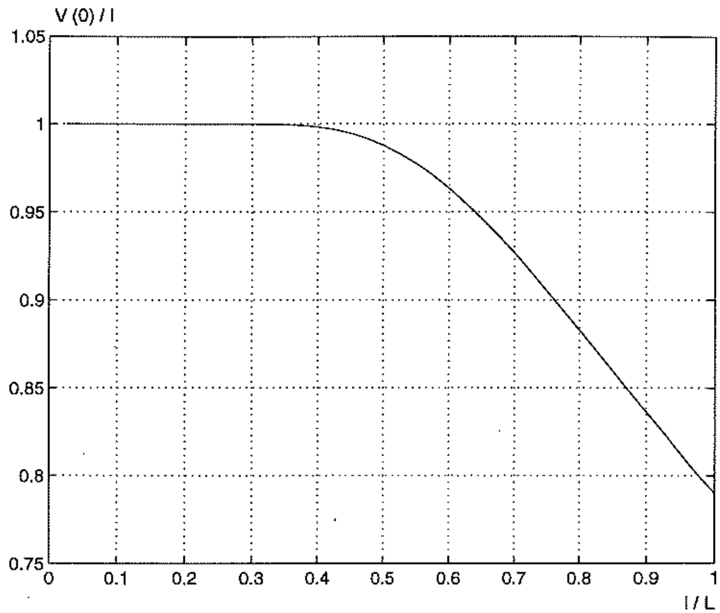

Figure 4.7.2. *Representative volume* $V(0) = \int exp(-\pi z^2/\ell^2)dz$.

As for the attentuation function

$$w = e^{-\frac{k|z-x|}{\ell}}, \qquad (4.7-7)$$

we find that

$$V(x) = \frac{\ell}{k}\{2 - exp(-\frac{k}{\ell}(\frac{L}{\ell} + x)) - exp(\frac{k}{\ell}(\frac{L}{2} - x))\}. \qquad (4.7-8)$$

and hence the choice $k = 2$ complies with (4.7-3). The function $\tilde{w}(z, x)$ is illustrated in Figure 7.4.3 for the same value of ℓ as was used for the corresponding Gaussian parameter, i.e. $\ell = 0.0157$m, whereas $V(0)/\ell$ is shown in Figure 4.7.4 as function of ℓ/L. For $\ell/L < 0.2$ it is seen that $V(0)$ is accureately approximated by (4.7-5)

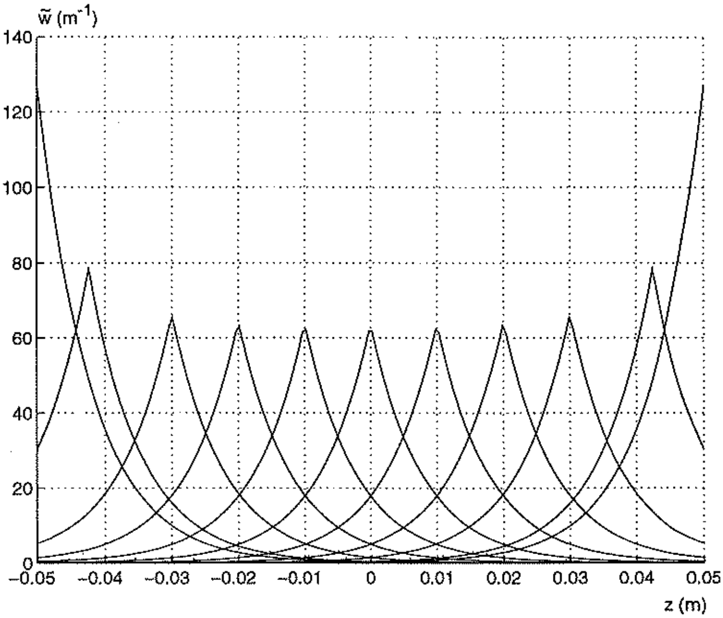

Figure 4.7.3. *The function* $\tilde{w}(z,x) = 1/V(x)exp(-2|z-x|/\ell)$.

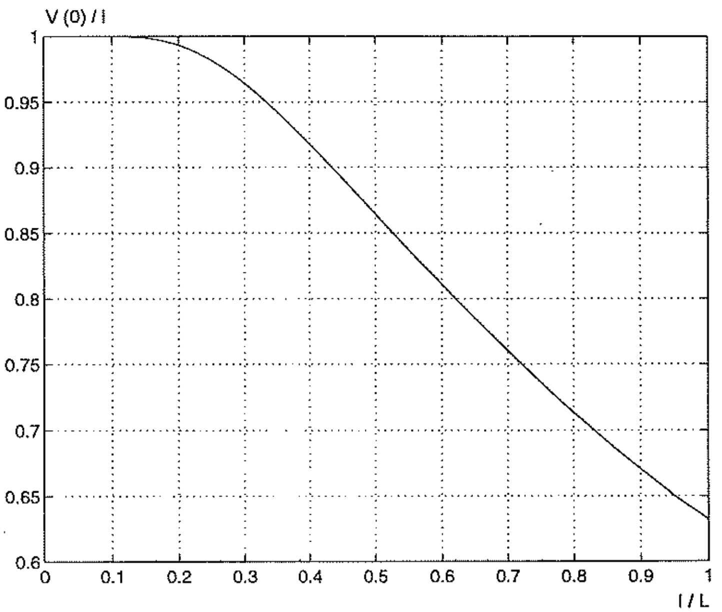

Figure 4.7.4. *Representative volume* $V(0)/\ell = 1 - exp(-L/\ell)$.

After these preliminaries we return to the strain softening bar. The strain hardening modulus is taken as $H = -0.01E$, while the reduced initial yield stress in the middle of the bar is $\sigma_y = 1.4$ MPa.

Assume that strain softening is initiated at a point x_0, such that

$$\dot{\epsilon}^p(x) = \dot{\kappa}(x) = \dot{\lambda}(x) = \dot{B}\delta(x - x_0), \tag{4.7-9}$$

where $\delta(x)$ is the Dirac delta function and where B depends only on time. By choosing $w^p = w^h = w$ the nonlocal plastic strain rate becomes

$$< \dot{\epsilon}^p >= \dot{B}\int_{-L/2}^{L/2} \tilde{w}(z,x)\delta(x - x_0)dz = \dot{B}\tilde{w}(x_0, x), \tag{4.7-10}$$

where $\tilde{w}(z,x)$ is defined by (3.6-19). Since $f = 0$ at x_0, we observe in view of (4.7-9) and (4.7-10) that

$$\dot{\sigma} = H < \dot{\kappa} > (x_0) = H\ \dot{B}\tilde{w}(x_0, x_0) \tag{4.7-11}$$

and hence (assuming that $w(0) = 1$)

$$\dot{B} = V(x_0)\frac{\dot{\sigma}}{H}. \tag{4.7-12}$$

Using $(4.6\text{-}53)_1$, (4.7-11) and (4.7-12) we note that the relationship between strain rate and stress rate becomes

$$\dot{\epsilon} = \dot{\sigma}(\frac{1}{E} + \frac{1}{H}V(x_0)\tilde{w}(x_0, x)), \tag{4.7-13}$$

valid for all $x \in [-L/2,\ L/2]$.

Remark 7.1. Again using that $\tilde{w}(x_0, x_0) = 1/V(x_0)$ we conclude from (4.7-13) that

$$\dot{\sigma} = \frac{H}{E + H}E\ \dot{\epsilon} \tag{4.7-14}$$

during loading. Hence the general nonlocal elastic-plastic response function reduces to

$$\mathcal{K} = \frac{H}{E + H} \tag{4.7-15}$$

in this degenerated case.□

Localization is initiated at the centre of the bar, i.e. at $x_0 = 0$. The width of the localized zone is

$$b = 2\ x_b, \tag{4.7-16}$$

where x_b is a solution to the equation $\dot{\epsilon} = 0$, i.e.

$$V(0)\tilde{w}(0,x) + \frac{H}{E} = 0, \tag{4.7-17}$$

according to (4.7-13). By the choice (4.7-1) of attenuation function, it is seen that (4.7-17) explicitly reads

$$\left.\begin{aligned} &\frac{V(0)}{V(x)} e^{-\dfrac{\pi x^2}{\ell^2}} + \frac{H}{E} = 0, \\ &V(x) = \int_{-L/2}^{L/2} e^{-\dfrac{\pi (z-x)^2}{\ell^2}} dz, \end{aligned}\right\} \tag{4.7-18}$$

in accordance with (3.6-17) and (3.6-19). The solution of (4.7-18) is graphically shown in Figure 4.7.5 for $\ell = 0.0157$ m.

