Chebyshev polynomials

Complexities in the complex plane

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– CENTRUM SCIENTIARUM MATHEMATICARUM –

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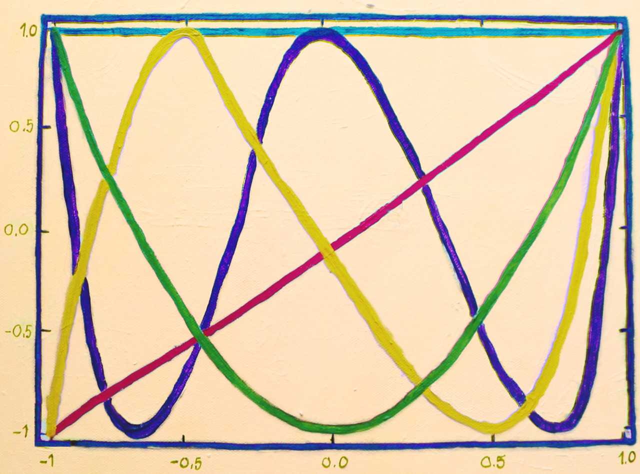

Lund University
Faculty of Engineering
Centre for Mathematical Sciences
Mathematics

Chebyshev polynomials

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Complexities in the complex plane

by Olof Rubin

LUND UNIVERSITY

Thesis for the degree of Doctor of Technology

Thesis advisor: Docent Jacob S. Christiansen

Co-advisors: Docent Mikael Persson Sundqvist and Docent Frank Wikström

Faculty opponent: Professor Klaus Schiefermayr

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Chebyshev polynomials: Complexities in the complex plane

Abstract

This thesis investigates Chebyshev polynomials in the complex plane which trace their origins to Chebyshev's foundational work in the mid-19th century and later extensions by Faber into the complex plane. The thesis begins with a historical review, highlighting key developments and results in the field. For this reason, the initial section provides a comprehensive overview of classical results. This recapitulation extends into the present day, highlighting several recent findings that have shed new light on classical research problems. The latter part focuses on original research conducted by the author together with collaborators. Our analysis encompasses Chebyshev polynomials on star graphs, lemniscates, and a comprehensive examination of weighted Chebyshev polynomials on the unit interval. A central aspect of the research presented in this thesis is based on the integration of numerical methods with rigorous mathematical proofs. Notably, several hypotheses presented in the research papers were formulated through insights gained from numerical experiments.

Key words

Chebyshev Polynomials, Extremal Polynomials, Widom Factors, Zeros of Polynomials

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MADE IN SWEDEN

Dedicated to
my wife and children
whose love provides me with constant motivation

Contents

List of publications

This thesis is based on the following publications and work in progress, referred to by their Roman numerals:

I **Extremal polynomials and sets of minimal capacity**

J. S. Christiansen, B. Eichinger and O. Rubin
Constructive Approximation (2024)

II **Chebyshev polynomials corresponding to a vanishing weight**

A. Bergman and O. Rubin
Journal of Approximation Theory, **301** (2024), 106048

III **Computing Chebyshev polynomials using the complex Remez algorithm**

O. Rubin
arXiv, 2405.05067v2 (2024)

IV **Chebyshev polynomials related to Jacobi weights**

J. S. Christiansen and O. Rubin
arXiv, 2409.02623 (2024)

V **Chebyshev polynomials and circular arcs**

J. S. Christiansen, B. Eichinger O. Rubin and M. Zinchenko
Working paper

VI **Large Widom factors**

J. S. Christiansen and O. Rubin
Working paper

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Additional publications by the author

The following articles are based on work related to my Master's degree project at Lund University and are not connected to my work concerning Chebyshev polynomials. For this reason they are not included in this thesis.

VII **On perturbation of operators and Rayleigh-Schrödinger coefficients**

M. Carlsson and O. Rubin
Complex Analysis and Operator Theory, **18** (2024), 47

VIII **A Hilbert-space variant of Geršgorin's circle theorem**

M. Carlsson and O. Rubin
Mathematische Nachricthen, **297** (2024), 3095–3106

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Last but certainly not least, I want to thank my wife, Felicia, the human embodiment of Athena, for being my unwavering companion throughout this journey. Quite literally, this thesis would not have been possible without you. I love our spontaneous life together and how every moment we share is filled with laughter. In hindsight, it's no surprise that we decided to get married on just a month's notice, right in the midst of writing this thesis.

Popular summary in English

The mid 18th century marks the dawn of the industrial revolution. To a great extent this can be attributed to the invention of the steam engine. An early development was the *Newcomen atmospheric engine*, developed in 1712. Harnessing the power generated by steam, it could produce mechanical motion. While this early steam engine was of some industrial importance it was not until James Watt's invention of the now-titled *Watt steam engine* in 1776 that the industrial revolution was a fact. His contribution was of such significant importance in enabling rapid industrialization that today his name is synonymous with the industrial revolution

One of the mechanical problems encountered in the construction of a steam engine involves the transfer of the straight-line motion generated by a piston to a mechanical rotation. As it turns out, it is difficult to design a construction which provides the lossless transfer between such forms of motion. Typically the simplest of constructions will create a deviation in the straight-line motion resulting in friction and consequently – in the long run – it will lead to rapid decay of the mechanical components. The first attempt in devising a mechanism that produces approximate straight-line motion from rotational motion was due to James Watt in 1784, an invention which we today call *Watt's linkage*. His construction deviated by about 1 part in 4000 from the desired straight-line motion.

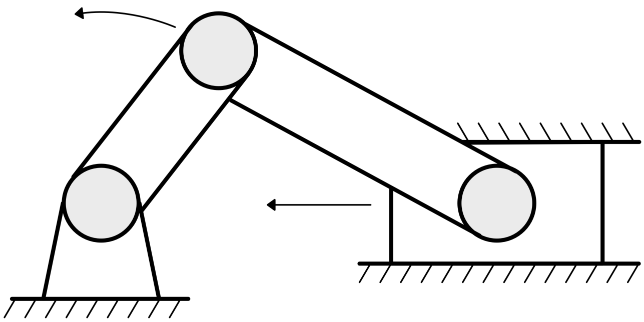


Figure 1: A mechanical linkage

Our story commences with Pafnuty Chebyshev (1821-1894) and his treatise *Théorie des mécanismes connus sous le nom de parallélogrammes* from 1854 in which he investigated the problem encountered in motion transfer introduced by Watt. From a mechanical perspective, Chebyshev's considerations amounted to devising a linkage whose deviation had an error of only about 1 part in 8000 from a straight-line motion. What is perhaps of greater significance is the related mathematical theory that he developed in order to study mechanical motion. The so-called theory of best approximation, introduced by Chebyshev, is central within mathematics and can loosely be described as trying to provide simple descriptions of complicated objects with the smallest error possible. The complicated object, henceforth referred to as a function, may not be explicitly computable but by carefully considering certain properties of the function one hopes to be able to decompose it into simple parts which are computable. With the introduction of computational machines it is hard to overstate the importance of such mathematical methods. One of the foundations of numerical computing concerns the possibility of decomposing a complicated function into an approximated form.

To be somewhat precise, the mathematical problem that Chebyshev used to describe the mechanical motion transfer was that of placing certain nodes on an interval. Let $a_1, a_2, \dots, a_n$ – the nodes – be some values between -1 and 1. For any point x within this interval it is possible to form the product of the distances between x and the nodes. This produces

$$|x - a_1| \times |x - a_2| \times \cdots \times |x - a_n|.$$

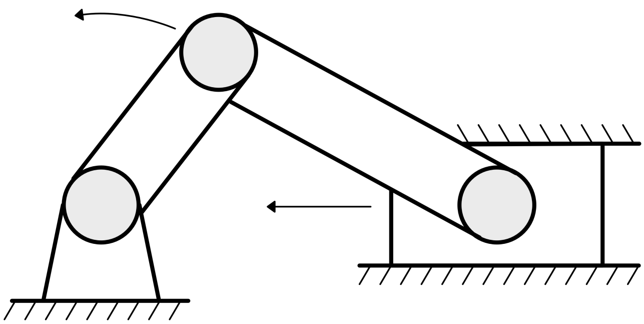
The question is how to choose $a_1, a_2, \dots, a_n$ in such a way that the resulting product deviates from 0 with as small an error as possible. This seemingly innocent problem gives rise to several complicated questions and has further consequences to the study of electromagnetic field theory as it is intimately tied to the theory of logarithmic potentials. This motivates a study of these minimization problems in much greater generality by moving the considerations into the two-dimensional plane. Such considerations date back to the works of Georg Faber from 1919.

In this thesis we will build upon the consideration of P. Chebyshev and G. Faber, together with the many generalizations developed thereafter and try to answer some open questions. In particular, how do geometric restrictions on the nodes affect the positioning of the minimizing configuration and what is the resulting deviation from 0?

Populärvetenskaplig sammanfattning på svenska

Vid sekelskiftet mellan 1600- och 1700-talet lades grunden för den industriella revolutionen. Dess uppkomst kan till stor del tillskrivas uppfinnandet av ångmaskinen. En tidig modell var Newcomens atmosfäriska maskin som togs fram 1712. Denna möjligjorde utnyttjandet av ångkraft för att producera mekanisk rörelse. En betydande förbättring av ångmaskinens effektivitet vilket bidrog till dess kommersiella genombrott 1776 gjordes av James Watt. Hans bidrag var av sådan betydelse att hans namn numera är synonymt med den industriella revolutionen.

Ett av de mekaniska problem som uppstår i en sådan konstruktion är överföringen av den linjära rörelse som genereras av en kolv till en mekanisk rotation. Det visar sig vara svårt att utforma en konstruktion som ger förlustfri överföring. Oftast skapar de enklaste konstruktionerna en oönskad avvikelse i den linjära rörelsen vilket leder till friktion och följaktligen till slitage hos de mekaniska komponenterna. Det första försöket att utforma en mekanism som producerar en ungefärlig linjär rörelse från en roterande rörelse utvecklades 1784 av James Watt och kallades senare för *Watt's parallellrörelse*. Hans konstruktion avvek med cirka 1 del på 4000 från den önskade linjära rörelsen.



Figur 2: En mekanisk Länkning

Vår berättelse tar vid med Pafnuty Chebyshev (1821-1894) och hans artikel *Théorie des mécanismes connus sous le nom de parallélogrammes* från 1854, där han undersökte problemet relaterat till rörelseöverföring som Watt introducerat. Ur ett mekaniskt perspektiv bestod Chebyshevs överväganden i att utforma en länkning vars avvikelse var felaktig med endast cirka 1 del på 8000 från den önskade linjära rörelsen. Av större betydelse är den relaterade matematiska teorin som han utvecklade för att studera mekanisk rörelse. Chebyshev introducerade den så kallade teorin om bästa approximationer vilken idag upptar en central roll inom matematiken. Löst formulerat handlar det om att försöka att ge en enkel beskrivning till ett komplicerat objekt på ett sådant sätt att avvikelsen blir så liten som möjligt. Det komplicerade objektet som vi hänvisar till som en funktion kanske inte är explicit beräkningsbar men genom att noggrant överväga vissa egenskaper hos funktionen hoppas man kunna dela upp den i enkla delar som är beräkningsbara. I och med introduktionen av beräkningsmaskiner under 1900-talet är det svårt att överskatta betydelsen av sådana matematiska metoder. En av de teoretiska grunderna för numerisk beräkning handlar om att dela upp en komplicerad funktion i en approximation bestående av enkla delar.

För att vara något mer specifik handlade det matematiska problem som Chebyshev övervägde i relation till det mekaniska problemet om att placera vissa noder på ett intervall. Låt $a_1, a_2, \dots, a_n$ vara några värden, så kallade noder, mellan -1 och 1. Givet en godtycklig punkt x inom detta intervall bildas produkten av avstånden mellan x och noderna. Detta ger

$$|x - a_1| \times |x - a_2| \times \cdots \times |x - a_n|.$$

Vad Chebyshev övervägde var att välja punkterna $a_1, a_2, \dots, a_n$ på ett sådant sätt att den resulterande produkten avviker från 0 med så liten felmarginal som möjligt. Detta tillsynes oskyldiga problem visar sig i sin tur ge upphov till invecklade följdfrågor med icke-triviala svar. Inom fysiken har sådana studier viktiga konsekvenser inom studiet av elektromagnetisk potentialteori eftersom produkter som ovan är intimt knutna till teorin om logaritmiska potentialer. Detta motiverar en generaliserad studie av dessa minimiseringsproblem i mycket större allmänhet vidgade till det tvådimensionella planet. Ursprunget till sådana överväganden kan tillskrivas Georg Faber 1919.

I denna avhandling kommer vi att bygga vidare på de frågor som ställdes av P. Chebyshev och G. Faber tillsammans med de många generaliseringar som utvecklats och försöka besvara några öppna frågor rörande de så kallade Chebyshevpolynomen. Hur påverkar geometrisksa begränsningar på noderna deras optimala positionering och vad är den resulterande avvikelsen från 0?

Chebyshev polynomials: Complexities in the complex plane

1 Introduction

We begin by considering the classical theory of Chebyshev polynomials relative to real sets.

1.1 The inception of approximation theory

In 1854 [1] Pafnuty Chebyshev introduced us to the problem of "best approximation". His problem formulation amounts to the following:

Problem 1. *Given a continuous function $f : [-1,1] \to \mathbb{R}$ and a natural number $n \in \mathbb{N}$ determine (real) parameters $a_0, \dots, a_{n-1}$ such that*

$$\max_{x\in[-1,1]} |f(x) - a_0 - a_1 x - \cdots - a_{n-1}x^{n-1}|$$

is minimal.

In modern terms, this minimal deviation represents the distance between the function f and the space of polynomials of degree at most $n-1$, measured with respect to the supremum norm on the interval $[-1,1]$. As a consequence of Weierstraß' approximation theorem, proven in [2], we know that this minimal deviation tends to zero as $n \to \infty$ for any fixed continuous function f. A construction of Bernstein, detailed in [3], provides such a sequence explicitly. In [1], Chebyshev has a different focus as his aim is directed toward finding the actual minimum rather than a sufficiently close approximant. Curiously, the considerations of Chebyshev predates Weierstraß' by more than thirty years. In order to study this general problem, Chebyshev simplifies the conditions through a consideration of Taylor series and ends up at the following reduced problem.

Problem 2. *Given a natural number $n \in \mathbb{N}$ determine (real) parameters $a_0, \dots, a_{n-1}$ such that*

$$\max_{x\in[-1,1]} |x^n - a_0 - a_1 x - \cdots - a_{n-1}x^{n-1}|$$

is minimal.