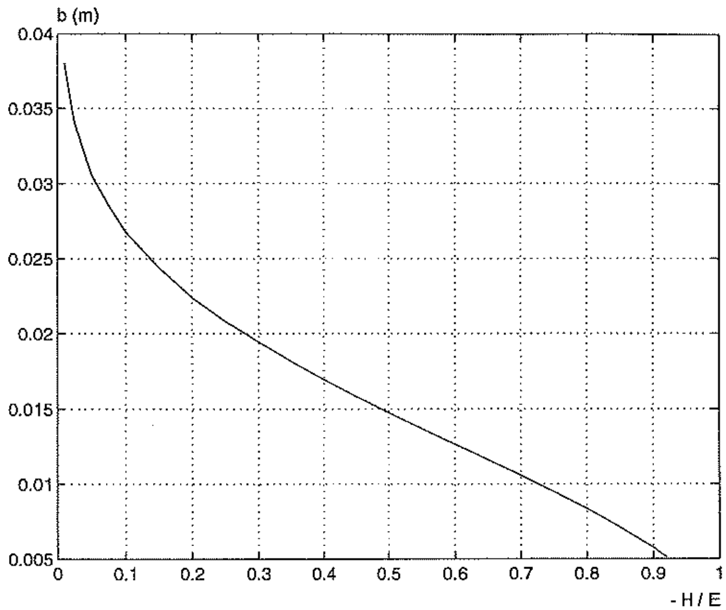

Figure 4.7.5. *Width of localized zone.*

If $\ell/L < 0.3$, then $V(x) = V(0)$ almost everywhere (but for a narrow zone close to the boundary) and hence (4.7-18) in view of (4.7-3) has the closed form solution

$$x_b = \pm\ell(\frac{1}{\pi}ln\frac{E}{-H})^{1/2}. \qquad (4.7-19)$$

Since here $\ell/L = 0.157$, clearly (4.7-19) will be an acceptable solution, as is easily checked by comparison with the graph in Figure 4.7.5. For $-H/E = 0.05$ it is found that $b = 0.0314 = 2\ell$.

Integration of (4.7-13) for $x_0 = 0$ provides us with the total displacement rate of the bar

$$\Delta\dot{u} = \dot{u}(\frac{L}{2}) - \dot{u}(-\frac{L}{2}) =$$

$$= \dot{\sigma}\int_{-L/2}^{L/2}(\frac{1}{E} + \frac{1}{H}V(0)\tilde{w}(0,x))dx = \dot{\sigma}(\frac{L}{E} + \frac{\ell_{ch}}{H}), \qquad (4.7-20)$$

where

$$\ell_{ch} = V(0)\int_{-L/2}^{L/2}\tilde{w}(0,x)dx, \qquad (4.7-21)$$

is a characteristic length which, for a given length of the bar, depends on the character of the attenuation function alone. Since the representative volume $V(x)$ deviates appreciably from $V(0)$ only at the boundary,

$$\int_{-L/2}^{L/2}\tilde{w}(0,x)dx \approx 1 \qquad (4.7-22)$$

for very rapidly decaying attenuation functions ($\ell/L << 1$). For the exponentially decaying functions (4.7-1) and (4.7-7) it turns out that (4.7-22) is in fact accurately satisfied if $\ell/L < 0.3$, as seen from Figure 4.7.6 for the Gaussian distribution function. In that case (4.7-21) may be replaced by

$$\ell_{ch} = V(0). \qquad (4.7-23)$$

and hence, since also (4.7-5) applies (cf. Figure 4.7.2),

$$\ell_{ch} = \ell, \qquad (4.7-24)$$

independent of length of the bar.

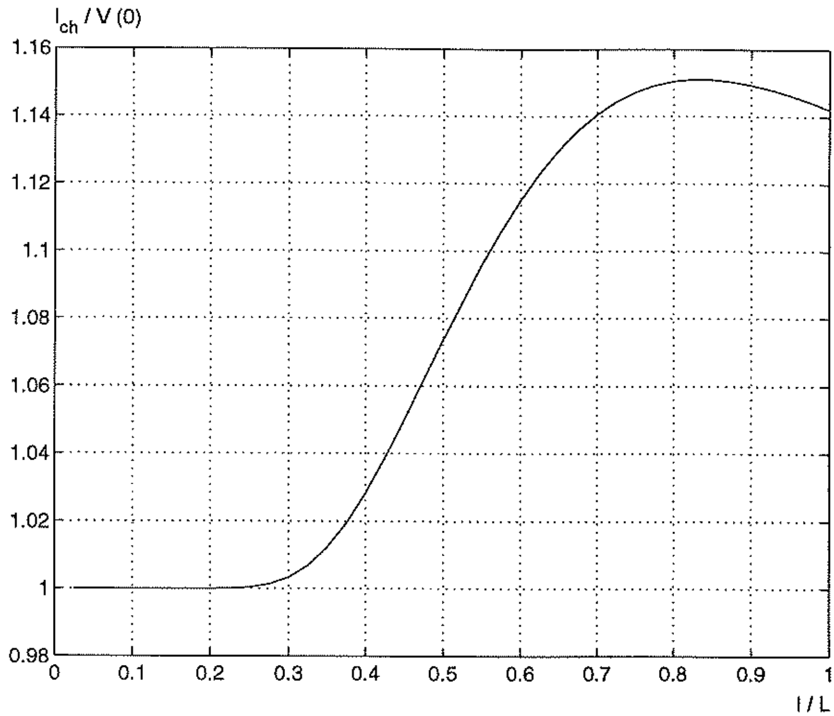

Figure 4.7.6. *Characteristic length ℓ_{ch} for Gaussian distribution function,* $\ell_{ch}/V(0) = \int \tilde{w}(0, z)dz$.

By its construction it is observed that the integral of $\tilde{w}(0, x)$ in fact deviates very little from unit value. hence (4.7-22) is approximately valid even if ℓ/L is close to one.

It has been assumed that localization is initiated at the centre of the bar ($x_0 = 0$). Clearly (4.4-17), (4.7-20) and (4.7-21) remain valid if $V(0)\tilde{w}(0, x)$ is replaced by $V(x_0)\tilde{w}(x_0, x)$. In Figure 4.7.7 the function $\ell_{ch}/V(x_0) = \int \tilde{w}(x_0, x)dx$ is shown for five different values of the quotient ℓ/L. For small values of ℓ/L (very rapidly decaying Gaussian function), we observe that the integral of $\tilde{w}(x_0, x)$ deviates from unity only in small regions close to the boundaries.

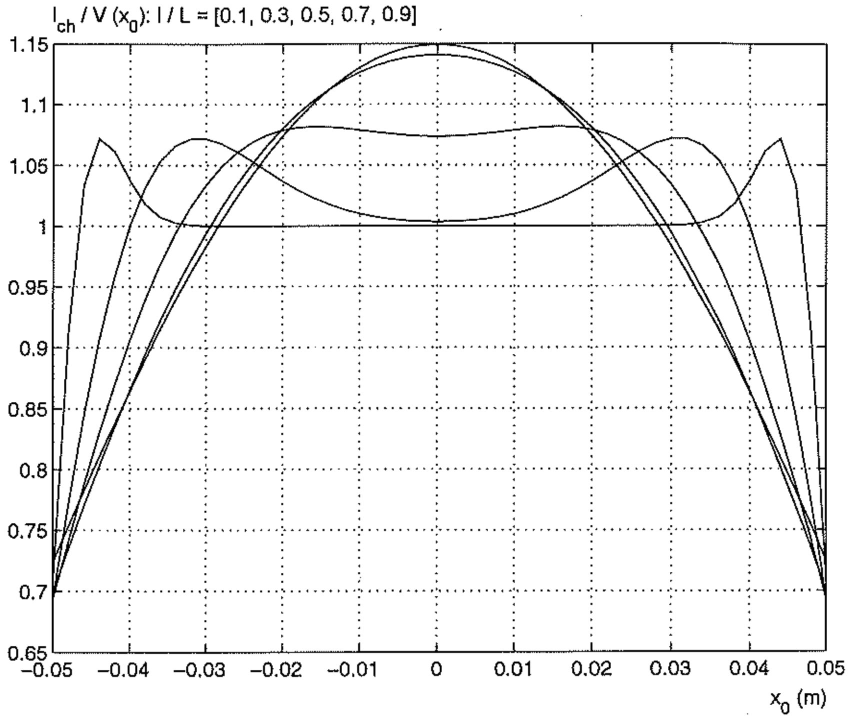

Figure 4.7.7. *The function* $\int \tilde{w}(x_0, x)dx$.