He further shows that the (as it turns out) unique solution to Problem 2 is given by the coefficients of the polynomial

$$a_0 + \cdots + a_{n-1}x^{n-1} = x^n - \frac{1}{2^n}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right].$$

Suitably rephrased,

$$T_n(x) = \frac{1}{2^n}\left[(x+\sqrt{x^2-1})^n + (x-\sqrt{x^2-1})^n\right] \tag{1}$$

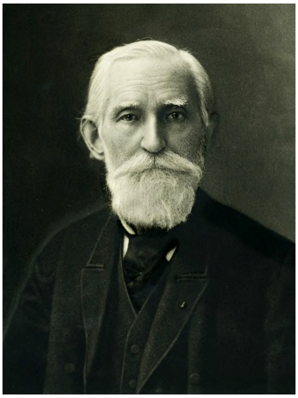


Figure 3: Pafnuty Chebyshev (1821-1894).

is the monic polynomial of degree n, which minimizes the supremum norm on $[-1, 1]$. The use of the letter T to denote these polynomials stems from the French transliteration Tschébycheff. Today, the polynomial (1) is referred to as the Chebyshev polynomials of the first kind of degree n. An alternative representation is obtained by writing $x = \cos\theta \in [-1, 1]$ in which case

$$x \pm \sqrt{x^2 - 1} = \cos\theta \pm i \sin\theta.$$

This leads us to the (well-known) formula

$$T_n(x) = \frac{e^{in\theta} + e^{-in\theta}}{2^n} = 2^{1-n} \cos n\theta. \tag{2}$$

This alternative representation illustrates the first occurrence – but certainly not the last – of the fact that minimizing polynomials tend to alternate between extremal points. As it turns out, this is a characterizing property of best approximations with respect to the supremum norm. From (2) we gather that the polynomials T_n alternate between 2^{1-n} and -2^{1-n} at $n + 1$ consecutive points on $[-1, 1]$.

While Chebyshev's intentions in [1] was to perform a theoretical study of a mechanical problem related to Watts parallelogram theory, his mathematical investigations are considerably deepened in [4]. Here he introduces the following problem:

Problem 3. *Given a continuous function* $f : [-1, 1] \to \mathbb{R}$*, a polynomial* P *which is strictly positive on* $[-1, 1]$ *and a natural number* $n \in \mathbb{N}$ *determine (real) parameters* $a_0, \ldots, a_{n-1}$ *such that*

$$\max_{x \in [-1,1]} \left[|f(x) - a_0 - a_1 x - \cdots - a_{n-1} x^{n-1}| / P(x) \right]$$

is minimal.

Again, of special interest to Chebyshev is the reduced case where $f(x) = x^n$. In this particular case, he determines the exact solution of the problem, something we will get back to. A detailed summary of the works of Chebyshev related to approximation theory can be found in [5]. After the groundbreaking work of Chebyshev, these problems have been extended in a variety of different directions. In particular we mention the works of Markov [6, 7], Borel [8], Faber [9], Akhiezer [10, 11], Bernstein [12, 13] and Widom [14]. For a historical overview detailing the early development of the subject by Chebyshev's students we refer the reader to [15]. The main purpose of this thesis is to investigate properties of minimal polynomials in the complex plane as introduced by Faber [9], these are also called Chebyshev polynomials and incorporates the classical Chebyshev polynomials on an interval as special cases.

This thesis is written in two parts. In the first introductory part, sections 1-3, classical theory is presented and in some cases the proofs are modified versions of previous proofs. The authors intention with such an inclusion is to detail the historical development of the subject. Classical methods turn out to be a good tool in our analysis. Another motivation for writing an introductory text solely treating Chebyshev polynomials in the complex plane is that such a text seems to be lacking from the existing literature.

In the second part of the thesis, Section 4, the research papers of the author are discussed together with results which do not appear in scientific publications.

2 Chebyshev polynomials - a background

2.1 An extension of the concept, existence and uniqueness

This section covers classical material, serving as an introduction to Chebyshev polynomials in the complex plane. Here we will settle questions concerning existence and uniqueness of solutions to the problems suggested by Chebyshev, where a monic minimizer is sought. For a recent, more general account of weighted Chebyshev polynomials in the complex plane, we refer the reader to [16]. Let $\mathsf{E} \subset \mathbb{C}$ denote a compact subset of the complex plane and $w : \mathsf{E} \to [0, \infty)$ a continuous function on E which is non-zero at infinitely many points of E. Implicitly, this assumption necessitates that E contains infinitely many points. The restriction that w is assumed to be continuous is made solely for convenience, enabling us to consider maximal values rather than least upper bounds. For any given natural number $n \in \mathbb{N}$ we introduce the quantity

$$t_n(\mathsf{E}, w) := \inf_{a_0,\dots,a_{n-1}\in\mathbb{C}} \max_{z\in\mathsf{E}} \left[w(z) \left|z^n + a_{n-1}z^{n-1} + \cdots + a_0\right|\right]. \tag{3}$$

Notice the that we have replaced the minus signs in Problem 3 with plus signs. While this is merely a change in perspective, it signifies that we are now focusing on minimal monic polynomials rather than approximating monomials using polynomials of lower degree. At the outset it is not clear that minimizing parameters exist. We begin by showing that this infimum is indeed a minimum. It is clear that

$$\|wP\|_{\mathsf{E}} := \max_{z\in\mathsf{E}} \left[w(z) \left|P(z)\right|\right]$$

defines a norm on the finite dimensional space of polynomials of degree at most n. Since norms on finite dimensional spaces are equivalent (see e.g. [17, Theorem III.3.1]) we conclude the existence of a positive constant $C > 0$ such that

$$\begin{aligned} C^{-1} \max_{z\in\mathsf{E}} &\left[w(z) \left|a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0\right|\right] \\ &\leq \sum_{j=0}^{n} |a_j| \leq C \max_{z\in\mathsf{E}} \left[w(z) \left|a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0\right|\right] \end{aligned} \tag{4}$$

for any choice of coefficients $a_0, \dots, a_{n-1}, a_n$. Now take sequences $\{a_j^{(k)}\}_k$ for $j = 0, \dots, n-1$ such that

$$\lim_{k\to\infty} \max_{z\in\mathsf{E}} \left[w(z) \left|z^n + a_{n-1}^{(k)} z^{n-1} + \cdots + a_0^{(k)}\right|\right] = t_n(\mathsf{E}, w)$$

It is clear from (4) that each sequence $\{a_j^{(k)}\}_k$ is bounded. Bolzano-Weierstraß' theorem implies that there exists a subsequence k_l with the property that $\{a_j^{(k_l)}\}$ is convergent for every $j = 0, \dots, n-1$. We introduce the limits a_j^* for $j = 0, \dots, n-1$ so that

$$\lim_{k_l\to\infty} a_j^{(k_l)} = a_j^*.$$

Again, (4) implies that

$$\max_{z\in \mathsf{E}} \left[w(z) \left| (a_{n-1}^{(k_l)} - a_{n-1}^*)z^{n-1} + \cdots + (a_0^{(k_l)} - a_0^*) \right| \right] \le C \sum_{j=0}^{n-1} |a_j^{(k_l)} - a_j^*| \to 0$$

as $k_l \to \infty$ and therefore we finally conclude that

$$\max_{z\in \mathsf{E}} \left[w(z) \left| z^n + a_{n-1}^* z^{n-1} + \cdots + a_0^* \right| \right] = t_n(\mathsf{E}, w).$$

This establishes the existence of a minimizer. We proceed by showing that such a minimizer is unique. The following lemma will be useful in this regard.

Lemma 1. *Let $a_0^*, \ldots, a_{n-1}^*$ be such that*

$$\max_{z\in \mathsf{E}} \left[w(z) \left| z^n + a_{n-1}^* z^{n-1} + \cdots + a_0^* \right| \right] = t_n(\mathsf{E}, w) \tag{5}$$

then there are (at least) $n + 1$ points $z_1, \ldots, z_{n+1}$ in E such that

$$\left[w(z_j) \left| z_j^n + a_{n-1}^* z_j^{n-1} + \cdots + a_0^* \right| \right] = t_n(\mathsf{E}, w).$$

Proof. We argue by contradiction. Let

$$T(z) = z^n + a_{n-1}^* z^{n-1} + \cdots + a_0^*$$

where the coefficients are chosen as to satisfy (5). Assume, in order to derive a contradiction, that there are fewer than $n + 1$ points in E where $w|T|$ attains the value $t_n(\mathsf{E}, w)$. Denote these points with $z_1, \ldots, z_m$, so that $m < n + 1$. Using Lagrange interpolation, we can find a polynomial of degree at most $n - 1$ denoted with Q, such that $Q(z_j) = T(z_j)$ for $j = 1, \ldots, m$. We consider the perturbed polynomial

$$T(z) - \varepsilon Q(z)$$

where $\varepsilon > 0$. This difference is a monic polynomial of degree n since Q has degree at most $n - 1$. The triangle inequality applied to the absolute value of the difference at a point z ensures us that

$$w(z)|T(z) - \varepsilon Q(z)| \le (1 - \varepsilon) w(z)|T(z)| + \varepsilon w(z)|T(z) - Q(z)|. \tag{6}$$

Since $T(z_j) - Q(z_j) = 0$ we can find a $\delta > 0$ with the property that if

$$z \in U_\delta := \bigcup_{j=1}^{m} \{\zeta : |\zeta - z_j| < \delta\}$$

then

$$w(z)|T(z) - Q(z)| < \frac{t_n(\mathsf{E}, w)}{2}.$$

If $z \in U_\delta$ we obtain from (6) that

$$w(z)|T(z) - \varepsilon Q(z)| \leq \left(1 - \frac{\varepsilon}{2}\right) t_n(\mathsf{E}, w). \tag{7}$$

On the other hand there exists some $0 < \rho < 1$ such that if $z \in \mathsf{E} \cap U_\delta^c$ then $w(z)|T(z)| \leq (1-\rho)t_n(\mathsf{E}, w)$. Here is a subtle yet crucial point: the choice of ρ does not need to account for ε, as it depends solely on δ. We conclude that, uniformly for $z \notin U_\delta$,

$$w(z)|T(z) - \varepsilon Q(z)| \leq (1-\rho)t_n(\mathsf{E}, w) + \varepsilon\|wQ\|_{\mathsf{E}}. \tag{8}$$

Combining (7) and (8) it is clear that by letting $\varepsilon > 0$ be sufficiently small, we can obtain the inequality

$$\|w(T - \varepsilon Q)\|_{\mathsf{E}} < t_n(\mathsf{E}, w)$$

but this contradicts the assumed minimality of T. □

With Lemma 1 at hand, we can now easily show that there is only one monic polynomial whose weighted deviation from zero is the smallest on a given compact set.

Theorem 1. *Let* $\mathsf{E} \subset \mathbb{C}$ *denote a compact set and* $w : \mathsf{E} \to [0, \infty)$ *a continuous function which is non-zero at infinitely many points of* E. *For every natural number* $n \in \mathbb{N}$ *there exists a unique monic polynomial denoted*

$$T_n^{\mathsf{E},w}(z) = z^n + a_{n-1}^* z^{n-1} + \cdots + a_0^*$$

such that

$$\|wT_n^{\mathsf{E},w}\|_{\mathsf{E}} = t_n(\mathsf{E}, w).$$

This is the so-called weighted Chebyshev polynomial of degree n with respect to the set E and the weight function w.

Proof. The existence of a minimizer has already been established. To prove the uniqueness of the minimizer we assume that there are two monic polynomials of degree n, denoted $T^{(1)}$ and $T^{(2)}$, satisfying

$$\|wT^{(1)}\|_{\mathsf{E}} = t_n(\mathsf{E}, w) = \|wT^{(2)}\|_{\mathsf{E}}.$$

Their average is denoted $T = \frac{1}{2}(T^{(1)} + T^{(2)})$. By the triangle inequality

$$\|wT\|_{\mathsf{E}} \leq \frac{1}{2}\|wT^{(1)}\|_{\mathsf{E}} + \frac{1}{2}\|wT^{(2)}\|_{\mathsf{E}} = t_n(\mathsf{E}, w).$$

On the other hand, since T is a monic polynomial of degree n, the reverse inequality $\|wT\|_{\mathsf{E}} \geq t_n(\mathsf{E}, w)$ follows by definition of t_n. It turns out that the average polynomial T is actually a minimizer. Lemma 1 implies the existence of $n+1$ distinct points $z_1, \dots, z_{n+1}$ such that

$$w(z_j)|T(z_j)| = t_n(\mathsf{E}, w).$$

Since

$$\begin{aligned} w(z_j)|T(z_j)| &= w(z_j)\frac{|T^{(1)}(z_j) + T^{(2)}(z_j)|}{2} \\ &\leq \frac{1}{2}w(z_j)|T^{(1)}(z_j)| + \frac{1}{2}w(z_j)|T^{(2)}(z_j)| \leq t_n(\mathsf{E}, w) \end{aligned}$$

equality holds throughout. But this is only possible if $\arg T^{(1)}(z_j) = \arg T^{(2)}(z_j)$ and

$$|T^{(1)}(z_j)| = t_n(\mathsf{E}, w) = |T^{(2)}(z_j)|$$

for all $j = 1, \dots, n+1$.

As a consequence the difference $T^{(1)} - T^{(2)}$ is a polynomial of degree at most $n-1$ that vanishes at the $n+1$ points $z_1, \dots, z_{n+1}$. This can occur if and only if the difference is constantly equal to 0. Therefore $T^{(1)} = T^{(2)}$ and we have established the existence of a unique minimizer, henceforth denoted $T_n^{\mathsf{E},w}$. □

If the risk of confusing the reader is low and the set E associated with the weight function w is clearly implied from the weight function we will simply use the notation T_n^w for the corresponding weighted Chebyshev polynomial. Alternatively, in the case where the weight is given by $w = 1$ we will use the notation T_n^{E}. The notation $\mathbb{D}$ will be used for the open unit disk and $\mathbb{T}$ for the unit circle. We will further use $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ to denote the Riemann sphere. Chebyshev polynomials corresponding to compact sets in the complex plane are only known explicitly for a very narrow class of sets and weights. To provide at least one example, we show that

$$T_n^{\mathbb{T}}(z) = z^n. \tag{9}$$

Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ be any monic polynomial. Then $P(z)/z^n$ is an analytic function away from the origin and has the value 1 at infinity. By the maximum modulus principle applied to the domain $\{z : |z| > 1\}$ we find that

$$\|P\|_{\mathbb{T}} \geq 1.$$

Since the polynomial z^n saturates this lower bound we find that $T_n^{\mathbb{T}}(z) = z^n$. This example further shows that the number of extremal points of a Chebyshev polynomial can be infinite. This is not the case however if the set E is a real subset as we will show in Theorem 2 below. To illustrate the difference between real and complex sets we now shift our attention to the rich structure exhibited by real Chebyshev polynomials and begin by proving that Chebyshev polynomials corresponding to real sets are have real coefficients.

Lemma 2. *Let* $\mathsf{E} \subset \mathbb{R}$ *be compact and* $w : \mathsf{E} \to [0, \infty)$ *be a continuous function. The coefficients of* $T_n^{\mathsf{E},w}$ *are real.*

Proof. The inequality

$$|w(x)\,\mathrm{Re}(T(x))| = |\,\mathrm{Re}(w(x)T(x))| \leq |w(x)T(x)|$$

holds for any polynomial T and $x \in \mathbb{R}$. Since further $\mathrm{Re}(T)$ attains the same values on $[-1, 1]$ as a polynomial with real coefficients, the result follows from the uniqueness of the minimizer. $\square$

As we already saw, without proof, the Chebyshev polynomials on an interval have the representation

$$T_n^{[-1,1]}(x) = 2^{1-n}\cos(n\theta)$$

where $x = \cos\theta$. As such if

$$x_{n-j} = \cos\left(\frac{\pi j}{n}\right)$$

for $j = 0, \cdots, n$ then $-1 = x_0 < x_1 < \cdots < x_n = 1$ and

$$T_n^{[-1,1]}(x_j) = 2^{1-n}(-1)^{n-j}.$$

The Chebyshev polynomial $T_n^{[-1,1]}$ alternates between $\pm 2^{1-n}$. This property is in fact characterizing for best approximations in the real setting, however, this realization took a long time to develop. Although Chebyshev described this phenomenon the first person to fully this characterization was Kirchberger [18], in 1902. The first complete proof was published by Borel in [8]. Achieser states in [15, p. 7] that Markov gave a proof in a series of lectures around 1905 that first appeared in print in 1948 [7].

Theorem 2 (Borel 1905 [8], Markov [7]). *Let* $\mathsf{E} \subset \mathbb{R}$ *be compact and* $w : \mathsf{E} \to [0, \infty)$ *be a continuous function which is non-zero at infinitely many points of* E. *A monic polynomial* T *of degree n coincides with the Chebyshev polynomial* $T_n^{\mathsf{E},w}$ *if and only if there are* $n + 1$ *points in* E *denoted by* $x_0 < x_1 < \cdots < x_n$ *such that*

$$w(x_j)T(x_j) = (-1)^{n-j}\|wT\|_{\mathsf{E}}.$$

Proof. Assume that wT has fewer than $n + 1$ sign changes between consecutive extremal points. It is then possible to find $a_1 < a_2 < \ldots < a_m$ with $m < n + 1$ such that:

- $\mathsf{E} \subset \bigcup_{k=1}^{m} [a_k, a_{k+1}]$,

- every (a_k, a_{k+1}) contains at least one extremal point of wT on E,
- all extremal points of wT on (a_k, a_{k+1}) have the same sign. Moreover, the sign of these extremal points alternates between adjacent intervals.

We consider the perturbed polynomial

$$T(x) - \varepsilon Q(x),$$

where

$$Q(x) = \pm \prod_{k=1}^{m-1} (x - a_k)$$

and the sign in front of the product is chosen so that Q has the same sign on (a_k, a_{k+1}) as the corresponding extremal points of wT. We can then essentially repeat the argument as in Lemma 1. Indeed there exists a $\delta > 0$ such that if U_δ denotes the δ-neighborhood around the extremal points of wT then $T(x)$ and $Q(x)$ both have the same sign in U_δ. We find that if $x \in \mathsf{E} \cap \overline{U_\delta}$ then

$$\begin{aligned} w(x)|T(x) - \varepsilon Q(x)| &\leq (1-\varepsilon) w(x)|T(x)| + \varepsilon w(x)|T(x) - Q(x)| \\ &< (1-\varepsilon)\|wT\|_{\mathsf{E}} + \varepsilon\|wT\|_{\mathsf{E}} = \|wT\|_{\mathsf{E}}. \end{aligned}$$

On the other hand if $x \in \mathsf{E} \cap U_\delta^c$ then there exists some $0 < \rho < 1$ depending on δ such that

$$w(x)|T(x) - \varepsilon Q(x)| \leq (1-\rho)\|wT\|_{\mathsf{E}} + \varepsilon\|wQ\|_{\mathsf{E}}.$$

By choosing $\varepsilon > 0$ sufficiently small, we ensure that $\|w(T - \varepsilon Q)\|_{\mathsf{E}} < \|wT\|_{\mathsf{E}}$. This implies that any minimizer must have at least $n+1$ alternating points.

We now turn to proving that if a polynomial exhibits an alternation set, then it is indeed the minimizer. Assume that there are $n+1$ points in E ordered as $x_0 < x_1 < \cdots < x_n$, and let T be a monic polynomial of degree n that satisfies

$$w(x_j)T(x_j) = \sigma(-1)^{n-j}\|wT\|_{\mathsf{E}},$$

where $\sigma \in \{-1, 1\}$. If $\|wT\|_{\mathsf{E}} > t_n(\mathsf{E}, w)$, then we have

$$\operatorname{sign} w(x_j)\left\{T(x_j) - T_n^{\mathsf{E},w}(x)\right\} = \sigma(-1)^{n-j}.$$

By the intermediate value theorem there exists at least n zeros of $T - T_n^{\mathsf{E},w}$. However, this difference is a polynomial of degree at most $n-1$ which implies that this is the zero polynomial contradicting the fact that $\|wT\|_{\mathsf{E}} > t_n(\mathsf{E}, w)$. To see that $\sigma = 1$ follows from the observation that $T_n^{\mathsf{E},w}$ has all its n zeros (which must be simple) between the alternating points. Since $T_n^{\mathsf{E},w}$ is positive to the right of its final zero we conclude that the final extremal point in the alternating set must be positive. $\square$

It readily follows from Theorem 2 that

$$T_n^{[-1,1]}(x) = 2^{1-n} \cos n\theta.$$

In order to provide further explicit examples we introduce the family of Jacobi weights for $\alpha \geq 0$ and $\beta \geq 0$

$$w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}, \quad x \in [-1,1]. \tag{10}$$

Using the transformation $x = \cos\theta$ together with Theorem 2 it is possible to conclude that

$$\begin{aligned} T_n^{w^{(1/2,1/2)}}(x) &= 2^{-n}\frac{\sin(n+1)\theta}{\sin\theta}, \\ T_n^{w^{(0,1/2)}}(x) &= 2^{-n}\frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}, \\ T_n^{w^{(1/2,0)}}(x) &= 2^{-n}\frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}, \end{aligned}$$

for $x \in [-1,1]$, see e.g. [19]. These are the normalized Chebyshev polynomials of the second, third, and fourth kind. Much of Paper IV consists of providing a fine analyze of the Chebyshev polynomials corresponding to Jacobi weights.

These examples illustrate that Theorem 2, also called the "Alternation theorem", can be used to determine Chebyshev polynomials of real sets explicitly, something we will see further examples of shortly. Its use can further provide insight into the asymptotic behavior of Chebyshev polynomials corresponding to real sets, see e.g. [12, 13, 20]. The fact that the Alternation theorem does not extend in the complex setting is one reason for the fact that Chebyshev polynomials corresponding to complex sets are less understood than their real counterparts.

2.2 Chebyshev polynomials corresponding to real sets

We begin this section by considering Markov's generalization of Problem 3, in the case where $f(x) = x^n$. We will detail the solution he provided, as discussed in [6]. Our goal is to ultimately derive a general result on weighted Chebyshev polynomials, as established by Bernstein in [13]. To achieve this, we will provide a detailed account of the steps leading up to the proof presented in Appendix A of Paper I. This approach allows us to bypass the analysis of the asymptotic formulas for orthogonal polynomials discussed by Bernstein in [12]. Consequently, our proof is shorter than Bernstein's, though it does not address the asymptotically alternating properties of orthogonal polynomials associated with weights on $[-1,1]$.

Figure 4: Andrei A. Markov (1856-1922).

Let $a_k \in \overline{\mathbb{C}} \setminus [-1,1]$ and form the weight function $w : [-1,1] \to (0,\infty)$ defined by

$$w(x) = \left[\prod_{k=1}^{2m}\left(1 - \frac{x}{a_k}\right)\right]^{-1/2}. \tag{11}$$

We require that $\prod_{k=1}^{2m}\left(1-\frac{x}{a_k}\right) > 0$. The case of an odd number of factors can be handled by taking $a_{2m} = \infty$ and $|a_k| < \infty$ for $k = 1,2,\dots,2m-1$. Let $z \in \mathbb{T}$ and $\rho_k \in \mathbb{D}$ be defined implicitly through the equations

$$x = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad \text{and} \quad a_k = \frac{1}{2}\left(\rho_k + \frac{1}{\rho_k}\right) \quad \text{for} \quad k = 1,2,\dots,2m. \tag{12}$$

The following result is due to Markov in [6]. However, the explicit representation for wT_n^w that we use can be found in [11, Appendix A] where the proof is left to the reader. For the sake of completeness in our presentation, we include a proof here.

Theorem 3 (Markov 1884, [6]). *Let $w : [-1,1] \to (0,\infty)$, $\{a_k : k = 1,\dots,2m\}$, $\{\rho_k : k = 1,\dots,2m\}$ be as in* (11) *and* (12). *For positive integers $n > m$,*

$$w(x)T_n^w(x) = 2^{-n}\prod_{k=1}^{2m}\sqrt{1+\rho_k^2}\left(z^{m-n}\prod_{k=1}^{2m}\sqrt{\frac{1-z\rho_k}{z-\rho_k}} + z^{n-m}\prod_{k=1}^{2m}\sqrt{\frac{z-\rho_k}{1-z\rho_k}}\right) \tag{13}$$

and

$$t_n([-1,1],w) = 2^{1-n}\exp\left\{\frac{1}{\pi}\int_{-1}^{1}\frac{\log w(x)}{\sqrt{1-x^2}}dx\right\}. \tag{14}$$

We remark that computation of integrals of the form

$$\frac{1}{\pi}\int_{-1}^{1}\frac{\log w(x)}{\sqrt{1-x^2}}dx$$

can be handled using the machinery of potential theory as we will see in the following section. In particular we will heavily rely on Lemma 3 for computations, whose proof we postpone.

Lemma 3. *For any $z \in \mathbb{C}$, we have*

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\log|x-z|}{\sqrt{1-x^2}}dx = \log\frac{\left|z+\sqrt{z^2-1}\right|}{2}, \tag{15}$$

where $z+\sqrt{z^2-1}$ maps $\mathbb{C}\setminus[-1,1]$ conformally onto $\mathbb{C}\setminus\overline{\mathbb{D}}$. In particular, for $z\in[-1,1]$ the integral is constantly equal to $-\log 2$.

Proof of Theorem 3. The proof rests upon showing that the function exhibited in (13) defines the product of a monic polynomial in x of degree n with the weight function w. We further show that this function possesses the specified alternating qualities made precise in Theorem 2. To begin, we consider a branch of the function $\Upsilon:\mathbb{C}\to\mathbb{C}$ defined by

$$\Upsilon(z) = \prod_{k=1}^{2m}\sqrt{z-\rho_k}.$$

The branch can be specified if we let $\Upsilon(z) = z^m(1+o(1))$ as $z\to\infty$. We introduce the function

$$\begin{aligned}\Phi_n(z) &= \frac{1}{2}\left(z^{2m-n}\frac{\Upsilon(1/z)}{\Upsilon(z)} + z^{n-2m}\frac{\Upsilon(z)}{\Upsilon(1/z)}\right)\Big/ w\left(\frac{z+z^{-1}}{2}\right)\\ &= \frac{1}{2}\left(z^{m-n}\prod_{k=1}^{2m}\sqrt{\frac{1-z\rho_k}{z-\rho_k}} + z^{n-m}\prod_{k=1}^{2m}\sqrt{\frac{z-\rho_k}{1-z\rho_k}}\right)\Big/ w\left(\frac{z+z^{-1}}{2}\right)\end{aligned} \tag{16}$$

and claim that Φ_n is a polynomial in x. To see this note that

$$w\left(\frac{z+z^{-1}}{2}\right) = z^m\prod_{k=1}^{2m}\left(\frac{(z-\rho_k)(1-\rho_k z)}{1+\rho_k^2}\right)^{-1/2}.$$

Substituting this expression into (16) yields

$$\Phi_n(z) = \frac{1}{2}\frac{1}{\prod_{k=1}^{2m}\sqrt{1+\rho_k^2}}\left(z^{-n}\prod_{k=1}^{2m}(1-z\rho_k) + z^{n-2m}\prod_{k=1}^{2m}(z-\rho_k)\right).$$

Represented this way, it is clear that Φ_n is a rational function in z which is analytic away from 0 and ∞.

From the definition in (16) we see that $\Phi_n(z) = \Phi_n(1/z)$ implying that $\Phi_n(x+\sqrt{x^2-1})$ has well-defined real-valued limit values as the complex variable x approaches $[-1,1]$ from either side with respect to the complex plane. Schwarz reflection principle [21, Theorem IX.1.1] implies that $\Phi_n(x)$ is entire.

By letting $z \to \infty$ it is clear that $\Phi_n(z)/z^n$ has the finite limit

$$\frac{1}{2\prod_{k=1}^{2m}\sqrt{1+\rho_k^2}}$$

Moving our considerations back to the variable x, we find that $\Phi_n(x+\sqrt{x^2-1})$ must be a polynomial of degree n in x with leading coefficient

$$2^{n-1}\Big/\prod_{k=1}^{2m}\sqrt{1+\rho_k^2}.$$

The polynomial

$$2^{-n}\prod_{k=1}^{2m}\sqrt{1+\rho_k^2}\left(z^{m-n}\prod_{k=1}^{2m}\sqrt{\frac{1-z\rho_k}{z-\rho_k}}+z^{n-m}\prod_{k=1}^{2m}\sqrt{\frac{z-\rho_k}{1-z\rho_k}}\right)\Big/w(x) \tag{17}$$

is necessarily monic in the variable x, and as we shall see, actually equal to T_n^w. Indeed, the only remaining task is to verify the alternating behavior of (17) when multiplied with $w(x)$. Note that for $|z|=1$

$$\left|z^{n-m}\prod_{k=1}^{2m}\sqrt{\frac{z-\rho_k}{1-z\rho_k}}\right|\leq 1$$

and hence the function defined in (17) is upper bounded by

$$2^{1-n}\prod_{k=1}^{2m}\sqrt{1+\rho_k^2} \tag{18}$$

whenever $x\in[-1,1]$. Since w is a real function, any a_k that has non-negative imaginary part must appear together with its complex conjugate. This ensures that (18) is positive. Let z traverse the upper part of the unit circle from 1 to -1. This corresponds to x going from 1 to -1. The maximal value from (18) is attained precisely when

$$\arg\left(z^{n-m}\prod_{k=1}^{2m}\sqrt{\frac{z-\rho_k}{1-z\rho_k}}\right)=0 \mod \pi.$$

Let

$$f(z)=\left(z^{n-m}\prod_{k=1}^{2m}\sqrt{\frac{z-\rho_k}{1-z\rho_k}}\right)^2=z^{2n-2m}\prod_{k=1}^{2m}\frac{z-\rho_k}{1-z\rho_k}.$$

This function is holomorphic in the unit disk and has $2n$ zeros inside. As z traverses the upper half-circle the image $f(z)$ will wrap around the origin $2n$ times, see the discussion following [21, Theorem V.3.4]. But this implies that as z goes from 1 to -1 along the upper half-circle, the value of

$$\arg\left(z^{n-m}\prod_{k=1}^{2m}\sqrt{\frac{z-\rho_k}{1-z\rho_k}}\right)$$

goes from 0 to $n\pi$. Consequently, the function in (17) has $n+1$ alternating points where it attains the value (18) on $[-1,1]$ when multiplied with w. Theorem 2 implies that T_n^w and (17) coincide. Using Lemma 3 we conclude that

$$\prod_{k=1}^{2m}\sqrt{1+\rho_k^2} = \exp\left\{\frac{1}{\pi}\int_{-1}^{1}\frac{\log w(x)}{\sqrt{1-x^2}}dx\right\}$$

and the proof is complete. □

The solution of Problem 3 due to Chebyshev in [4], for the case $f(x) = x^n$, can be deduced from Theorem 3 by letting $a_{2k} = a_{2k+1}$ for every $k = 1, \dots, m$. The next major development succeeding Markov's results concerning weighted Chebyshev polynomials on $[-1,1]$ can be attributed to Bernstein who between 1930 and 1931 presented a two part series of articles where he provided the precise asymptotic behavior of both orthogonal and Chebyshev polynomials with respect to weight functions on $[-1,1]$. This presented a remarkable shift in the focus of the subject. While Chebyshev and Markov had been investigating precise formulas for weighted Chebyshev polynomials, Bernstein instead analyzed their asymptotic behavior. It should be stressed however that Faber, already in 1919 [9], had generalized the notion of Chebyshev polynomials to considerations in the complex plane and provided a detailed study of their asymptotics for analytic Jordan domains. We refrain from detailing these studies here as we will discuss them thoroughly in Section 3. Bernstein considered the general case of weights w on $[-1,1]$ which are assumed to be merely Riemann integrable and showed that the two expressions in (14) are still asymptotically equivalent as $n \to \infty$. Astonishingly, his results are also valid for weights having zeros on $[-1,1]$. Typically, determining the behavior of Chebyshev polynomials corresponding to vanishing weights becomes much more difficult. Indeed, the results of Chebyshev and Markov are valid for weights which are reciprocals of positive polynomials. Such weights can only poorly approximate vanishing weights. We remark that the existence and uniqueness of a minimizing monic polynomial $T_n^{\mathsf{E},w}$ can be shown for any bounded measurable function $w : \mathsf{E} \to [0,\infty)$ which is non-zero on a set consisting of at least $n+1$ distinct points, see [16]. It is important to note that in this case the maximum value from (3) may not be attained and one needs to replace it with a supremum.