Remark 7.4. We recall that the function $\int \tilde{w}(x_0, x)dx$, graphically illustrated in Figure 4.7.7, appears frequently in Section 3.6, where it is simply denoted by $\beta = \{\bar{1}\}$, cf *Remark 6.1* in Section 3.6.3.□

From (4.7-20) is obtained the stress-displacement relation

$$\sigma = \begin{cases} \dfrac{E}{L}\Delta u, & 0 < \Delta u < \dfrac{L\sigma_y}{E}, \\[2ex] \sigma_y + \dfrac{E}{L(1 + \dfrac{E}{H}\dfrac{\ell_{ch}}{L})}(\Delta u - \dfrac{L\sigma_y}{E}), & \Delta u > \dfrac{L\sigma_y}{E}, \end{cases} \tag{4.7-25}$$

where σ_y is the reduced initial yield stress at the centre of the bar. The stress-displacement relation (4.7-25) is shown in Figure 4.7.8 for $\ell = 0.0157$ m.

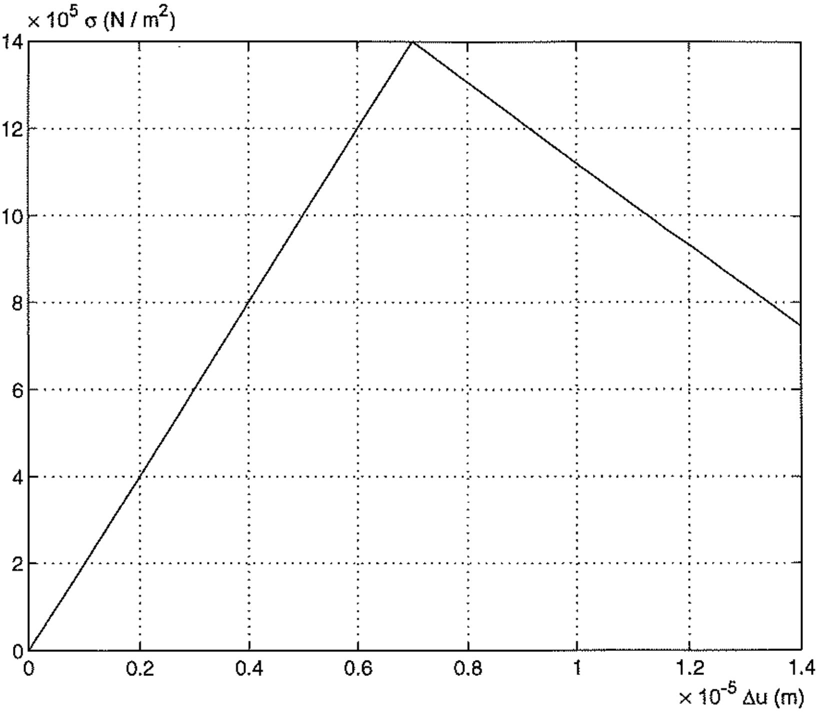

Figure 4.7.8. *Stress versus total displacement* ($\ell/L = 0.157$).

Using (4.7-13) and (4.7-25) the strain distribution can now be determined for different values of total displacement of the bar. The result is shown in Figure 4.7.9 for values of stress corresponding to $\Delta u = (0.8, 1.0, 1.2)\times10^{-5}$ m.

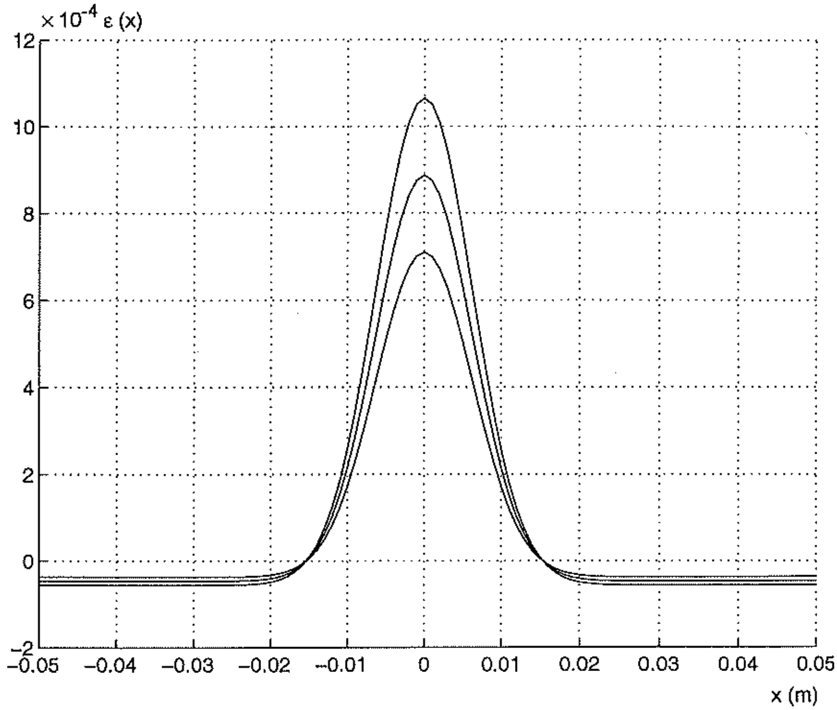

Figure 4.7.9. *Strain distribution along the bar for three different values of total displacement* ($\ell/L = 0.157$).

For $\ell/L = 0.157$ the width of the localized zone has been calculated by solving numerically the integral equation (4.7-18) for different values of $-H/E$. In particular $b = 0.0314$ m ($= 2\ell$) if $-H/E = 0.05$, which is verified by the strain distribution shown in Figure 4.7.9. As expected, the width of the zone remains constant when the loading is increased.

Remark 7.3. The analysis above highlights the difference between permanent deformation and plastic flow in nonlocal plasticity. We recall that the stress point never reaches the yield surface except for the single point at $x = 0$. Yet it appears clearly that permanent deformation remains within the entire localized zone if stress is relaxed to zero. □

As discussed previously, and in view of (4.7-17) it is evident that the width of the localized zone (for given L) depends merely on the quotient H/E and the parameter ℓ. For a specific material and a given choice of attenuation function (say Gaussian), it is therefore the parameter ℓ which determines the width of the zone. In other words ℓ is a constitutive parameter, the value of which is related to the physical properties of the material.

In view of (3.4-10), (3.6-109), (3.6-110) and (3.6-116), the specific dissipation per unit length is determined by

$$\check{D}^p = -\{(\int \frac{\partial \tilde{\psi}}{\partial < \epsilon^p >}(z)\tilde{w}(x,z)dz)\dot{\epsilon}^p + (\int \frac{\partial \hat{\psi}}{\partial < \kappa >}(z)\tilde{w}(x,z)dz)\dot{\kappa}\}, \qquad (4.7-26)$$

where now

$$\rho_0 \tilde{\psi} = \frac{1}{2}E(\epsilon - < \epsilon^p >)^2 + \frac{1}{2}H \ < \kappa >^2 \ , \ E + H > 0. \qquad (4.7-27)$$

Hence, by virtue of (4.6-53),

$$\check{D}^p(x) = \{(\int_{-L/2}^{L/2} \sigma(z)\tilde{w}(x,z)dz)\dot{\epsilon}^p - (\int_{-L/2}^{L/2} H < \kappa > \tilde{w}(x,z))dz)\dot{\kappa}\}$$

$$= (\int_{-L/2}^{L/2} \tilde{w}(x,z)\sigma_y dz)\dot{\lambda}(x), \qquad (4.7-28)$$

since $f = 0$ during loading. Using (4.7-9), (4.7-28) and (3.4-11), it then follows that the total dissipation in the bar is given by

$$\mathcal{D}^p = \sigma_y \dot{B} \int_{-L/2}^{L/2} \int_{-L/2}^{L/2} \tilde{w}(x,z)\delta(x - x_0)dxdz = \sigma_y \dot{B} \int_{-L/2}^{L/2} \tilde{w}(x_0, z)dz. \qquad (4.7\text{- }29)$$

Hence, for $x_0 = 0$ and with the use of (4.7-12),

$$\mathcal{D}^p = \frac{\sigma_y \dot{\sigma}}{H} V(0) \int_{-L/2}^{L/2} \tilde{w}(0,z)dz = \frac{\sigma_y \dot{\sigma}}{H} \ell_{ch}, \qquad (4.7\text{- }30)$$

where (4.7-21) has been used to obtain the second equality.

At failure, the total amount of dissipation at separation is given by $2G_c$, where G_c is the fracture energy per unit area. Hence, since $\dot{\sigma} = -\sigma_y$ at failure,

$$\frac{\sigma_y^2}{-H}\ell_{ch} = 2G_c, \qquad (4.7-31)$$

or

$$\ell_{ch} = \frac{-H}{E}\lambda_c, \qquad (4.7-32)$$

where

$$\lambda_c = \frac{2G_c E}{\sigma_y^2} \qquad (4.7-33)$$

is the characteristic length of the material (see Gustafsson 1985)[4.7-1]. If (4.7-24) is valid $\ell = \ell_{ch}$, if not we must solve the integral equation (4.7-21) (for a given length of the specimen) to obtain the value of the parameter ℓ corresponding to the given value of $\ell_{ch} = -H/E\lambda_c$.