Theorem 4 (Bernstein 1931 [13]). *Suppose $\alpha_k \in \mathbb{R}$ and $b_k \in [-1,1]$ for $k = 0, 1, \dots, m$. Consider a weight function $w : [-1,1] \to [0,\infty)$ of the form*

$$w(x) = w_0(x)\prod_{k=0}^{m}|x-b_k|^{\alpha_k}, \tag{19}$$

Figure 5: Sergei N. Bernstein (1880-1968).

where w_0 is Riemann integrable and satisfies $1/M \leq w_0(x) \leq M$ for some constant $M \geq 1$. Then

$$t_n([-1,1],w) \sim 2^{1-n} \exp\left\{\frac{1}{\pi}\int_{-1}^{1} \frac{\log w(x)}{\sqrt{1-x^2}}dx\right\} \tag{20}$$

as $n \to \infty$.

Given two functions $f, g : \mathbb{N} \to \mathbb{C}$, such that $g(n) \neq 0$ for sufficiently large n, we use the notation $f \sim g$ as $n \to \infty$ to denote

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = 1.$$

A proof which is a modification of Bernstein's proof in [13] and circumvents the analysis of orthogonal polynomials can be found in the appendix of Paper I.

Apart from Bernstein's formula in Theorem 4, few asymptotic results have been established for Chebyshev polynomials associated with weights which vanish on parts of the interval $[-1,1]$. If $w > 0$ holds almost everywhere then the nth root asymptotics are known. In this case

$$\lim_{n\to\infty} \left(T_n^{[-1,1],w}(z)\right)^{1/n} = \frac{z+\sqrt{z^2-1}}{2}$$

holds for $z \notin [-1,1]$, see [22]. The precise asymptotical behavior – so-called Szegő-Widom asymptotics – of $T_n^{[-1,1],w}$ on $\mathbb{C} \setminus [-1,1]$ were determined in [23] in the case where w is strictly positive. In [24, 25] the asymptotical behavior of $T_n^{[-1,1],w}$ on $[-1,1]$ was determined for positive smooth weight functions. This was done using Chebyshev's result for weights w which are given as reciprocals of polynomial which are strictly positive on $[-1,1]$ together with a polynomial approximation argument. We will later see that Theorem 4 has applications to the analysis of Chebyshev polynomials corresponding to sets in the complex plane.

Figure 6: Naum Achieser (1901-1980).

Following in Bernstein's footsteps, Achieser considered in 1933 the case of weighted Chebyshev polynomials with respect to disjoint intervals of the form $\mathsf{E}(a,b) = [-1,-a] \cup [b,1]$ with $0 < a, b < 1$. In the particular case where $b = a$, we write $\mathsf{E}(a,b) = \mathsf{E}(a)$.

Theorem 5 (Achieser 1933 [10]). *For any $n \in \mathbb{N}$*

$$t_{2n}(\mathsf{E}(a)) = 2^{1-2n}(1-a^2)^n. \tag{21}$$

As $n \to \infty$

$$t_{2n+1}(\mathsf{E}(a)) \sim 2^{-2n}(1-a^2)^{n+\frac{1}{2}}\sqrt{\frac{1+a}{1-a}}.$$

In fact Achieser provides the full asymptotic formula for any choice of $0 < a, b < 1$ including the possible effects of a weight function. This generalizes Bernstein's formula and we refer the reader to [11, Appendix E] for details. The emerging pattern is that in the generic case when a and b differ, the sequence $\{t_n(\mathsf{E}(a,b))\}$ has a full interval of limit points rather than just two points. A recent proof of Theorem 5 using elliptic functions is given in [26], we will provide a novel proof based on Theorem 4. Theorem 5 points out a recurring phenomenon in the world of Chebyshev polynomials associated to compact sets which have several components. The limit behavior of $t_n(\mathsf{E}, w)$ may differ along different subsequences. This was studied in detail by Widom in [14].

Proof of Theorem 5. Due to symmetry of $\mathsf{E}(a)$ and uniqueness of the corresponding Chebyshev polynomial we obtain $T_n^{\mathsf{E}(a)}(-x) = (-1)^n T_n^{\mathsf{E}(a)}(x)$. Formulated differently, $T_{2n}^{\mathsf{E}(a)}$ is an even polynomial and $T_{2n+1}^{\mathsf{E}(a)}$ is an odd polynomial. We write

$$T_{2n}^{\mathsf{E}(a)}(x) = x^{2n} + \sum_{k=0}^{n-1} a_k x^{2k} = P_n^{\mathsf{E}(a)}(x^2),$$
$$T_{2n+1}^{\mathsf{E}(a)}(x) = x^{2n+1} + \sum_{k=0}^{n-1} b_k x^{2k+1} = xQ_n^{\mathsf{E}(a)}(x^2)$$

where $Q_n^{\mathsf{E}(a)}$ and $P_n^{\mathsf{E}(a)}$ are nth degree monic polynomials. By changing the variable from x to $t = x^2$ we find that $Q_n^{\mathsf{E}(a)}$ and $P_n^{\mathsf{E}(a)}$ are the nth degree monic minimizer of the expressions

$$\max_{t\in[a^2,1]} \left|w(t)\left(t^n + \alpha_{n-1}t^{n-1} + \dots + \alpha_0\right)\right|$$

with $w(t) = \sqrt{t}$ and $w(t) = 1$ respectively. By Theorem 2 one immediately concludes that $P_n^{\mathsf{E}(a)}$ can be explicitly represented by

$$P_n^{\mathsf{E}(a)}(t) = T_n^{[-1,1]}\left(\frac{2t-1-a^2}{1-a^2}\right)\left(\frac{1-a^2}{2}\right)^n.$$

We may therefore conclude that

$$t_{2n}(\mathsf{E}(a)) = 2^{1-2n}(1-a^2)^n.$$

To determine the odd norms we consider the change of variables $\xi = \frac{2}{1-a^2}\left(t - \frac{1+a^2}{2}\right) \Leftrightarrow \frac{1-a^2}{2}\xi + \frac{1+a^2}{2}$. This yields

$$t_{2n+1}(\mathsf{E}(a)) = \min_{\beta_0,\dots,\beta_{n-1}} \max_{\xi\in[-1,1]} \left(\frac{1-a^2}{2}\right)^n \left|\sqrt{\frac{1-a^2}{2}\xi + \frac{1+a^2}{2}}\left(\xi^n + \sum_{k=0}^{n-1}\beta_k\xi^k\right)\right|.$$

Theorem 4 provides the precise asymptotical formula

$$t_{2n+1}(\mathsf{E}(a)) \sim 2^{1-n}\left(\frac{1-a^2}{2}\right)^n \exp\left\{\frac{1}{\pi}\int_{-1}^{1}\frac{\log\sqrt{\frac{1-a^2}{2}\xi + \frac{1+a^2}{2}}}{\sqrt{1-\xi^2}}d\xi\right\}$$

as $n \to \infty$. The integral is effectively computed using Lemma 3 and we find that

$$\exp\left\{\frac{1}{\pi}\int_{-1}^{1}\frac{\log\sqrt{\frac{1-a^2}{2}\xi + \frac{1+a^2}{2}}}{\sqrt{1-\xi^2}}d\xi\right\} = \frac{1+a}{2}.$$

In conclusion,

$$t_{2n+1}(\mathsf{E}(a)) \sim 2^{1-n}\left(\frac{1-a^2}{2}\right)^n \frac{1+a}{2}. \qquad \square$$

We will later see that Theorem 5 has applications to the study of Chebyshev polynomials corresponding to arcs in the complex plane. This naturally motivates us to lift our considerations to the complex plane.

Figure 7: Georg Faber (1877-1966).

3 Lifting the considerations to the complex plane

3.1 Relating Chebyshev polynomials to conformal mappings

The first broadening of the concept of Chebyshev polynomials to the setting of the complex plane is due to Faber who is also responsible for their naming, see [9]. He begins his investigation by letting T_n^{E} be defined as the monic minimizer of degree n with respect to the supremum norm on a compact set E in $\mathbb{C}$. He then goes on with mentioning some easy cases where these polynomials can be explicitly determined. Without proof, he states the following result, although he claims that it is "just as obvious" as the determination of (9). We choose to include a proof here since it illustrates a reasoning that is central to estimates of Chebyshev polynomials in the complex plane, an argument that will be reused several times throughout.

Theorem 6 (Faber 1919 [9]). *Let $P(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_0$. If*

$$\mathsf{E}(r) = \{z : |P(z)| = r^m\}$$

for $r > 0$, then

$$T_{nm}^{\mathsf{E}(r)}(z) = P(z)^n$$

for any $n \in \mathbb{N}$.

The set $\mathsf{E}(r)$ is called a lemniscate. If $r > 0$ is sufficiently small then $\mathsf{E}(r)$ will contain as many components as the number of distinct zeros of P. For large enough r on the other hand, $\mathsf{E}(r)$ will consist of one component.

Proof. As alluded to by Faber, the proof follows along the same lines as the proof used to show (9). Indeed let Q be any monic polynomial of degree nm and form the quotient

$$\varphi(z) = \frac{Q(z)}{P(z)^n}.$$

On the set $\{z : |P(z)| > r^m\}$ the function φ is analytic since all zeros of P lie inside $\{z : |P(z)| < r^m\}$. Analyticity extends to the point at infinity since

$$\lim_{z\to\infty} \varphi(z) = 1.$$

By the maximum principle applied to the unbounded component we find that

$$1 \leq \|\varphi\|_{\mathsf{E}(r)} = \frac{\|Q\|_{\mathsf{E}(r)}}{r^{nm}}.$$

We conclude that $\|Q\|_{\mathsf{E}(r)} \geq r^{nm}$ holds for any monic polynomial Q of degree nm. Since $\|P^n\|_{\mathsf{E}(r)} = r^{nm}$ the uniqueness of the minimizer implies that $T_{nm}^{\mathsf{E}(r)} = P(z)^n$. □

Faber does not address the issue of determining Chebyshev polynomials for degrees other than nm. As we will later see the analysis of these can be quite involved, see also [27, Theorem 3.2]. One of the main points of Faber's article is to show that the classical Chebyshev polynomials $T_n^{[-1,1]}$ are also Chebyshev polynomials in the extended sense to ellipses in the complex plane with focii at ± 1. For a recent generalization of this result see [28, Theorem 1.4].

Perhaps of even greater influence on subsequent research in approximation theory, he demonstrated that a class of polynomials, introduced by him in 1903 and now known as Faber polynomials [29], can be used to construct sequences of polynomials that asymptotically achieve the same minimal norm as Chebyshev polynomials. We proceed with explaining this chain of ideas. For this reason, let E denote a compact set and let Ω_{E} denote the unbounded component of the complement of E with respect to the Riemann sphere $\overline{\mathbb{C}}$. We note that the maximum principle implies that $T_n^{\mathsf{E}} = T_n^{\partial\Omega_{\mathsf{E}}}$. This shows that it is of no importance in what follows whether we consider Chebyshev polynomials on a set or its corresponding outer boundary. With the additional assumption that Ω_{E} is simply connected, the Riemann mapping theorem implies the existence of a map $\Phi : \Omega_{\mathsf{E}} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ satisfying

$$\Phi'(\infty) := \lim_{z\to\infty} \frac{\Phi(z)}{z} > 0, \tag{22}$$

see e.g. [21, Theorem VII.4.2]. It follows that Φ has the Laurent series expansion

$$\Phi(z) = \Phi'(\infty)z + a_0 + a_{-1}z^{-1} + \cdots \tag{23}$$

at infinity. The Faber polynomial of degree n corresponding to E is denoted with F_n^{E} and is defined as the polynomial satisfying

$$F_n^{\mathsf{E}}(z) = \left(\frac{\Phi(z)}{\Phi'(\infty)}\right)^n + O(z^{-1}) \tag{24}$$

as $z \to \infty$. It is clear that F_n^{E} defined this way is a monic polynomial of degree n. If $\mathsf{E}(r) = \{z : |\Phi(z)| = r\}$ for $r > 1$ denotes the level curve parametrized in the positive direction, then for every z satisfying $r < |\Phi(z)| < R$ it can easily be seen through an analysis of the corresponding Laurent series that

$$\Phi'(\infty)^n F_n^{\mathsf{E}}(z) = \frac{1}{2\pi i}\int_{\mathsf{E}(R)} \frac{\Phi(\zeta)^n}{\zeta - z} d\zeta$$

and therefore

$$\begin{aligned} \Phi(z)^n &= \frac{1}{2\pi i}\int_{\mathsf{E}(R)} \frac{\Phi(\zeta)^n}{\zeta - z} d\zeta - \frac{1}{2\pi i}\int_{\mathsf{E}(r)} \frac{\Phi(\zeta)^n}{\zeta - z} d\zeta \\ &= \Phi'(\infty)^n F_n^{\mathsf{E}}(z) - \frac{1}{2\pi i}\int_{\mathsf{E}(r)} \frac{\Phi(\zeta)^n}{\zeta - z} d\zeta. \end{aligned}$$

The argument used to prove Theorem 6 has the following adaptation. Let Q be any monic polynomial of degree $n \in \mathbb{N}$. Then

$$\varphi(z) = \Phi'(\infty)^n \frac{Q(z)}{\Phi(z)^n}$$

is analytic on Ω_{E} and $\varphi(\infty) = 1$. Since $|\Phi(z)| \to 1$ as $z \to \partial\Omega_{\mathsf{E}}$, the maximum modulus theorem implies that

$$1 \leq \|\varphi\|_{\mathsf{E}} = \Phi'(\infty)^n \|Q\|_{\mathsf{E}}$$

and hence we obtain the lower bound

$$\frac{1}{\Phi'(\infty)^n} \leq \|Q\|_{\mathsf{E}}. \tag{25}$$

Faber, in [9], provides the following argument to show that if the boundary of E is smooth then $\Phi'(\infty)^n t_n(\mathsf{E}) \sim 1$ as $n \to \infty$. Assume that E is the closure of an analytic Jordan domain or, equivalently formulated, Φ extends analytically to some neighborhood of $\partial\Omega$. Then

$$F_n^{\mathsf{E}}(z)\Phi'(\infty)^n = \Phi(z)^n + \frac{1}{2\pi i}\int_{\mathsf{E}(r)} \frac{\Phi(\zeta)^n}{\zeta - z} d\zeta$$

for some r satisfying $0 < r < 1$. We immediately conclude that if $z \in \mathsf{E} = \mathsf{E}(1)$ then there exists some $C > 0$ independent of n such that

$$|F_n^{\mathsf{E}}(z)\Phi'(\infty)^n| \leq 1 + Cr^n.$$

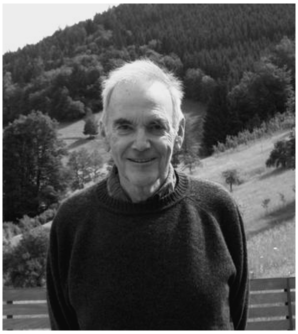


Figure 8: Harold Widom (1932-2021). Photo by Renate Schmid.

By combining this with (25) we find that

$$1 \leq \Phi'(\infty)^n t_n(\mathsf{E}) \leq 1 + Cr^n$$

and since $0 < r < 1$ we obtain the first half of the following theorem.

Theorem 7 (Faber 1919 [9]). *Let* E *denote the closure of an analytic Jordan domain with exterior conformal map* $\Phi : \Omega_{\mathsf{E}} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ *as in* (23) *then*

$$t_n(\mathsf{E})\Phi'(\infty)^n \sim 1 \tag{26}$$

as $n \to \infty$. *Furthermore*

$$T_n^{\mathsf{E}}(z)\Phi'(\infty)^n\Phi(z)^{-n} = 1 + o(1) \tag{27}$$

as $n \to \infty$ *uniformly on closed subsets of* Ω_{E}.

Proof. We are left to prove that the left hand side of (27) converges locally uniformly to 1. For this reason, note that the functions

$$\varphi_n(z) = T_n^{\mathsf{E}}(z)\Phi'(\infty)^n\Phi(z)^{-n}$$

are analytic on Ω_{E} and attain the value 1 at infinity. Since we further have that $\|\varphi_n\|_{\mathsf{E}} \leq 1 + Cr^n$ for some $C > 0$ and $0 < r < 1$, Montel's theorem (see e.g. [21, Theorem VII.2.9]) implies that $\{\varphi_n\}$ is a normal family in Ω_{E}. Since any convergent subsequences of $\{\varphi_n\}$ must converge to the constant function 1 locally uniformly on Ω_{E} we conclude that so does the full sequence. $\square$

It is actually possible to show that (27) extends to the boundary of E, see e.g. [14, Section 2]. At the time, Faber's analysis greatly advanced the understanding of minimal polynomials in the complex plane. His results were extended in a 1969 paper by Harold Widom [14]. Not only did Widom lift the regularity assumption on the boundary of E but the greatest advancement that he provided was that he also carried out a detailed study of the case where E consists of several components which are Jordan curves of class C^{2+}. A curve is of class C^{2+} if its arc-length parametrization is a twice continuously differentiable function such that its second derivative satisfies a Hölder condition. In this case, certain obvious modifications are needed. Not only is there no conformal map Φ from Ω_{E} to $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ but even with a generalized notion of such a map, the lower bound (25) is no longer optimal. Without getting into details, Widom introduces polynomials of the form

$$W_n(z) := \frac{1}{2\pi i} \int_{\Gamma} \frac{\Phi(\zeta)^n f_n(\zeta)}{\zeta - z} d\zeta$$

where Φ is a generalization of the exterior conformal map of E and f_n solves a minimal problem associated with the set E. If E has one component then $f_n = 1$ but otherwise $\|f_n\|_{\mathsf{E}} \geq 1$. Widom then shows, in a similar fashion to how we showed that (25) holds, that

$$\|f_n\|_{\mathsf{E}} \Phi'(\infty)^{-n} \leq t_n(\mathsf{E})$$

but more importantly he also proves that

$$t_n(\mathsf{E}) \leq \|\Phi'(\infty)^{-n} W_n\|_{\mathsf{E}} \sim \|f_n\|_{\mathsf{E}} \Phi'(\infty)^{-n} \tag{28}$$

as $n \to \infty$, see [14, Theorem 8.3]. As a consequence he concludes

$$t_n(\mathsf{E}) \sim \|f_n\|_{\mathsf{E}} \Phi'(\infty)^{-n}, \quad n \to \infty.$$

This result also extends to the case of weight functions on the boundary. In order to describe this we limit ourselves to the case where the set has one component. The reason for this limitation is to avoid having to deal with multi-valuedness. Assume that E is the closure of a Jordan domain, and $w : \partial\mathsf{E} \to (0, \infty)$ is a continuous function. The Dirichlet problem on $\Omega_{\mathsf{E}} = \overline{\mathbb{C}} \setminus \mathsf{E}$ with boundary data $\log w$ on $\partial\mathsf{E}$ has a unique solution, see e.g. [30, Corollary 4.1.8]. We conclude the existence of a harmonic function ω on Ω_{E} such that $\omega(z) \to \log w(\zeta)$ as $z \to \zeta \in \partial\mathsf{E}$. Let $\tilde{\omega}$ denote the harmonic conjugate which vanishes at infinity then the function

$$R(z) = \exp\Big(\omega(z) + i\tilde{\omega}(z)\Big), \tag{29}$$

satisfies $R(\infty) > 0$ and $|R(z)| = w(z)$ on $\partial\mathsf{E}$. If $\Phi : \Omega_{\mathsf{E}} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ denotes the exterior conformal map and Q is any monic polynomial of degree n then by the maximum principle

$$\max_{\zeta \in \partial\mathsf{E}} w(\zeta)\, |Q(\zeta)| = \max_{\zeta \in \partial\mathsf{E}} \left| \frac{Q(\zeta)R(\zeta)}{\Phi(\zeta)^n} \right| \geq \lim_{z \to \infty} \left| \frac{Q(z)R(z)}{\Phi(z)^n} \right| = \Phi'(\infty)^{-n} R(\infty).$$

Therefore $t_n(\mathsf{E}, w) \geq \Phi'(\infty)^{-n} R(\infty)$. On the other hand Widom, in [14, Theorem 8.3], shows the following generalization of Faber's result which allows for zeros of the weight function.

Theorem 8 (Widom 1969 [14]). *Let* E *be the closure of a Jordan domain with* C^{2+} *boundary with exterior conformal map* $\Phi : \Omega_{\mathsf{E}} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ *and associate to* E *an upper semicontinuous weight function* $w : \partial\,\mathsf{E} \to [0, \infty)$ *such that*

$$\int_{\partial \mathsf{E}} \log w(z)|dz| > -\infty.$$

Then as $n \to \infty$

$$t_n(\mathsf{E}, w)\Phi'(\infty)^n R(\infty)^{-1} \sim 1 \tag{30}$$

and

$$T_n^{\mathsf{E},w}(z) = \Phi'(\infty)^{-n}\Phi(z)^n R(\infty)\Big(R(z)^{-1} + o(1)\Big) \tag{31}$$

holds uniformly on closed subsets of Ω_{E}*. The function* R *is defined in* (29).

Widom shows this result in a much more general setting and allows for multiple components.

After having fully extended what can be proven for unions of Jordan curves with C^{2+} regularity and associated weights, Widom turns to the consideration of Jordan arcs. In the case where $\mathsf{E} = [-1, 1]$, the exterior conformal map is given by

$$\Phi(z) = z + \sqrt{z^2 - 1}$$

and hence $\Phi'(\infty) = 2$. Based on the fact that

$$t_n([-1, 1]) = 2^{-n+1} = 2\Phi'(\infty)^{-n}$$

Widom conjectures that the asymptotical behavior given in (28) is still valid for arcs if one first multiplies the right-hand side by 2. And this is true if all arcs of E are contained on the real line, i.e. E is a finite union of closed intervals, see [14, Theorem 11.5]. This turns out to be false in general. This doesn't even hold when E consists of a single Jordan arc as was shown through a counter example of Thiran and Detaille in [31]. They showed that this fails for circular arcs. We will return to an in-depth considerations of circular arcs in Section 4 and we will show how Theorem 4 can be applied to determine the asymptotical behavior of the quantity t_n in such cases.

We end this section with a discussion on Faber polynomials and how their properties can be used to better understand Chebyshev polynomials. Assume again that E is compact connected and Ω_{E} denotes its unbounded complement respect to the Riemann sphere $\overline{\mathbb{C}}$. Let $\Phi : \Omega_{\mathsf{E}} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ be the conformal map of the form specified in (23). If E is the closure of a Jordan domain with $C^{1+\alpha}$ boundary, for some $0 < \alpha < 1$, then [32, Lemma 1.3] implies that there exists some $C > 0$ so that

$$\left\|F_n^{\mathsf{E}}\Phi'(\infty)^n - \Phi^n\right\|_{\overline{\Omega}_{\mathsf{E}}} \leq C\frac{\log n}{n^{\alpha}}. \tag{32}$$

See also [33, Theorem 2, p.68].

Theorem 9 (Suetin 1971 [32]). *Let* E *denote the closure of a Jordan domain bounded by a curve of class* $C^{1+\alpha}$*, then there exists some* $C > 0$ *such that*

$$1 \leq \Phi'(\infty)^n t_n(\mathsf{E}) \leq 1 + C\frac{\log n}{n^\alpha}. \tag{33}$$

In particular

$$t_n(\mathsf{E})\Phi'(\infty)^n \sim 1$$

as $n \to \infty$.

Proof. It follows from (32) that

$$\frac{1}{\Phi'(\infty)^n} \leq \|T_n^{\mathsf{E}}\|_{\mathsf{E}} \leq \|F_n^{\mathsf{E}}\|_{\mathsf{E}} \leq \frac{1}{\Phi'(\infty)^n}\left(1 + C\frac{\log n}{n^\alpha}\right)$$

and we conclude the result. □

Assuming that E is merely a convex set then [34, Theorem 2] implies that

$$|\Phi'(\infty)^n F_n^{\mathsf{E}} \circ \Phi^{-1}(w) - w^n| \leq 1 \tag{34}$$

if $|w| \geq 1$. From this we may conclude the following.

Theorem 10 (Kövari-Pommerenke 1967 [34]). *Let* E *denote a compact convex set then*

$$\Phi'(\infty)^n t_n(\mathsf{E}) \leq 2.$$

Proof. Consider the set $\mathsf{E}(r) = \{z : |\Phi(z)| = r\} = \{\Phi^{-1}(w) : |w| = r\}$ for $r > 1$. From (34) we conclude that if $z \in \mathsf{E}(1+\varepsilon)$ for $\varepsilon > 0$ then

$$|\Phi'(\infty)^n F_n^{\mathsf{E}}(z)| \leq |\Phi(z)|^n + 1 \leq 1 + (1+\varepsilon)^n.$$

By the maximum principle and the minimality of t_n it follows that

$$\Phi'(\infty)^n t_n(\mathsf{E}) \leq \Phi'(\infty)^n \|F_n^{\mathsf{E}}\|_{\mathsf{E}} \leq \Phi'(\infty)^n \|F_n^{\mathsf{E}}\|_{\mathsf{E}(1+\varepsilon)} \leq 1 + (1+\varepsilon)^n.$$

By letting $\varepsilon \to 0$ we conclude the result. □

We note that equality holds for the convex set $\mathsf{E} = [-1, 1]$.

3.2 Approximating using potential theory

Here we will give a more general meaning to the quantity $\Phi'(\infty)$ that appeared throughout the previous section and show that a generalization of this quantity exists for every compact set E. In particular, the lower bound (25) extends to any compact set. For this reason we make a digression into potential theory and refer the reader to [30] for details. Other references detailing potential theory can be found in [35, 36]. We begin by stating certain facts.

Throughout this section let E denote a compact subset of $\mathbb{C}$ and introduce the notation Ω_{E} for the unbounded component of $\overline{\mathbb{C}} \setminus \mathsf{E}$ and $\mathscr{M}(\mathsf{E})$ for the space of probability measures which have support contained in E. Given $\mu \in \mathscr{M}(\mathsf{E})$ we define the potential function U^μ corresponding to μ via the formula

$$U^\mu(z) = \int_{\mathsf{E}} \log \frac{1}{|z-\zeta|} d\mu(\zeta).$$

This is actually the negative of the potential function as defined in [30], however, this has to do with our preference for energy minimization rather than maximization. We define the energy functional as

$$\mathscr{E}(\mu) = \int_{\mathsf{E}} U^\mu(z) d\mu(z).$$

A set E is called polar if

$$\inf_{\mu \in \mathscr{M}(\mathsf{E})} \mathscr{E}(\mu) = \infty.$$

For any non-polar set, there exists a unique measure $\mu_{\mathsf{E}} \in \mathscr{M}(\mathsf{E})$ such that

$$\mathscr{E}(\mu_{\mathsf{E}}) = \inf_{\mu \in \mathscr{M}(\mathsf{E})} \mathscr{E}(\mu),$$

see [30, Theorems 3.3.2, 3.7.6]. This measure is called the equilibrium measure relative to E and is supported on the outer boundary of E. As an example, we consider the closed unit disk $\overline{\mathbb{D}}$. Since the corresponding equilibrium measure, $\mu_{\overline{\mathbb{D}}}$ is unique it must also be rotationally invariant and hence $\mu_{\overline{\mathbb{D}}} = \frac{d\theta}{2\pi}$. Using conformal mappings, it is possible to relate equilibrium measures between different sets. Let E_1 and E_2 denote two compact sets with unbounded complements Ω_{E_1} and Ω_{E_2}. Assume that there exists a meromorphic function $\Phi : \Omega_{\mathsf{E}_1} \to \Omega_{\mathsf{E}_2}$ such that $\Phi(\infty) = \infty$ and that Φ extends continuously to $A_1 \subset \partial\Omega_{\mathsf{E}_1}$. The subordination principle [30, Theorem 4.3.8] says that

$$\mu_{\mathsf{E}_2}(\Phi(A_1)) \geq \mu_{\mathsf{E}_1}(A_1)$$

with equality if Φ is a homeomorphism between $\Omega_{\mathsf{E}_1} \cup A_1$ and $\Omega_{\mathsf{E}_2} \cup \Phi(A_1)$. With this result we can show that

$$\mu_{[-1,1]} = \frac{1}{\pi} \frac{dx}{\sqrt{1-x^2}}. \tag{35}$$

Indeed for $0 \le \alpha < \beta \le \pi$ let $\Gamma_{\alpha,\beta} = \{e^{i\theta} : \alpha \le |\arg\theta| \le \beta\}$. The so-called Joukowski map $\Phi(w) = \frac{1}{2}(w + w^{-1})$, whose name originates from [37], is a conformal map between $\overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ and $\overline{\mathbb{C}} \setminus [-1,1]$ satisfying $\Phi(\infty) = \infty$. Furthermore $\Phi(\Gamma_{\alpha,\beta}) = [\cos\beta, \cos\alpha]$. We conclude from the subordination principle that

$$\mu_{[-1,1]}([\cos\beta, \cos\alpha]) \ge \frac{\beta - \alpha}{\pi}.$$

On the other hand by summing this inequality over several disjoint intervals whose union is $[-1,1]$ we see that $\mu_{[-1,1]}$ is absolutely continuous with respect to Lebesgue measure and also that equality must hold since $\mu_{[-1,1]} \in \mathscr{M}([-1,1])$. Therefore

$$\mu_{[-1,1]}([\cos\beta, \cos\alpha]) = \frac{\beta - \alpha}{\pi} = \int_{\cos\beta}^{\cos\alpha} \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} dx$$

from which (35) follows.