4.7.2 Numerical examples

In Section 4.6.4 the nonlocal generalized Euler procedure was demonstrated for two types of yield functions of von Mises type. Here the integration technique will be applied to the problem of the strain softening bar, solved previously by analytical methods for the case when the yield function varied linearly with stress.

Only forward Euler technique has been employed. The final system of equations to be solved is given by (4.3-36) (or equivalently by (4.3-38)).

The linear yield function is constituted by (4.6-55), for which the explicit form of (4.3-36) is given by (4.6-60), whereas for the nonlinear yield function defined by (4.6-64)$_3$, the corresponding system of equations to be solved is obtained by substituting (4.6-73) into (4.6-36).

The quasi-local continuous tangential stiffness matrix defined by (4.4-8) is used for the elastic-plastic stiffness matrix in the Newton-Raphson scheme (4.3-12)-(4.3-14).

With regard to the constitutive assumption (4.6-53)$_1$ we observe that (3.5-22) applies. Upon replacing $\mathbf{R}$ by $\partial f/\partial\sigma$ we thus obtain[4.7-2]

$$\check{\mathcal{K}} = 1 - \pi\beta^p E\left(\frac{\partial f}{\partial \sigma}\right)^2, \; E = 2\mu, \qquad (4.7-34)$$

where we have also used (3.6-2), (3.6-41) and (3.6-42)$_2$.

[4.7-1]Since (4.7-24) is valid for the bar under consideration ($-H/E = 0.05$, $\ell/L = 0.157$), it follows from (4.7-32) that $\lambda_c \approx 0.3$ m. For mortar λ_c is typically 0.25 m, for concrete 0.2 m – 2.0 m and for wood 0.02 m.

[4.7-2]The function β^p in the right-hand side of (4.7-34) is defined by (3.6-25), and shown graphically in Figure 4.7.7 for an attenuation function of Gaussian type.

For linear g we thus obtain

$$\left.\begin{aligned} \check{\mathcal{K}} &= 1 - \pi\beta^p E, \\ \frac{1}{\pi} &= \beta^p E + \beta^h H, \end{aligned}\right\} \tag{4.7 − 35}$$

where (3.6-52) and (3.6-54) has been used to evaluate the function π. If $w^\beta = w^h = w$, (4.7-35) is replaced by

$$\check{\mathcal{K}} = 1 - \frac{E}{E+H} = \frac{H}{E+H}. \tag{4.7 − 36}$$

Hence the quasi-local continuous tangential stiffness matrix may be written in the form

$$\left.\begin{aligned} &\check{\mathcal{K}}\mathcal{L} = E(1 - \pi\beta\ E), \\ &\pi\beta = \begin{cases} \dfrac{1}{E+H} & \text{if } g = 0,\ \hat{g} > 0,\ (\dot{\lambda} > 0), \\ 0 & \text{otherwise},\ (\dot{\lambda} = 0). \end{cases} \end{aligned}\right\} \tag{4.7 − 37}$$

Similarly, for nonlinear g,

$$\left.\begin{aligned} \check{\mathcal{K}} &= 1 - 4\beta^p\pi\sigma^2, \\ \frac{1}{\pi} &= 4E\overline{\{\sigma\}}_p\sigma + \beta^h r, \end{aligned}\right\} \tag{4.7 − 38}$$

while the correspondence of (4.7-36) reads

$$\left.\begin{aligned} &\check{\mathcal{K}}\mathcal{L} = E(1 - \pi 4\beta^p\ \sigma^2), \\ &\pi = \begin{cases} \dfrac{1}{4E\overline{\{\sigma\}}_p\sigma + \beta^h r} & \text{if } \dot{\lambda} > 0, \\ 0 & \text{if } \dot{\lambda} = 0. \end{cases} \end{aligned}\right\} \tag{4.7 − 39}$$

Remark 7.4. If we disregard the boundary effects induced by the attenuation functions ($\beta^p = \beta^h \approx 1$), β^p and β^h may be omitted in (4.7-38) and $\overline{\{\sigma\}}_p$ replaced by $\overline{[\sigma]}_p$.□

From (3.6-48) we obtain finally the functional stress rate,

$$\frac{1}{\rho_0}\mathcal{H}_s = E(\beta^p\dot{\epsilon}^p - <\dot{\epsilon}^p>), \tag{4.7 − 40}$$

appearing in the integrand of the external psuedo forces defined by (4.3-13).

The Gaussian distribution function (4.7-1) has been used for two different values of the quotient ℓ/L (= 0.157 and 0.134, respectively).

The value of the constant function r (corresponding to the strain hardening modulus) has been taken as $r = -1.6 \cdot 10^{22}$ $(\mathrm{N/m^2})^3$ and the reduced initial yield stress at the centre of the bar as $\sigma_y = 1.7$ MPa.

Loading is effected by successively increasing the displacement at one end of the bar, the other end kept fixed. The result of the computations is demonstrated graphically; in Figures 4.7.10-11 with regard to the linear yield function and in Figures 4.7.12-14 with regard to the nonlinear one[4.7-3].

Figure 4.7.10 shows stress versus total displacement of the bar for $\ell/L = 0.157$ and $\ell/L = 0.314$. The number of elements in the finite element discretization is $n = 100$. Figures 4.7.11 show the corresponding strain distribution for increasing values of total displacement Δu of the bar. Convergence is very accurate.

Discretization with $n = 200$ produces curves impossible to distinguish from those presented here. In addition, for $\ell/L = 0.157$ the result is in perfect agreement with the analytical solution (see Figures 4.7.8-9).

Figure 4.7.12 shows the stress-displacement curves when the yield function depends quadratically on stress, again for the two values $\ell/L = 0.157$ and $\ell/L = 0.314$, while the corresponding strain distribution is illustrated in Figure 4.7.13. For $\ell/L = 0.157$ the width of the localized zone does not seem to differ appreciably from that obtained for the linear yield function, i.e. $b \approx 2\ell$. For $\ell/L = 0.314$, however, we observe that the width of the localized zone is less than 2ℓ.

Figure 4.7.14 demonstrates the excellent convergence properties of the finite element solutions, clearly being objective with respect to the mesh.

[4.7-3]The computational work has been performed by L. Strömberg, Division of Solid Mechanics at Lund University.

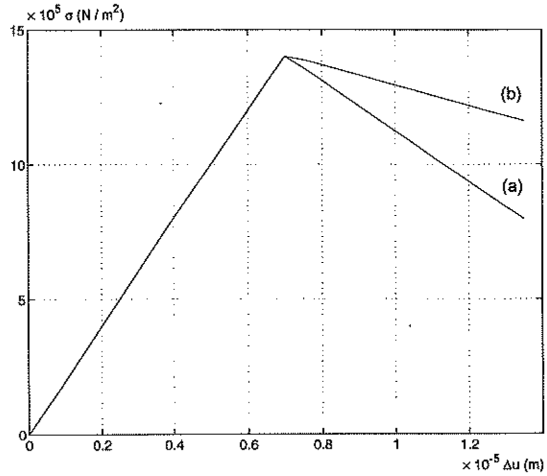

Figure 4.7.10. *Linear yield function. Stress vs total displacement of the bar* $(n = 100)$*: (a)* $\ell/L = 0.517$*; (b)* $\ell/L = 0.314$.

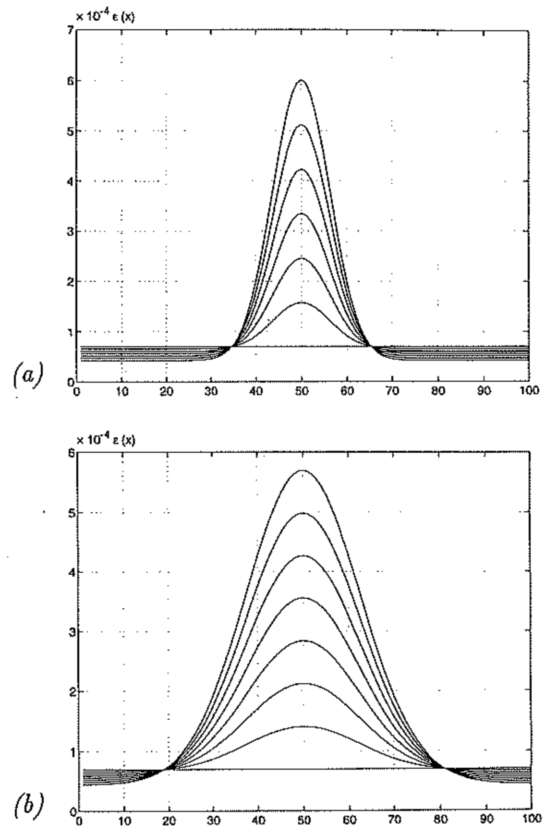

Figure 4.7.11. *Linear yield function. Strain distribution along the bar* $(n = 100)$*:* *(a)* $\ell/L = 0.157$*; (b)* $\ell/L = 0.314$.