There is a more general notion called the harmonic measure which for any point of the complement of the non-polar compact set E defines a measure. The equilibrium measure is simply its value at infinity.

The capacity of a set is defined through the formula

$$\mathrm{Cap}(\mathsf{E}) = e^{-\mathscr{E}(\mu_{\mathsf{E}})}$$

and is equal to 0 if the set E is polar. It is a conformal invariant in the sense that if E_1 and E_2 denote two compact sets with associated unbounded components Ω_{E_1} and Ω_{E_2} such that there exists a conformal map

$$\Phi : \Omega_{\mathsf{E}_1} \to \Omega_{\mathsf{E}_2}$$

with $\Phi(z) = \alpha z + O(1)$ as $z \to \infty$, $\alpha \neq 0$, then [30, Theorem 5.2.3]

$$|\alpha| \, \mathrm{Cap}(\mathsf{E}_1) = \mathrm{Cap}(\mathsf{E}_2). \tag{36}$$

A proof of the following useful formula concerning the capacity of polynomial preimages can be found in [30]. If Q is a polynomial of exact degree n with leading coefficient a then [30, Theorem 5.2.3]

$$\mathrm{Cap}(Q^{-1}(\mathsf{E})) = \left(\frac{\mathrm{Cap}(\mathsf{E})}{|a|}\right)^{1/n}. \tag{37}$$

A related concept which can be used to determine the capacity of a set is its associated Green's function. If E is again a compact set of positive capacity and $\widetilde{\Omega}_{\mathsf{E}}$ is a component of $\overline{\mathbb{C}} \setminus \mathsf{E}$ such that $\partial\widetilde{\Omega}_{\mathsf{E}}$ is non-polar, then there exists a unique function $G_{\widetilde{\Omega}_{\mathsf{E}}} : \widetilde{\Omega}_{\mathsf{E}} \times \widetilde{\Omega}_{\mathsf{E}} \to (0, \infty]$ such that:

- $G_{\widetilde{\Omega}_{\mathsf{E}}}(\cdot, w)$ is harmonic and bounded on closed subsets of $\widetilde{\Omega}_{\mathsf{E}} \setminus \{w\}$,
- As $z \to w$

$$G_{\widetilde{\Omega}_{\mathsf{E}}}(z, w) = \begin{cases} \log|z| + O(1), & w = \infty, \\ -\log|z-w| + O(1), & w \neq \infty; \end{cases} \tag{38}$$

- $G_{\widetilde{\Omega}_{\mathsf{E}}}(z, w) \to 0$ as $z \to \zeta$ for all $\zeta \in \partial\widetilde{\Omega}_{\mathsf{E}}$ except, possibly, outside a set of capacity 0.

If a certain behavior is valid outside of a set of capacity 0 then we say that it holds quasi-everywhere. The function $G_{\widetilde{\Omega}_{\mathsf{E}}}(\cdot, w)$ is called Green's function at w corresponding to $\widetilde{\Omega}_{\mathsf{E}}$. In the case where $w = \infty$ and Ω_{E} denotes the unbounded component of E we simply write $G_{\Omega_{\mathsf{E}}}(z, \infty) =: G_{\mathsf{E}}(z)$ which defines a function on the unbounded component Ω_{E}. The behavior at ∞ is explicit in terms of capacity. Indeed [30, Theorem 5.2.1] implies that

$$G_{\mathsf{E}}(z) = \log|z| - \log \mathrm{Cap}(\mathsf{E}) + o(1), \qquad \text{as } z \to \infty. \tag{39}$$

We further have from [30, Theorem 4.4.2] that

$$G_{\mathsf{E}}(z) = \mathscr{E}(\mu_{\mathsf{E}}) - U^{\mu_{\mathsf{E}}}(z). \tag{40}$$

By combining this with (35) we can prove Lemma 3 which found applications in the previous sections.

Proof of Lemma 3. Using (35) we recognize the relation

$$-U^{\mu_{[-1,1]}}(z) = \frac{1}{\pi}\int_{-1}^{1} \frac{\log|x-z|}{\sqrt{1-x^2}}dx.$$

It is easy to verify that

$$G_{[-1,1]}(z) = \log|z + \sqrt{z^2-1}|$$

from the characterizing properties of Green's function. We therefore conclude that $\mathscr{E}([-1,1]) = \log 2$ and if $z \notin [-1,1]$ we gather from (40) that

$$\frac{1}{\pi}\int_{-1}^{1} \frac{\log|x-z|}{\sqrt{1-x^2}}dx = \log|z+\sqrt{z^2-1}| - \log 2.$$

This proves the relation if $z \notin [-1,1]$.

Since both sides of (40) are lower semicontinuous on $\mathbb{C}$ and agree outside a set with 2-dimensional Lebesgue measure zero they coincide on all of $\mathbb{C}$, see [30, Theorem 2.7.5]. □

The relation in (37) can be used to get lower bounds for Chebyshev polynomials. Extending Faber's lower bound (25) is the so-called Szegő inequality from [38].

Figure 9: Gábor Szegő (1895-1985).

Theorem 11 (Szegő 1924 [38]). *Let* $\mathsf{E} \subset \mathbb{C}$ *denote a compact set then*

$$\mathrm{Cap}(\mathsf{E})^n \leq t_n(\mathsf{E}). \tag{41}$$

Proof. It is clear that the lemniscatic set

$$\mathsf{E}_n := \{z : |T_n^{\mathsf{E}}(z)| \leq t_n(\mathsf{E})\}$$

contains E. Since Cap increases under set inclusions we gather that

$$\mathrm{Cap}(\mathsf{E}) \leq \mathrm{Cap}(\mathsf{E}_n).$$

Recalling that the capacity and radius of a disk coincide the result follows from (37) since

$$\mathrm{Cap}(\mathsf{E}_n) = t_n(\mathsf{E})^{1/n}. \qquad \square$$

A similar proof can be employed to prove a version of Szegő's inequality for real sets that was shown by Schiefermayr in [39] using different means than presented here.

Theorem 12 (Schiefermayr 2008 [39]). *Let* $\mathsf{E} \subset \mathbb{R}$ *denote a compact set, then*

$$2\,\mathrm{Cap}(\mathsf{E})^n \leq t_n(\mathsf{E}). \tag{42}$$

Proof. The polynomial T_n^{E} is real and therefore

$$\mathsf{E} \subset \mathsf{E}_n := \{x : -t_n(\mathsf{E}) \leq T_n^{\mathsf{E}}(x) \leq t_n(\mathsf{E})\}.$$

Again, (37) implies that

$$\mathrm{Cap}(\mathsf{E}_n) = \mathrm{Cap}([-t_n(\mathsf{E}), t_n(\mathsf{E})])^{1/n} = \left(\frac{t_n(\mathsf{E})}{2}\right)^{1/n}.$$

The result follows by monotonicity of capacity with respect to set inclusion. $\square$

Szegő's inequality, Theorem 11, can be related to (25). Indeed, in the case that E has simply connected complement with respect to the Riemann sphere, denoted Ω_{E} and $\Phi : \Omega_{\mathsf{E}} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ is the conformal map satisfying $\Phi(\infty) = \infty$ and $\Phi'(\infty) > 0$ then

$$\Phi'(\infty)\,\mathrm{Cap}(\mathsf{E}) = 1.$$

To see this, note that $G_{\mathsf{E}}(z) = \log|\Phi(z)| = \log|z| + \log|\Phi'(\infty)| + o(1)$ as $z \to \infty$ and by referring to the defining properties of the Green's function the result follows. As a further consequence we see that for $r > 1$

$$\mathsf{E}(r) = \{z : |\Phi(z)| = r\} = \{z : G_{\mathsf{E}}(z) = \log r\}.$$

The curves $\mathsf{E}(r)$ are therefore called the Green lines or equipotential lines corresponding to the set E.

Szegő actually proved another connection between $\mathrm{Cap}(\mathsf{E})$ and $t_n(\mathsf{E})$. These ideas had previously appeared in [9, 40].

Theorem 13 (Faber, Fekete and Szegő [9, 40, 38]). *Let $\mathsf{E} \subset \mathbb{C}$ be a compact set, then*

$$\lim_{n\to\infty} t_n(\mathsf{E})^{1/n} = \mathrm{Cap}(\mathsf{E}).$$

This limits how fast $t_n(\mathsf{E})$ can grow and a central point in the understanding of Chebyshev polynomials concerns which bounds can be placed upon the quantity

$$\mathscr{W}_n(\mathsf{E}) = \frac{t_n(\mathsf{E})}{\mathrm{Cap}(\mathsf{E})^n}, \tag{43}$$

the so-called Widom factor of degree n corresponding to E. This choice of naming stems from [41] where examples of compact sets E such that the sequence $\{\mathscr{W}_n(\mathsf{E})\}$ grows subexponentially are exhibited. The fact that the sets considered there are of Cantor-type should be stressed. We saw previously, that certain regularity conditions on the boundary of E guarantees that $\{\mathscr{W}_n(\mathsf{E})\}$ is bounded. In fact if E is convex then Theorem 10 says that $\mathscr{W}_n(\mathsf{E}) \leq 2$ for all $n \in \mathbb{N}$ and if E is the closure of a Jordan domain whose boundary curve is $C^{1+\alpha}$ then Theorem 9 implies that

$$\lim_{n\to\infty} \mathscr{W}_n(\mathsf{E}) = 1.$$

From Szegő's inequality we gather that the latter is the smallest possible limit. It is an open question what regularity conditions may be relaxed while still guaranteeing that $\mathscr{W}_n$ is asymptotically minimal, that is $\lim_{n\to\infty} \mathscr{W}_n(\mathsf{E}) = 1$. The existence of closures of Jordan domains E where $\liminf_{n\to\infty} \mathscr{W}_n(\mathsf{E}) > 1$ are known. Examples include Julia sets as can be deduced from [42]. Such matters are considered in Paper VI. It should be noted that in these known cases the boundary curves are nowhere differentiable.

We end this section by showing the elementary fact that Widom factors are invariant under dilations and translations.

Theorem 14. *Let* $\mathsf{E} \subset \mathbb{C}$ *denote a compact non-polar set. If* $\alpha \in \mathbb{C} \setminus \{0\}$ *and* $b \in \mathbb{C}$ *then*

$$\mathcal{W}_n(\alpha \mathsf{E} + b) = \mathcal{W}_n(\mathsf{E}).$$

Proof. First of all (43) is well-defined since $\operatorname{Cap} \mathsf{E} > 0$ by assumption. Furthermore from (36) we gather that

$$\operatorname{Cap}(\alpha \mathsf{E} + b) = |\alpha| \operatorname{Cap}(\mathsf{E}).$$

On the other hand, by the uniqueness of the Chebyshev polynomial,

$$\alpha^n T_n^{\alpha \mathsf{E}+b}\left(\frac{z-b}{\alpha}\right) = T_n^{\mathsf{E}}(z).$$

Therefore $t_n(\alpha \mathsf{E} + b) = |\alpha|^n t_n(\mathsf{E})$ and so we see that

$$\mathcal{W}_n(\alpha \mathsf{E} + b) = \frac{t_n(\alpha \mathsf{E} + b)}{\operatorname{Cap}(\alpha \mathsf{E} + b)^n} = \frac{|\alpha|^n t_n(\mathsf{E})}{|\alpha|^n \operatorname{Cap}(\mathsf{E})} = \mathcal{W}_n(\mathsf{E}). \qquad \square$$

The Widom factor does not depend on the size of a set but rather its topological, geometric and potential-theoretic properties.

3.3 The deviation of Chebyshev polynomials

Widom, in his 1969 article [14], greatly expanded the understanding of Chebyshev polynomials related to disjoint unions of closures of Jordan domains. His analysis was complete in this case. If E is the closure of a Jordan domain with sufficiently smooth boundary then his results says that

$$\lim_{n\to\infty} \mathcal{W}_n(\mathsf{E}) = 1,$$

which asymptotically saturates the lower bound of Szegő, see Theorem 11. We further saw in Theorem 6 examples of sets where, at least for a subsequence, $\mathcal{W}_n(\mathsf{E}) = 1$ holds. Indeed, assuming that P is a monic polynomial of degree m and $r > 0$ then (37) implies that with $\mathsf{E}(r) = \{z : |P(z)| = r^m\}$

$$\operatorname{Cap}(\mathsf{E}(r)) = \operatorname{Cap}(P^{-1}\{z : |z| \le r^m\}) = r.$$

On the other hand $T_{nm}^{\mathsf{E}(r)} = P(z)^n$ and therefore

$$\mathcal{W}_{nm}(\mathsf{E}(r)) = \frac{t_{nm}(\mathsf{E}(r))}{\operatorname{Cap}(\mathsf{E}(r))^{nm}} = \frac{r^{nm}}{r^{nm}} = 1.$$

A natural question concerns whether there are other examples of sets where Szegő's lower bound is saturated at least for a subsequence. The answer turns out to be no.

Theorem 15 (Christiansen, Simon and Zinchenko 2020 [27]). *Let* $\mathsf{E} \subset \mathbb{C}$ *be a compact set with unbounded complement* Ω_{E}. *Fix* n_0. *Then* $\mathscr{W}_{n_0}(\mathsf{E}) = 1$ *if and only if there is a polynomial* P*, of degree* n_0 *such that*

$$\partial\Omega_{\mathsf{E}} = \{z : |P(z)| = 1\}.$$

As we see, sets saturating Szegő's lower bound are precisely lemniscates. If we again let $\mathsf{E}(r) = \{z : |P(z)| = r^m\}$, where $\deg P = m$, then $\mathsf{E}(r)$ will be an analytic Jordan curve if $r > 0$ is large enough. For such values of r, Theorem 7 implies that $\mathscr{W}_n(\mathsf{E}(r)) \to 1$ as $n \to \infty$. The critical case occurs when $r = r_0$ is the smallest value for which $\mathsf{E}(r)$ is connected. In this case, $\mathsf{E}(r_0)$ will no longer be a Jordan curve as it will contain a point of self intersection. Classical theory is insufficient to determine the limit points of $\mathscr{W}_n(\mathsf{E}(r_0))$. A crucial part of our study will be considerations of the lemniscatic family

$$\{z : |z^m - 1| = r^m\}$$

where $r > 0$. The critical case occurs when $r = 1$. As we show in Paper II, asymptotic minimization still holds in this case.

Theorem 6 has been generalized much further by Kamo and Borodin in [42] where the generalization appeared as the main lemma.

Theorem 16 (Kamo and Borodin (1994) [42]). *Let* $\mathsf{E} \subset \mathbb{C}$ *be compact and let* P *denote a monic polynomial of degree* m. *Then*

$$T_{nm}^{P^{-1}(\mathsf{E})} = T_n^{\mathsf{E}} \circ P \tag{44}$$

and

$$\mathscr{W}_{nm}(P^{-1}(\mathsf{E})) = \mathscr{W}_n(\mathsf{E}).$$

A recent proof of this can be found in [27]. Using (44) it is possible to get further examples of explicit Chebyshev polynomials by simply taking polynomial preimages of compact sets. Moreover, if P is a monic polynomial of degree m and $\mathsf{E} \subset \mathbb{R}$, then for any n, Schiefermayr's inequality (42) implies that

$$\mathscr{W}_{nm}(P^{-1}(\mathsf{E})) = \mathscr{W}_n(\mathsf{E}) \geq 2.$$

As such, Theorem 16 gives us several examples of sets in the complex plane where, a subsequence of the Widom factors are lower bounded by 2. Again, a natural question for such sets is to determine the remaining limit points of the Widom factors. In Paper I we will particularly focus on the case

$$\{z : z^m \in [-2, 2]\}$$

which form star shaped sets centered at the origin.

As we briefly mentioned before, Thiran and Detaille produced a counter example to Widom's conjecture that the Widom factors corresponding to a Jordan arc should converge to 2 as the degree increases. Their counter example was provided by the family of sets $\{\Gamma_\alpha : 0 < \alpha < \pi\}$, where Γ_α is the circular arc

$$\Gamma_\alpha := \{z : |\arg z| \leq \alpha\}.$$

They established in [31] that

$$\mathscr{W}_n(\Gamma_\alpha) \sim 2\cos^2(\alpha/4),$$

as $n \to \infty$. Their example shows that any number between 1 and 2 is a possible limit point of Widom factors corresponding to a Jordan arc. In Paper I we show that 2 is a limit point in the arc setting precisely when the arc is a straight line segment.

An analogue of Theorem 15 for the saturation of Schiefermayr's lower bound exists for real subsets and is due to Totik.

Theorem 17 (Totik 2011, 2014 [43, 44]). *Let* $\mathsf{E} \subset \mathbb{R}$ *and fix* $n \in \mathbb{N}$. *Then*

$$\mathscr{W}_n(\mathsf{E}) = 2$$

if and only if there exists some polynomial P *of degree* n *such that*

$$P^{-1}([-1, 1]) = \mathsf{E}.$$

Furthermore

$$\lim_{n\to\infty} \mathscr{W}_n(\mathsf{E}) = 2$$

if and only if E *is an interval.*

A new proof of the first part of Theorem 17 is provided in [27]. One of the main points of Paper I is to investigate sets in the complex plane where $\mathscr{W}_n(\mathsf{E}) \to 2$ as $n \to \infty$.

A related question concerns placing upper bounds on Widom factors. Such an upper bound, which is independent of the degree n, is called a *Totik-Widom bound* in [28]. In the real setting such a Totik-Widom bound is provided in [20]. To describe this, we introduce the concept of a *Parreau-Widom set*. These are the sets $\mathsf{E} \subset \mathbb{C}$ such that

$$\mathrm{PW}(\mathsf{E}) := \sum_{\{z:\nabla G_{\mathsf{E}}(z)=0\}} G_{\mathsf{E}}(z) < \infty.$$

In words, this quantity is equal to the sum of the critical values of the corresponding Green's function with a pole at infinity. It is clear that finitely connected sets are examples of Parreau-Widom sets. A Parreau-Widom set always satisfies a Totik-Widom bound.

Theorem 18 (Christiansen, Simon and Zinchenko 2017 [20, 45]). *Let* $\mathsf{E} \subset \mathbb{R}$ *be a regular (in the sense of potential theory) Parreau-Widom set. Then*

$$\mathcal{W}_n(\mathsf{E}) \leq 2\exp(\mathrm{PW}(\mathsf{E})). \tag{45}$$

Extending these considerations to the full generality of the complex plane remains an area of research in its early stages and it is an open question whether such bounds can be extended. In [46] these concepts were investigated for subsets of the unit circle. The first step in generalizing (45) to the complex plane is to demonstrate that compact connected sets have bounded Widom factors. This question was originally posed as an interesting problem in [47, Problem 4.4], and it was initially claimed that D. Wrase had provided an example of a compact connected set with unbounded Widom factors. However, recent findings in [48] have cast doubt on this claim. After nearly 50 years of being considered settled, this question now appears to be open once again.

Upper bounds for Widom factors associated with sets exhibiting specific structures in the complex plane exist, although without explicit constants. In order to describe one such result, we remark that two arcs meet at a a cusp point if the two angles formed at their intersection are 0 and 2π radians. If Ω is an open set such that $\partial\Omega$ consists of curves and arcs then an outward pointing cusp from Ω is a point $\zeta \in \partial\Omega$ which is a cusp point for $\partial\Omega$ and such that the sector forming the 2π-angle is contained in the complement of Ω.

Theorem 19 (Totik and Varga 2015 [49]). *Let* $\mathsf{E} \subset \mathbb{C}$ *be a compact set with unbounded complement* Ω_{E}. *If* $\partial\Omega_{\mathsf{E}}$ *is a finite union of Dini-smooth Jordan arcs, disjoint except possibly at their endpoints and such that* $\partial\Omega_{\mathsf{E}}$ *does not contain an outward cusp, then* E *satisfies a Totik-Widom bound.*

Examples of such sets include lemniscates. In this case the Totik-Widom bound can actually be made explicit.

Theorem 20 (Christiansen, Simon and Zinchenko [27]). *Let* P *be a monic polynomial of degree* m *and*

$$\mathsf{E}(r) = \{z : |P(z)| = r^m\}.$$

For any n

$$\mathcal{W}_n(\mathsf{E}(r)) \leq \max_{1\leq j\leq m} \mathcal{W}_j(\mathsf{E}(r)).$$

Proof. We have already seen that $\mathrm{Cap}(\mathsf{E}(r)) = r$. Any natural number can be expressed as $nm + l$ where $l \in \{0, 1, \dots, m-1\}$ and $n \in \mathbb{N}$. Therefore by making use of Theorem 6 we see that

$$t_{nm+l}(\mathsf{E}(r)) \leq t_l(\mathsf{E}(r))t_{nm}(\mathsf{E}(r)) = t_l(\mathsf{E}(r))\,\mathrm{Cap}(\mathsf{E}(r))^{nm} = \mathcal{W}_l(\mathsf{E}(r))\,\mathrm{Cap}(\mathsf{E}(r))^{nm+l}.$$

In conclusion,
$$\mathscr{W}_{nm+l}(\mathsf{E}(r)) \le \mathscr{W}_l(\mathsf{E}(r)) \le \max_{1\le j\le m} \mathscr{W}_j(\mathsf{E}(r)). \qquad \square$$

It follows from the proof that for a fixed l the mapping $n \mapsto \mathscr{W}_{nm+l}(\mathsf{E}(r))$ is decreasing. If r is large enough so that $\mathsf{E}(r)$ is an analytic curve then Theorem 6 implies that the limit is 1.

In [50, 48, 51] Totik-Widom bounds for sets with reduced boundary regularity were studied. To better understand these results we consider quasicircles and quasiconformal arcs. A quasicircle Γ is a Jordan curve such that any three points on the boundary satisfies the so-called *Ahlfors condition*: there exists some $A > 0$, such that if z_1, z_2 both belong to Γ then
$$|z_1 - z| + |z - z_2| \le A|z_1 - z_2|$$
whenever z lies on that subarc of Γ, with smallest diameter connecting z_1 and z_2, see e.g. [36, 52]. A quasidisk is the bounded component of the complement of a quasicircle. Examples of quasicircles include boundaries whose parametrization satisfies Lipschitz conditions but also fractal sets like the von Koch snowflake. A quasiconformal arc is any proper subarc of a quasicircle.

Theorem 21 (Andrievskii 17 [48, 51]). *If* $\mathsf{E} = \bigcup_{j=1}^m \mathsf{E}_j$ *where the sets* E_j *are mutually disjoint closed quasidisks and quasiconformal arcs then* E *satisfies a Totik-Widom bound.*

It is not at all clear what the least upper bound is. Andrievskii also considered the case where no regularity is present and concluded the following.

Theorem 22 (Andrievskii 17 [48]). *Let* $\mathsf{E} = \bigcup_{j=1}^m \mathsf{E}_j$ *where the sets* E_j *are mutually distjoint compact and connected sets that all satisfy* $\mathrm{diam}(\mathsf{E}_j) > 0$. *Then as* $n \to \infty$
$$\mathscr{W}_n(\mathsf{E}) = O(\log n).$$

These results highlight a significant distinction between Chebyshev and Faber polynomials for sets lacking boundary regularity. In fact, Gaier [53], building on an example by Clunie [54], demonstrated the existence of a quasicircle E such that there is a positive constant α for which the associated sequence of monic Faber polynomials $\{F_n^{\mathsf{E}}\}$ satisfies
$$\frac{\|F_{n_k}^{\mathsf{E}}\|}{\mathrm{Cap}(\mathsf{E})^{n_k}} > {n_k}^{\alpha}$$
for an increasing sequence n_k. On the other hand, Theorem 21 demonstrates that E satisfies a Totik-Widom bound, while Theorem 22 shows that the growth rate n^α is not possible for Widom factors of single-component sets. We emphasize that the purported counter-example to connected compact sets satisfying Totik-Widom bounds, as claimed in [47],

is based on the very construction referenced in [54]. Andrievskii's result shows that this example does not provide the necessary counterexample.

We remark that there are known examples of quasidisks for which $\limsup_{n\to\infty} \mathcal{W}_n(\mathsf{E}) > 1$, see [42] and Paper VI. Consequently, sufficient conditions to ensure that Widom factors corresponding to closed Jordan domain converge to 1 require some regularity of the bounding curve.

3.4 The zeros of Chebyshev polynomials

The final section of this background on Chebyshev polynomials focuses on the behavior of their zeros. For a fixed degree n, almost nothing is known about the precise location of the zeros of T_n^{E}. Many times, the asymptotical zero distribution is the interesting object to study. The following result constitutes an exception.

Theorem 23 (Fejér (1922) [55]). *Let $\mathsf{E} \subset \mathbb{C}$ be a compact set and $w : \mathsf{E} \to [0, \infty)$ a non-negative weight function which is non-zero on at least n points. All zeros of $T_n^{\mathsf{E},w}$ lie in* $\mathrm{cvh}(\mathrm{supp}(w))$ *– the convex hull of the support of the weight function.*

Proof. If w has precisely n points in its support then the Chebyshev polynomial is uniquely determined to be the polynomial with all its zeros coinciding with the supporting set.

We consider the case where w has at least $n + 1$ points of support. In order to obtain a contradiction, assume that $T_n^{\mathsf{E},w}(z) = \prod_{k=1}^{n}(z - a_k)$ and that $a_1 \notin \mathrm{cvh}(\mathrm{supp}(w))$. The Hahn-Banach Theorem, see e.g. [17, Theorem III.6.2], tells us that there exists a line which naturally decomposes $\mathbb{C} \setminus L$ into two connected components, one containing $\mathrm{supp}(w)$ and one containing a_1. If a_1^* denotes the orthogonal projection of a_1 onto L then

$$|z - a_1^*| < |z - a_1|$$

holds for every $z \in \mathrm{supp}(w)$. Consequently

$$w(z)|z - a_1^*| \prod_{k=2}^{n} |z - a_k| \le w(z) \prod_{k=1}^{n} |z - a_k| = w(z)|T_n^{\mathsf{E},w}(z)|$$

with strict inequality on $\mathrm{supp}(w) \setminus \{a_k \mid k = 2, \dots, n\}$. Since w contains at least $n+1$ points in its support we conclude that $(z - a_1^*) \prod_{k=2}^{n}(z - a_k)$ is a monic polynomial of degree n with smaller weighted supremum norm than $T_n^{\mathsf{E},w}$ which is a contradiction. □

The remaining results we consider in this section provide information on the asymptotical behavior of the zeros of the Chebyshev polynomials as the degree goes to infinity. The first study with this flavor was performed on partial sums of Taylor series by Jentzsch in [56] and later substantially extended by Szegő in [38]. They were both interested in describing the zero distribution of partial sums of power series of analytic functions.

If P is a polynomial of exact degree n with zeros at $z_1, \dots, z_n$, counting multiplicity, then we define the normalized zero counting measure of P as the probability measure

$$\nu(P) = \frac{1}{n}\sum_{j=1}^{n} \delta_{z_j} \tag{46}$$

where δ_z denotes the Dirac measure at z.

Theorem 24 (Jentzsch (1916) [56], Szegő (1922) [57]). *Let*

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \tag{47}$$

and the nth partial sum of f be given as

$$P_n(z) = \sum_{k=0}^{n} a_k z^k.$$

If f has radius of convergence $0 < r < \infty$ then there is a subsequence of degrees $\{n_m\}_m$ such that

$$\nu(P_{n_m}) \xrightarrow{*} \left.\frac{d\theta}{2\pi}\right|_{\{z:|z|=r\}} \tag{48}$$

as $n_m \to \infty$.

To be precise concerning the accreditation of this result, Jentzsch showed in [56] that every point of $\{z : |z| = r\}$ was a limit point of the zeros of the partial sums. Szegő extended Jentzsch' result in [57] by showing that for a subsequence of $\{P_n\}$ the corresponding zeros distribute in an equidistributed fashion in any circular sector with respect to the corresponding angle and that these zeros approach the circle determined by the radius of convergence $|z| = r$. See also [58, §2.1] for a potential theoretic proof of Theorem 24. The proof stated there is based on [58, Theorem 2.1.1] a simplification dealing with Chebyshev polynomials that we now formulate.

Theorem 25 (Blatt, Saff and Simkani (1988) [59]). *Let $\mathsf{E} \subset \mathbb{C}$ be a compact set with $\mathrm{Cap}(\mathsf{E}) > 0$ such that the unbounded component Ω_{E} of $\mathbb{C} \setminus \mathsf{E}$ has a boundary which is regular (in the sense of potential theory). If*

$$\nu(T_n^{\mathsf{E}})(A) \to 0 \tag{49}$$

as $n \to \infty$ for any closed set A contained in the union of the bounded components of $\mathbb{C} \setminus \overline{\Omega_{\mathsf{E}}}$ then

$$\nu(T_n^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}} \tag{50}$$

as $n \to \infty$.

In the special case where the bounded components of the complement are void, the following result trivially follows from Theorem 25.

Corollary 1 (Blatt, Saff and Simkani (1988) [59]). *Suppose that* $\mathsf{E} \subset \mathbb{C}$ *is a compact set with* $\mathrm{Cap}(\mathsf{E}) > 0$*, connected complement and empty interior. Then*

$$\nu(T_n^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}}, \quad \text{as} \quad n \to \infty. \tag{51}$$

We intend to prove Theorem 25 by combining recent simplified proofs. As a first step, we show that the support of any limit measure of $\nu(T_n^{\mathsf{E}})$ is contained in $\partial\Omega_{\mathsf{E}}$. It is always so that most zeros of T_n^{E} approach the polynomially convex hull of E as the following result shows.

Theorem 26 (Blatt, Saff and Simkani (1988) [59]). *Let* $\mathsf{E} \subset \mathbb{C}$ *be a compact set with* $\mathrm{Cap}(\mathsf{E}) > 0$ *then*

$$\nu(T_n^{\mathsf{E}})(A) \to 0$$

for any closed subset A *in the unbounded component* Ω_{E} *of* $\mathbb{C} \setminus \mathsf{E}$.