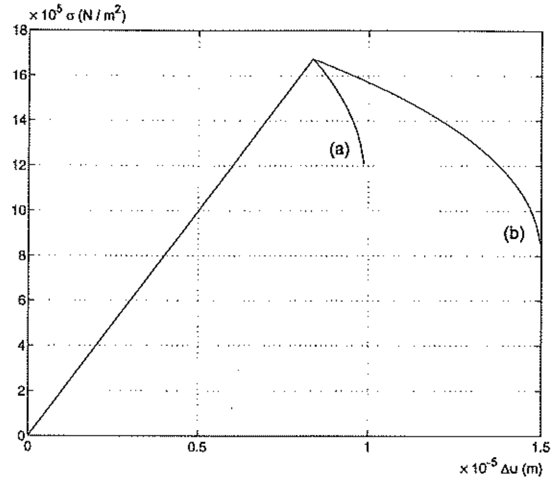

Figure 4.7.12. *Nonlinear yield function. Stress vs. total displacement of the bar* $(n = 100)$*: (a)* $\ell/L = 0.157$*; (b)* $\ell/L = 0.314$.

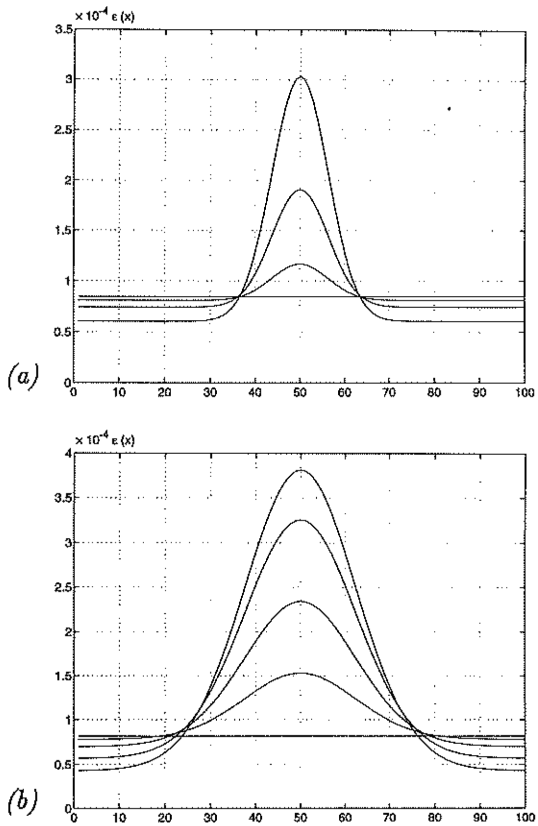

Figure 4.7.13. *Nonlinear yield function. Strain distribution along the bar* $(n = 100)$*:* *(a)* $\ell/L = 0.157$*; (b)* $\ell/L = 0.134$.

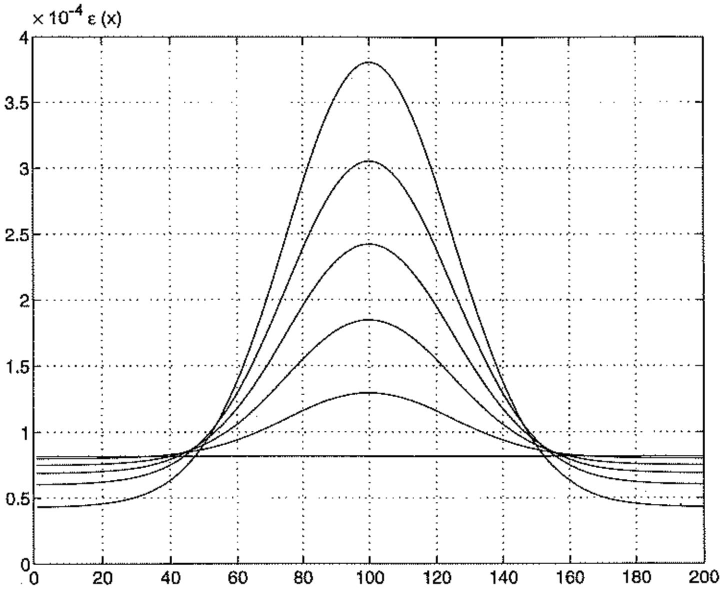

Figure 4.7.14 *Nonlinear yield function. Strain distribution along the bar* $(n = 200)$: $\ell/L = 0.134$.

4.8 Concluding remarks

Localization in a strain softening solid is associated with material instability and loss of ellipticity in quasi-static problems. From a computational point of view this appears to be a crucial problem due to numerical instabilities and mesh sensitivity. Conventional continuum models have essential deficiencies, if used to describe strain softening behaviour. One main drawback is that the amount of dissipation at failure, nonphysically is predicted to vanish, another that finite element solutions become nonobjective with respect to the mesh size.

Only results[4.8−1] from one dimensional analysis are available, but nethertheless some important conclusions may be drawn.

It is claimed that nonlocal plasticity models are well adapted to describe the essential features of strain softening, including that of localization. The nonlocal approach provides in a natural way for the introduction of a constitutive characteristic length, being that material parameter which mainly controls the development of the localized zone. It is also important to note that the problem of achieving computational objectivity seems to be inherently solved by the nonlocal concept.

[4.8−1] In agreement with those by e.g. Bazant and Feng-Bao Lin (1988) and Belytschko and Lasry (1989).

4.8.1 Discussion and conclusions

Various numerical techniques commonly used in local plasticity have been extended to comply with the nonlocal concept. The crucial problem is that of satisfying the consistency condition, which in nonlocal plasticity corresponds to an integral equation defined throughout the region of loading points. As a consequence, the integration of the rate equations must be performed simultaneously for a set of interacting loading points.

The quasi-local integration technique in Section 4.6.5 may be regarded as an uncorrected forward Euler procedure, which will probably lead to a violation of the yield criterion. A safe integration technique cannot avoid the problem of simultaneously solving a coupled system of equations to comply with the consistency condition.

At present the numerical experience is not comprehensive enough to provide for accurate predictions of the reliability of the one integration technique or the other. It is conjectured however, that the nonlocal backward Euler procedure outlined in Section 4.6.3 will converge very fast, possibly providing accurate results after only the first iteration, and hence there will be no need for repeatedly updating the matrix $\bar{\mathbf{Q}}$, which else appears to be a crucial problem.

Equilibrium iterations should be performed by employing the general Newton-Raphson scheme (4.3-12)-(4.3-14). It is believed that the convergence rate will not change considerably if ${}_{n+1}^{i-1}\tilde{\mathbf{F}}$ is replaced by the actual external forces ${}_{n+1}^{i-1}\mathbf{F}$ alone, the contribution from the external pseudo forces being neglected.

The quasi-local tangential stiffness tensor $\check{\mathcal{K}}\mathcal{L}$ should be used for the elastic-plastic stiffness matrix in the Newton-Raphson algorithm, evaluated either incrementally as in Section 4.4.1 or, as in Section 4.4.2, by using the consistency condition to solve for the function π in an average sense.

The analytical solution to the problem of the strain softening bar implies that the width of the localized zone depends on the length of the bar, although this size effect is negligible if $\ell/L < 0.3$, ℓ being the non-local parameter and L the length of the bar. For such values of the quotient ℓ/L, the width of the localized zone is entirely determined by the nonlocal constitutive parameter ℓ, Young's modulus and the strain hardening modulus, which is in agreement with the results from the finite element analysis.

The analytical solution predicts a finite amount of total dissipation, although plastic loading is confined to a region of vanishing size at the centre of the bar. Comparison with the total separation work at failure provides a relation between the nonlocal parameter ℓ and the characteristic length λ_c of the material (in general expressed by an integral equation for ℓ).

4.8.2 Future developments

The numerical strategies and algorithms proposed in Chapter 4 have merely been treated tentatively, and only results from one-dimensional analysis are available. Full-scale models based on these numerical procedures may certainly be applied to general two- or three-dimensional structures. However, the next step in the future development should involve a systematic evaluation of the nonlocal generalized Euler procedure as well as the quasi-local integration technique. In addition the convergence properties of the Newton-Raphson algorithm should be investigated for different choices of the elastic-plastic stiffness matrix, by analysing simple one- or two-dimensional equilibrium problems.

We recall that the nonlocal theory is derived for finitely deforming elastic-plastic bodies, and accordingly the finite element formulation should be extended to be capable of handling large deformations.

Ever since the pioneering work of Hill (1958) on uniqueness and stability in elastic-plastic solids and of Rudnicki and Rice (1975) and Rice (1976) on localization of plastic deformation, considerable interest has been focused on theories which treat localization as bifurcation from a state of homogeneous deformation. Investigating the possibility of finding analytical expressions for critical bifurcation directions in nonlocal strain softening solids would certainly be a challenging issue for future development.

The occurence of elastic-plastic coupling is an interesting feature of nonlocal plasticity. However, elastic-plastic coupling in nonlocal plasticity cannot alone explain the striking effect of material stiffness degradation (damage) for a material such as concrete, especially not in the softening region where the effect is even more pronounced. The idea of combining theories of damage and plasticity seems logical and is used by many researchers in the field of fracture mechanics, but a general nonlocal plasticity - damage theory has not yet been derived, and is thus another issue for future development.

Appendix A

Equations of balance in nonlocal continuum mechanics

This appendix contains a derivation of the equations of balance for nonlocal continua which leads to (2.3-1)-(2.3-4). Nonlocal constitutive theory is not treated here - we refer to Section 2.4 for a brief survey of the subject with respect to nonlocal elastic-plastic continua. As mentioned in Section 1.3 the formulation presented here is based on works of Edelen and Laws (1971) and Edelen (1976). For a detailed account of background and history we refer to Edelen (1976), where an extensive list of references is provided.

As stated in Section 1.3 the principle of local action is not valid in nonlocal continuum theories, [A.0-1] that is long-range interactions between a particle at place $\mathbf{X}$ and a particle at place $\mathbf{Z}$ contribute to the stress at $\mathbf{X}$. In linearized theory this may be exemplified with a nonlocal Hooke's law of the form

$$\sigma_{ij}(\mathbf{X}) = C_{ijkl}(\mathbf{X})\ e_{kl}(\mathbf{X}) + \int c_{ijkl}(\mathbf{X},\mathbf{Z})\ e_{kl}(\mathbf{Z})\ dV(\mathbf{Z}),$$

where the integration is extended all over the body.

Equations of balance will be postulated for the entire body (global relations), and will in general not be valid for arbitrary parts of the body (as is the case in local theory). Relations valid only for the body as a whole are called *nonlocal* and relations valid for arbitrary parts of it are called *local*.

A.1 Conservation of mass

Consider a body which occupies a region B of three-dimensional Euclidean point space at an initial time $t = 0$. We identify the body with the region and refer to the body

[A.0-1]In our terminology nonlocal continuum mechanics comprises theories which admit long-range (nonlocal) interactions. Accordingly, such categories of generalized continua as micropolar media or materials with gradient effects are not nonlocal.

itself as B. A material point (particle) is identified with its position X in the Euclidean point space at initial time. Hence the initial configuration of the body is taken as the reference configuration.

The motion of the material point is defined by

$$\mathbf{x} = \boldsymbol{\chi}(\mathbf{X}, t), \ \mathbf{X} = \boldsymbol{\chi}^{-1}(\mathbf{x}, t), \tag{A - 1}$$

where $\boldsymbol{\chi}$ is assumed to be a continuous function of its arguments and differentiable as much as desired. We interpret (A-1) as usual, that is $\mathbf{x}$ is the position of the material point with position $\mathbf{X}$ in the initial configuration. The deformation gradient associated with the motion is defined by

$$\mathbf{F} = \text{ Grad } \mathbf{x} = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial \mathbf{X}} \ , \tag{A - 2}$$

and it is assumed that the Jacobian

$$J = \det \ \mathbf{F} \tag{A - 3}$$

is strictly positive for all t. The velocity is defined by

$$\dot{\mathbf{x}}(\mathbf{X}, t) = \frac{\partial \boldsymbol{\chi}(\mathbf{X}, t)}{\partial t} \ , \tag{A - 4}$$

and the velocity field is given by

$$\dot{\mathbf{x}}(\mathbf{x}, t) = \dot{\mathbf{x}}(\boldsymbol{\chi}^{-1}(\mathbf{x}, t), t). \tag{A - 5}$$

The boundary of B is denoted ∂B, and $B(t)$ and $\partial B(t)$ denote the images of B and ∂B, respectively, under the motion given by (A-1).

Let $\rho_0 = \rho_0(\mathbf{X})$ denote the mass density in the initial configuration of the body B and correspondingly $\rho = \rho(\mathbf{x}, t)$ the mass density at time t in the configuration of the body specified by the motion (A-1). Global conservation of mass implies that

$$\frac{d}{dt} \int_{B(t)} \rho \ dV = 0 \tag{A - 6}$$

at any time t, while local conservation of mass implies that

$$\frac{d}{dt} \int_{P(t)} \rho \ dV = 0 \tag{A - 7}$$

at any time t for arbitrary parts P of the body B. We recall from local theory that (A-7) is equivalent to the statements [A.1-1]

$$\frac{\partial}{\partial t}(\rho J) = 0, \quad \rho J = \rho_0, \tag{A-8}$$

and

$$\dot{\rho} + \rho \text{div}\ \dot{\mathbf{x}} = 0, \tag{A-9}$$

representing material and spatial forms, respectively, of local conservation of mass. We emphasize that (A-7) is a local statement, although the equation involves an integral (in accordance with our previous discussion).

If we reject the assumption (A-7), then the global condition (A-6) is a nonlocal statement. To see what conclusions can now be drawn from the global statement of conservation of mass, we write (A-6) in the equivalent form

$$\int_B \frac{\partial}{\partial t}(\rho J) dV = 0, \tag{A-10}$$

where the integration in the original formulation has been converted to an integration over the initial configuration. Since the integration in (A-10) is over a fixed region the result given in (A-8) can no longer be deduced. However, we may replace the single nonlocal statement given by (A-10) with a set of two statements, one of those being local and one nonlocal, of the form

$$\frac{\partial}{\partial t}(\rho J) = \hat{\rho} J, \quad \int_B \hat{\rho} J\ dV = 0, \tag{A-11}$$

or equivalently, in spatial form

$$\dot{\rho} + \rho\ \text{div}\ \dot{\mathbf{x}} = \hat{\rho}, \quad \int_{B(t)} \hat{\rho} dv = 0. \tag{A-12}$$

The quantity $\hat{\rho}$ introduced in (A-11) will be referred to as the *localization residual* for mass. In order to get some physical interpretation of $\hat{\rho}$, we integrate $(\text{A-12})_1$ over an arbitrary part $P(t)$ of $B(t)$ and obtain the local statement

$$\int_{P(t)} (\dot{\rho} + \rho\ \text{div}\ \dot{\mathbf{x}}) dv = \int_{P(t)} \hat{\rho} dv, \tag{A-13}$$

[A.1-1] Converting the integration in (A-7) to an integration over the initial configuration leads to (A-8) and using the formula for differentiating a determinant leads to (A-9). Note that ρ in (A-8) designates a function of $\mathbf{X}$ and t, *i.e.* $\rho(\mathbf{x}, t) = \rho[\chi(\mathbf{X}, t), t] = \rho(\mathbf{X}, t)$ where we, with abuse of notations, have retained the same function symbol for the density.

or equivalently

$$\frac{d}{dt}\int_{P(t)}\rho\, dv=\int_{P(t)}\hat{\rho} dv=-\int_{B(t)-P(t)}\hat{\rho}\, dv, \qquad (A-14)$$

where we have used $(A\text{-}12)_2$ to derive the second equality. In view of $(A\text{-}14)_2$ we may interpret the localization residual $\hat{\rho}$ as the rate of production of mass of a 'particle' at place $\mathbf{x}$ due to the presence of the rest of the body. Since $\hat{\rho}$ has zero mean on $B(t)$ there is no net generation of mass for the body as a whole (in accordance with (A-6)). Hence, if there are regions within the body with production of mass, then there also must be regions with compensating destruction of mass.

Evidently the local relations (A-12)-(A-14) are not statements about conservation of mass in a usual sense, since $(A\text{-}12)_1$ must hold for *every* choice of $\hat{\rho}$ with zero mean on $B(t)$. Rather we may consider $(A\text{-}12)_1$ as an equivalence class of local statements for conservation of mass with respect to each integrable function $\hat{\rho}$, which satisfies $(A\text{-}12)_2$.

It is a matter of fact that mass usually is locally conserved in continuum theories, that is each part of a given body satisfies (A-7) and accordingly the localization residual $\hat{\rho}$ vanishes identically (*locally mass closed system*). This means that e.g. cavitating fluids are excluded and so are cavitation phenomena in various micromechanical continuum models of damaging materials. Although it is not difficult to treat the general case with $\hat{\rho}\neq 0$, we choose for the sake of simplicity to consider only locally mass closed systems in what follows.

A.2 Balance of linear momentum

The global statement of balance of linear momentum reads

$$\frac{d}{dt}\int_{B(t)}\rho\dot{\mathbf{x}}\, dv=\int_{B(t)}\rho\, \mathbf{f}\, dv+\int_{\partial B(t)}\mathbf{t}\, da, \qquad (A-15)$$

where $\mathbf{f}=\mathbf{f}(\mathbf{x},t)$ is the external body force field per unit mass (specific body force) and $\mathbf{t}=\mathbf{t}(\mathbf{x},t)|_{\mathbf{x}\,\epsilon\,\partial\, B(t)}$ is the surface traction per unit area applied on the boundary of the body at time t.

In classical local theory it is assumed that (A-15) also holds for an arbitrary part $P(t)$ of the body, where then $\mathbf{t}=\mathbf{t}(\mathbf{x},t)|_{\mathbf{x}\,\epsilon\,\partial\, P(t)}$ represents the traction vector on the boundary surface of the arbitrary part $P(t)$. Hence $\mathbf{t}=\mathbf{t}(\mathbf{x},t)$ is defined throughout $B(t)$ for all t. This assumption leads to the existence of the Cauchy stress tensor with properties providing for local balance of linear momentum.

In nonlocal theory we must start with the global statement (A-15), where $\mathbf{t}$ is defined only on the actual boundary $\partial B(t)$ of $B(t)$. We assume that $\partial B(t)$ constitutes a regular surface with a unit normal vector field $\mathbf{n} = \mathbf{n}(\mathbf{x},t)|_{\mathbf{x}\,\epsilon\,\partial\,B(t)}$, which is taken to be oriented from the interior to the exterior of $B(t)$. Clearly there exists a tensor field defined on $\partial B(t)$ such that its transvection with $\mathbf{n}$ gives the surface traction $\mathbf{t}$. What we need, however, is a tensor field defined for all $\mathbf{x}$ (and for all t) throughout $B(t)$ with such properties that it provides us with the possibility of converting the surface integral in (A-15) into a volume integral over $B(t)$. To comply with this, we therefore assume the existence of a tensor field $\mathbf{T}(\mathbf{x},t)$ defined on all of $B(t)$ such that

$$\mathbf{T}(\mathbf{x},t)|_{\mathbf{x}\,\epsilon\,\partial\,B(t)}\mathbf{n} = \mathbf{t}(\mathbf{x},t)|_{\mathbf{x}\,\epsilon\,\partial\,B(t)}. \qquad (\mathrm{A}-16)$$

Since $\mathbf{t}$ is only defined on $\partial B(t)$, it is obvious that this tensor field cannot be unique. Irrespective of this fact we refer to every tensor field $\mathbf{T}$, which satisfies (A-16) as a Cauchy stress tensor.

Using (A-16) and imploying the divergence theorem, we can write (A-15) in the form

$$\int_{B(t)} (\rho\ddot{\mathbf{x}} - \rho\,\mathbf{f} - \text{ div }\mathbf{T})dv = \mathbf{0}, \qquad (\mathrm{A}-17)$$

where it has been assumed that the body is locally mass closed [A.2−1]. The statement (A-17) can be localized by setting

$$\rho\ddot{\mathbf{x}} - \rho\,\mathbf{f} - \text{ div }\mathbf{T} = \rho\hat{\mathbf{f}} \qquad (\mathrm{A}-18)$$

for every localization residual $\hat{\mathbf{f}}$ which satisfies

$$\int_{B(t)} \rho\,\hat{\mathbf{f}}\,dv = \mathbf{0} \qquad (\mathrm{A}-19)$$

Integration of (A-18) over an arbitrary part $P(t)$ of $B(t)$ gives

$$\frac{d}{dt}\int_{P(t)} \rho\,\dot{\mathbf{x}}\,dv - \int_{P(t)} (\rho\,\mathbf{f} + \text{ div }\mathbf{T})dv = \int_{P(t)} \rho\,\hat{\mathbf{f}}\,dv = -\int_{B(t)-P(t)} \rho\,\hat{\mathbf{f}}\,dv, \quad (\mathrm{A}-20)$$

where (A-19) has been used to establish the second equality. The formulation $(\text{A-20})_2$ provides us with an interpretation of the localization residual field $\hat{\mathbf{f}}$ as a specific body force field acting upon a particle at place $\mathbf{x}$ because of the presence of the rest of the

[A.2−1] Recall that $d/dt(\int \psi\ \rho dv) = \int \dot{\psi}\rho dv$ for locally mass closed systems, where the time dependent function ψ may be a scalar, vector or tensor of any order.

body. We emphasize that the collection of vectors $\hat{\mathbf{f}}$ represents an *internal* body force field (e.g. mutual gravitational forces), while it is recalled that $\mathbf{f}$ represents an external body force field arising from sources outside the body.

A.3 Balance of rotational momentum

In the absence of external couple densities, the global equation of balance of rotational momentum is given by [A.3-1]

$$\frac{d}{dt}\int_{B(t)}\{\mathbf{x}\wedge\rho\dot{\mathbf{x}}dv = \int_{B(t)}\mathbf{x}\wedge\rho\mathbf{f}\ dv + \int_{\partial B(t)}\mathbf{x}\wedge\mathbf{t}\ da, \qquad (\mathrm{A}-21)$$

where the origin of the position vector $\mathbf{x}$ may be arbitrarily chosen, but such that it is stationary in the referential configuration (being an inertial frame). Using (A-16) to convert the surface integral into a volume integral, we conclude that

$$\int_{B(t)}\{\mathbf{x}\wedge\rho\ddot{\mathbf{x}} - \mathbf{x}\wedge\rho\mathbf{f} - \ \mathrm{div}\ (\mathbf{x}\wedge\mathbf{T})\} = \mathbf{0}, \qquad (\mathrm{A}-22)$$

where we have also taken advantage of the fact that the system is locally mass closed.

Localization of (A-22) gives

$$\mathbf{x}\wedge\rho\ddot{\mathbf{x}} - \mathbf{x}\wedge\rho\mathbf{f} - \ \mathrm{div}\ (\mathbf{x}\wedge\mathbf{T}) = \rho\hat{\mathbf{M}}, \qquad (\mathrm{A}-23)$$

where

$$\int_{B(t)}\rho\hat{\mathbf{M}}dv = \mathbf{0}. \qquad (\mathrm{A}-24)$$

It is a routine matter to prove the identity

$$\mathrm{div}\ (\mathbf{x}\wedge\mathbf{T}) = \mathbf{T}^{\mathrm{T}} - \mathbf{T} + \mathbf{x}\wedge\ \mathrm{div}\ \mathbf{T}. \qquad (\mathrm{A}-25)$$

Using (A-18) and (A-25), the relation (A-23) can be written in the form

$$\mathbf{x}\wedge\rho\hat{\mathbf{f}} + \mathbf{T} - \mathbf{T}^{\mathrm{T}} = \rho\hat{\mathbf{M}}. \qquad (\mathrm{A}-26)$$

Integration of (A-23) over an arbitrary part $P(t)$ of $B(t)$ yields

[A.3-1] The skew symmetric tensor $\mathbf{a}\wedge\mathbf{b} = \mathbf{a}\otimes\mathbf{b} - \mathbf{b}\otimes\mathbf{a}$ is the wedge product or exterior product of the vectors $\mathbf{a}$ and $\mathbf{b}$.

$$\frac{d}{dt}\int_{P(t)} \mathbf{x}\wedge\rho\dot{\mathbf{x}}dv - \int_{P(t)}\{\mathbf{x}\wedge\rho\mathbf{f} + \text{ div } (\mathbf{x}\wedge\mathbf{T})\}dv$$

$$= \int_{P(t)} \rho\hat{\mathbf{M}}\; dv = -\int_{B(t)-P(t)} \rho\hat{\mathbf{M}}\; dv, \qquad (A-27)$$

where we again have used the assumption of local conservation of mass, and in addition (A-24) to obtain the second equality. In view of $(A\text{-}27)_2$ we interpret the localization residual $\hat{\mathbf{M}}$ as an internal torque tensor which acts upon a particle at place $\mathbf{x}$ because of the rest of the body.

A.4 Balance of energy

In order to make a global statement of balance of energy, a set of nonmechanical quantities must be introduced. To this end, let $r(\mathbf{x},t)$ denote the rate at which nonmechanical energy is supplied to the body per unit mass (specific energy supply) and let $q(\mathbf{x},t)|_{\mathbf{x}\,\epsilon\,\partial\,B(t)}$ denote the rate at which nonmechanical energy is supplied to the body at its boundary per unit area. These supplies arise entirely from sources external to the body. Finally, let $e(\mathbf{x},t)$ denote such energy density per unit mass that $\int_{B(t)} \rho e\; dv$ is the total internal energy of the body. The global equation of balance of energy then reads

$$\frac{d}{dt}\int_{B(t)} \rho(\frac{1}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}} + e)dv = \int_{B(t)} \rho\mathbf{f}\cdot\dot{\mathbf{x}}\; dv + \int_{\partial B(t)} \mathbf{t}\cdot\dot{\mathbf{x}}\; da + \int_{B(t)} \rho r\; dv$$

$$+ \int_{\partial B(t)} q\; da. \qquad (A-28)$$

With arguments similar to those used when the Cauchy stress tensor was introduced (cf. (A-16)), we introduce a *heat flux vector* $\mathbf{q}(\mathbf{x},t)$ defined for all $\mathbf{x}$ in $B(t)$ such that

$$\mathbf{q}(\mathbf{x},t)|_{\mathbf{x}\,\epsilon\,\partial B(t)}\cdot\mathbf{n} = q(\mathbf{x},t)|_{\mathbf{x}\,\epsilon\,\partial B(t)}. \qquad (A-29)$$

Since $\mathbf{n}$ is the outward normal vector, we may interpret $\mathbf{q}(\mathbf{x},t)|_{\mathbf{x}\,\epsilon\,\partial B(t)}$ as the flux of nonmechanical energy that comes out of the body.

Proceeding as in the previous sections, we use the divergence theorem to rewrite (A-28) in the form

$$\int_{B(t)}\{\rho(\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}} + \dot{e}) - \rho(\mathbf{f}\cdot\dot{\mathbf{x}} + r) - \text{ div } (\mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}} + \mathbf{q})\}dv = 0, \qquad (A-30)$$

which can be localized by setting

$$\rho(\dot{\mathbf{x}}\cdot\ddot{\mathbf{x}}+\dot{e})-\rho(\mathbf{f}\cdot\dot{\mathbf{x}}+r)-\ \mathrm{div}\ (\mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}}+\mathbf{q})\}=\rho\hat{w} \qquad (\mathrm{A}-31)$$

for every $\hat{w}$ which satisfies

$$\int_{B(t)}\rho\hat{w}\ dv=0. \qquad (\mathrm{A}-32)$$

Using (A-18) and the identity

$$\mathrm{div}\ (\mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}})=\ \mathrm{div}\ \mathbf{T}\cdot\dot{\mathbf{x}}+\mathbf{T}\cdot\ \mathrm{grad}\ \dot{\mathbf{x}}, \qquad (\mathrm{A}-33)$$

it follows that (A-31) can be written

$$\rho\dot{e}-\mathbf{T}\cdot\ \mathrm{grad}\ \dot{\mathbf{x}}-\ \mathrm{div}\ \mathbf{q}-\rho r+\rho\dot{\mathbf{x}}\cdot\hat{\mathbf{f}}=\rho\hat{w}. \qquad (\mathrm{A}-34)$$

When (A-31) is integrated over an arbitrary $P(t)$ of $B(t)$ and (A-32) is used, it follows that

$$\frac{d}{dt}\int_{B(t)}\rho(\frac{1}{2}\dot{\mathbf{x}}\cdot\dot{\mathbf{x}}+e)dv-\int_{P(t)}\{\rho(\mathbf{f}\cdot\dot{\mathbf{x}}+r)+\ \mathrm{div}\ (\mathbf{T}^{\mathrm{T}}\dot{\mathbf{x}}+\mathbf{q})\}dv$$

$$=\int_{P(t)}\rho\hat{w}\ dv=-\int_{B(t)-P(t)}\rho\hat{w}\ dv, \qquad (\mathrm{A}-35)$$

and hence we interpret the localization residual $\hat{w}$ as the rate at which energy is supplied to a particle at place $\mathbf{x}$ due to the presence of the rest of the body.

A.5 Entropy

In all thermodynamic theories some statement of the *Second Law* is made [A.5-1]. In the mathematical theory of thermodynamics, one version or another of the *Clausius-Duhem inequality* is usually chosen to express the basic ideas of the law. A *global* statement of the inequality may be written in the form

$$\frac{d}{dt}\int_{B(t)}\rho\eta\ dv-\int_{\partial P(t)}\frac{q}{\theta}da-\int_{B(t)}\rho\frac{r}{\theta}\geq 0, \qquad (\mathrm{A}-36)$$

[A.5-1] The reader may consult Truesdell (1984) for a discussion of the origins of mathematical thermodynamics.

where $\eta(\mathbf{x},t)$ is the *specific entropy field* and $\theta(\mathbf{x},t) > 0$ the *temperature field.* As for the physical meaning, we note that (A-36) is a statement about the nonnegativity of the internal production of entropy of the entire body at any time.

As will soon be seen the global statement (A-36) has an essential deficiency as a starting point for a nonlocal theory of the type considered here. To understand this we postpone for a moment the development of the nonlocal formulation and look at local theory. If we apply (A-36) to strictly local bodies and use (A-29) and the divergence theorem, we obtain

$$\rho\dot{\eta} - \text{ div } (\frac{q}{\theta}) - \rho\frac{r}{\theta} \geq 0. \qquad (A-37)$$

Multiplication by θ gives the equivalent inequality

$$\theta(\rho\dot{\eta} - \text{ div } (\frac{q}{\theta}) - \rho\frac{r}{\theta}) \geq 0, \qquad (A-38)$$

which integrated over the region $B(t)$ yields

$$\int_{B(t)} \theta(\rho\dot{\eta} - \text{ div } (\frac{\mathbf{q}}{\theta}) - \rho\frac{r}{\theta})dv \geq 0. \qquad (A-39)$$

We note that (A-36) and (A-39) are of course not equivalent statements (unless the temperature field is uniform). The physical meaning of (A-39) is also basically different from that of (A-36). While (A-36) was a statement of the nonnegativity of entropy production, (A-39) is a statement of the nonnegativity of the internal production of heat (nonmechanical energy rate) of the entire body at any time.

Since this is not the place to discuss what physical idea the Second Law should give expression to (no general agreement exists), we refer to both (A-36) and (A-39) as statements of the Clausius-Duhem inequality.

The nonlocal thermodynamic theory considered here will be based on the equality (A-39). We recall that the Clausius-Duhem inequality (in one form or another) is of vital importance for the development of any constitutive thermodynamic theory. As mentioned already, we refer to Section 2.4 for a discussion of nonlocal constitutive theory (with respect to a special class of elastic-plastic materials). However, in order to motivate the choice of (A-39) as the fundamental inequality, we will complete this appendix by deriving a *reduced* global Clausius-Duhem inequality.[A.5-2] For that purpose we use the localized equation of balance of energy, (A-31), to eliminate the specific energy supply r from the left-hand side of (A-39), and obtain

[A.5-2] A corresponding local reduced Clausius-Duhem inequality is frequently used as the starting point for constitutive thermodynamic theories.

$$\int_{B(t)}(\theta\rho\dot{\eta} - \rho\dot{e} + \frac{\mathbf{q}\cdot\ \text{grad}\ \theta}{\theta} + \mathbf{T}\cdot\ \text{grad}\ \dot{\mathbf{x}} - \rho\dot{\mathbf{x}}\cdot\hat{\mathbf{f}})dv \geq 0, \qquad (A-40)$$

where (A-32) has also been used. If we introduce the *Helmholz free energy*

$$\psi = e - \theta\eta, \qquad (A-41)$$

we can write (A-40) in the form

$$\int_{B(t)}\{-\rho(\dot{\psi} + \dot{\theta}\eta) + \frac{\mathbf{q}\cdot\ \text{grad}\ \theta}{\theta} + \mathbf{T}\cdot\ \text{grad}\ \dot{\mathbf{x}} - \rho\dot{\mathbf{x}}\cdot\hat{\mathbf{f}})dv \geq 0. \qquad (A-42)$$

We note that the corresponding genuine local form reads

$$-\rho(\dot{\psi} + \dot{\theta}\eta) + \frac{\mathbf{q}\cdot\ \text{grad}\ \theta}{\theta} + \mathbf{T}\cdot\ \text{grad}\ \dot{\mathbf{x}} \geq 0, \qquad (A-43)$$

which is the form of the reduced Clausius-Duhem inequality usually derived preparatory to the treatment of local constitutive theory. Hence (A-42), which was derived from (A-39), reduces to the usual reduced Clausius-Duhem inequality when the body is strictly local. This observation, together with the fact that the localization residual $\hat{w}$ dropped out in the derivation of (A-40), are the main reasons for the choice of (A-39) as the basic axiom of our nonlocal thermodynamic theory.

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