Proof. We define the sequence of functions $\{h_n\}$ via

$$h_n(z) = \frac{1}{n} \log |T_n^{\mathsf{E}}(z)| + U^{\mu_{\mathsf{E}}}(z) + \frac{1}{n} \sum_{k=1}^{m_n} G_{\Omega_{\mathsf{E}}}(z, z_{k,n}) \tag{52}$$

where $z_{1,n}, \dots, z_{m_n,n}$ is an enumeration of the zeros of T_n^{E} that reside in Ω_{E}, counting multiplicity. From the properties of $G_{\Omega_{\mathsf{E}}}$ detailed in (38) we gather that the function h_n is harmonic in Ω_{E}. This harmonicity extends to the point at ∞. Furthermore, for quasi-every $\zeta \in \partial\Omega_{\mathsf{E}}$,

$$\limsup_{z \to \zeta} h_n(z) = \frac{1}{n} \log |T_n^{\mathsf{E}}(\zeta)| - \mathrm{Cap}(\mathsf{E}) \le \frac{1}{n} \log \|T_n^{\mathsf{E}}\|_{\mathsf{E}} - \mathrm{Cap}(\mathsf{E}) =: \varepsilon_n$$

where $\varepsilon_n \to 0$, as a result of Theorem 13. From the maximum principle, see e.g. [30, Theorem 3.6.9], we conclude that $h_n(z) \le \varepsilon_n$ for every $z \in \Omega_{\mathsf{E}}$. Since

$$\lim_{z \to \infty} \left(\frac{1}{n} \log |T_n^{\mathsf{E}}(z)| + U^{\mu_{\mathsf{E}}}(z) \right) = 0$$

the symmetry of $G_{\Omega_{\mathsf{E}}}$ provides us with the estimate

$$\varepsilon_n \ge h_n(\infty) = \frac{1}{n} \sum_{k=1}^{m_n} G_{\Omega_{\mathsf{E}}}(\infty, z_{n,k}) = \frac{1}{n} \sum_{k=1}^{m_n} G_{\mathsf{E}}(z_{n,k}) = \int_{\Omega_{\mathsf{E}}} G_{\mathsf{E}}(z) d\nu(T_n^{\mathsf{E}})(z).$$

In particular,

$$\lim_{n \to \infty} \int_{\Omega_{\mathsf{E}}} G_{\mathsf{E}}(z) d\nu(T_n^{\mathsf{E}})(z) = 0. \tag{53}$$

Let A denote any closed subset of Ω_{E}. Since A is compact, there exists some $c > 0$ such that $G_{\mathsf{E}}(z) \geq c$ for all $z \in A$. As a consequence

$$\nu(T_n^{\mathsf{E}})(A) \leq \frac{1}{c}\int_A G_{\mathsf{E}}(z)d\nu(T_n^{\mathsf{E}}) \leq \frac{1}{c}\int_{\Omega_{\mathsf{E}}} G_{\mathsf{E}}(z)d\nu(T_n^{\mathsf{E}})(z).$$

From (53) we gather that $\nu(T_n^{\mathsf{E}})(A) \longrightarrow 0$ as $n \to \infty$. $\square$

If (49) holds then any limit point measure of $\nu(T_n^{\mathsf{E}})$ must be supported on the boundary of the unbounded complement Ω_{E}. The equilibrium measure μ_{E} has this very property and so this is what we expect based on Theorem 25. To conclude the proof we need a certain minimality condition which is a consequence of the strong asymptotics of Chebyshev polynomials outside of the convex hull of E.

Theorem 27. *Let* E *be a compact set with positive capacity. Uniformly on compact subsets of* $\overline{\mathbb{C}} \setminus \operatorname{cvh}(\mathsf{E})$ *it holds that*

$$|T_n^{\mathsf{E}}(z)|^{1/n} \big/ \operatorname{Cap}(\mathsf{E}) \exp G_{\mathsf{E}}(z) \longrightarrow 1 \tag{54}$$

as $n \to \infty$.

Proof. As a consequence of Theorem 23, we conclude that the family of functions $\{h_n\}$ defined by

$$h_n(z) := \frac{1}{n}\log \|T_n^{\mathsf{E}}\|_{\mathsf{E}} + G_{\mathsf{E}}(z) - \frac{1}{n}\log|T_n^{\mathsf{E}}(z)|$$

is a family of harmonic functions on $\mathbb{C} \setminus \operatorname{cvh}(\mathsf{E})$. From (39) together with [30, Corollary 3.6.2] we gather that h_n extends harmonically at infinity with the value

$$h_n(\infty) = \frac{1}{n}\log\|T_n^{\mathsf{E}}\|_{\mathsf{E}} - \log \operatorname{Cap}(\mathsf{E}) = \frac{1}{n}\log \mathscr{W}_n(\mathsf{E}) \geq 0.$$

The inequality is a consequence of Theorem 11. Since h_n extends superharmonically to the unbounded component Ω_{E} of $\mathbb{C} \setminus \mathsf{E}$ and

$$\lim_{z \to \zeta} h_n(z) = \frac{1}{n}\log\|T_n^{\mathsf{E}}\|_{\mathsf{E}} - \frac{1}{n}\log|T_n^{\mathsf{E}}(\zeta)| \geq 0$$

for q.e. $\zeta \in \partial\Omega_{\mathsf{E}}$ we conclude from the extended minimality principle [30, Theorem 3.6.9] that $h_n(z) \geq 0$ for every $z \in \Omega_{\mathsf{E}}$. On the other hand $h_n(\infty) \longrightarrow 0$ due to Theorem 13. The final ingredient is supplied by a variant of Harnack's theorem [30, Theorem 1.3.10] which entails that these conditions are enough to guarantee that $h_n \longrightarrow 0$ locally uniformly. Combining these considerations gives us that

$$\exp(h_n(z)) = \frac{\|T_n^{\mathsf{E}}\|_{\mathsf{E}}^{1/n}\exp(G_{\mathsf{E}}(z))}{|T_n^{\mathsf{E}}(z)|^{1/n}} = \frac{\|T_n^{\mathsf{E}}\|_{\mathsf{E}}^{1/n}}{\operatorname{Cap}(\mathsf{E})}\frac{\operatorname{Cap}(\mathsf{E})\exp(G_{\mathsf{E}}(z))}{|T_n^{\mathsf{E}}(z)|^{1/n}} \longrightarrow 1$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \operatorname{cvh}(\mathsf{E})$. $\square$

While it is not true that $\nu(T_n^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}}$ as $n \to \infty$ for any compact set E with $\mathrm{Cap}(\mathsf{E}) > 0$, as exemplified by $\mathbb{T} = \mathsf{E}$, there is a general "sweeping" procedure which relates any limit point of $\nu(T_n^{\mathsf{E}})$ with μ_{E}. We recall the notation $\mathscr{M}(\mathsf{E})$ which denotes the family of probability measures on E. Again, Ω_{E} denotes the unbounded complement of E. If $\mu \in \mathscr{M}(\mathsf{E})$ then we say that μ^b is the "balayage" of μ to $\partial\Omega_{\mathsf{E}}$ if $\mu^b \in \mathscr{M}(\partial\Omega_{\mathsf{E}})$ and

$$U^{\mu}(z) = U^{\mu^b}(z) \tag{55}$$

for quasi-every $z \in \overline{\Omega}_{\mathsf{E}}$. The measure μ^b defined this way, is the unique measure in $\mathscr{M}(\partial\Omega_{\mathsf{E}})$ satisfying (55) with finite energy $\mathscr{E}(\cdot) < \infty$, see [60, Theorem 2.2]. Recall that μ_{E} always has its support contained in $\partial\Omega_{\mathsf{E}}$.

Returning to the example of the unit circle we know that $U^{\mu_{\mathbb{T}}}(z) = -\log|z|$ for $|z| \geq 1$ and since $U^{\delta_0}(z) = -\log|z|$ for $|z| > 0$ we conclude that $\mu_{\mathbb{T}}$ is the balayage of $\delta_0 = \nu(T_n^{\mathbb{T}})$. This observation can be significantly generalized as we now show, see also [28, Theorem 2.1].

Theorem 28 (Mhaskar and Saff (1991) [60]). *Let* E *denote a polynomially convex compact set with* $\mathrm{Cap}(\mathsf{E}) > 0$. *If* μ_∞ *denotes any limit point of* $\{\nu(T_n^{\mathsf{E}})\}$ *then* μ_∞ *is supported on* E *and for all* $z \in \Omega_{\mathsf{E}}$, *the complement of* $\mathbb{C} \setminus \mathsf{E}$, *it holds that*

$$U^{\mu_\infty}(z) = U^{\mu_{\mathsf{E}}}(z). \tag{56}$$

Proof. From Theorem 26 it immediately follows that any limit point μ_∞ of the sequence $\{\nu(T_n^{\mathsf{E}})\}$ is supported on the set E. As a consequence U^{μ_∞} is harmonic on Ω_{E}. Pick a subsequence n_k such that $\nu(T_{n_k}^{\mathsf{E}}) \xrightarrow{*} \mu_\infty$. Then

$$U^{\mu_\infty}(z) = \lim_{n_k\to\infty} \int_{\mathbb{C}} \log\frac{1}{|z-\zeta|}\, d\nu(T_{n_k}^{\mathsf{E}})(\zeta) = \lim_{n_k\to\infty} -\frac{1}{n_k}\log|T_{n_k}^{\mathsf{E}}(z)|.$$

Theorem 27 therefore implies that in a neighborhood of infinity

$$U^{\mu_\infty}(z) = -G_{\mathsf{E}}(z) - \log\mathrm{Cap}(\mathsf{E}) = U^{\mu_{\mathsf{E}}}(z)$$

and therefore the identity principle for harmonic functions implies that this equality persists on Ω_{E}. □

We are now in a position to finally prove Theorem 25.

Proof of Theorem 25. Let μ_∞ be any limit point of $\nu(T_n^{\mathsf{E}})$. Then Theorem 28 implies that μ_∞ is supported on the polynomially convex hull of E. On the other hand the condition that $\nu(T_n^{\mathsf{E}})(A) \to 0$ for any closed subset on the bounded components of $\mathbb{C} \setminus \partial\Omega_{\mathsf{E}}$ implies that the support of any limit point measure μ_∞ is contained in $\partial\Omega_{\mathsf{E}}$. The regularity (in the sense of potential theory) of $\partial\Omega_{\mathsf{E}}$ together with Theorem 28 and the lower-semicontinuity of potentials gives us that

$$U^{\mu_\infty}(\zeta) \leq \liminf_{z\to\zeta} U^{\mu_\infty}(z) = \liminf_{z\to\zeta} U^{\mu_{\mathsf{E}}}(z) \leq -\log \operatorname{Cap}(\mathsf{E}).$$

Since the support of μ_∞ lies on $\partial\Omega_{\mathsf{E}}$ and μ_∞ necessarily is a probability measure we conclude that

$$\mathscr{E}(\mu_\infty) = \int_{\mathbb{C}} U^{\mu_\infty}(\zeta)\,d\mu_\infty(\zeta) \leq -\log \operatorname{Cap}(\mathsf{E}).$$

The uniqueness of the minimizer μ_{E} for the energy potential now implies that $\mu_{\mathsf{E}} = \mu_\infty$. Since μ_∞ was an arbitrary limit point and $\nu(T_n^{\mathsf{E}})$ is limit point compact as a consequence of the Banach-Alaoglu Theorem, see e.g. [17, Theorem V.3.1], we finally conclude that

$$\nu(T_n^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}}$$

as $n \to \infty$. $\square$

In a completely analogous fashion to how we showed Theorem 26 we can prove the following simplification of [61, Theorem III.4.1] using Theorem 25.

Theorem 29 (Saff and Totik (1997) [61]). *Let* $\mathsf{E} \subset \mathbb{C}$ *be a compact set of positive capacity such that the unbounded component* Ω_{E} *of* $\mathbb{C} \setminus \mathsf{E}$ *has a boundary which is regular (in the sense of potential theory). If every bounded component of* $\mathbb{C} \setminus \partial\Omega_{\mathsf{E}}$ *contains a point* z_0 *such that*

$$\liminf_{n\to\infty} |T_n^{\mathsf{E}}(z_0)|^{1/n} \to \operatorname{Cap}(\mathsf{E}) \tag{57}$$

then $\nu(T_n^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}}$ *as* $n \to \infty$.

Proof. With the intent of applying Theorem 25 we show that the number of zeros on any closed subset of the bounded components of $\mathbb{C} \setminus \mathsf{E}$ is at most $o(n)$ as $n \to \infty$. Let $\widetilde{\Omega}_{\mathsf{E}}$ be a bounded component of $\mathbb{C} \setminus \mathsf{E}$. We define

$$h_n(z) = \frac{1}{n}\log|T_n^{\mathsf{E}}(z)| - \operatorname{Cap}(\mathsf{E}) + \frac{1}{n}\sum_{k=1}^{m_n} G_{\widetilde{\Omega}_{\mathsf{E}}}(z, z_{k,n})$$

where $z_{1,n}, \dots, z_{m_n,n}$ is an enumeration of the zeros of T_n^{E} contained in $\widetilde{\Omega}_{\mathsf{E}}$. We find that for all $\zeta \in \partial\widetilde{\Omega}_{\mathsf{E}}$ it holds that

$$\limsup_{z\to\zeta} h_n(z) = \frac{1}{n}\log|T_n^{\mathsf{E}}(\zeta)| - \operatorname{Cap}(\mathsf{E}) \leq \frac{1}{n}\log\|T_n^{\mathsf{E}}\|_{\mathsf{E}} - \operatorname{Cap}(\mathsf{E}) =: \varepsilon_n$$

where, as a consequence of Theorem 13, $\varepsilon_n \to 0$ as $n \to \infty$. From the maximum principle, we gather that

$$h_n(z) \leq \varepsilon_n$$

for all $z \in \widetilde{\Omega}_{\mathsf{E}}$. By taking $z = z_0$ we find that

$$\frac{1}{n}\sum_{k=1}^{m_n} G_{\widetilde{\Omega}_{\mathsf{E}}}(z_0, z_{k,n}) \leq \varepsilon_n + \mathrm{Cap}(\mathsf{E}) - \frac{1}{n}\log|T_n^{\mathsf{E}}(z_0)| = o(1)$$

as $n \to \infty$. On the other hand

$$\frac{1}{n}\sum_{k=1}^{m_n} G_{\widetilde{\Omega}_{\mathsf{E}}}(z_0, z_{k,n}) = \frac{1}{n}\sum_{k=1}^{m_n} G_{\widetilde{\Omega}_{\mathsf{E}}}(z_{k,n}, z_0) = \int_{\widetilde{\Omega}_{\mathsf{E}}} G_{\widetilde{\Omega}_{\mathsf{E}}}(z, z_0)\,d\nu(T_n^{\mathsf{E}})(z) \to 0$$

as $n \to \infty$. Any closed subset A of $\widetilde{\Omega}_{\mathsf{E}}$ is compact and hence, given such a set, there exists some $c > 0$ such that $G_{\widetilde{\Omega}_{\mathsf{E}}}(z, z_0) \geq c$. It now follows that

$$\begin{aligned}\limsup_{n\to\infty} \nu(T_n^{\mathsf{E}})(A) &\leq \limsup_{n\to\infty} \frac{1}{c}\int_A G_{\widetilde{\Omega}_{\mathsf{E}}}(z, z_0)\,d\nu(T_n^{\mathsf{E}})(z)\\ &\leq \limsup_{n\to\infty} \frac{1}{c}\int_{\widetilde{\Omega}_{\mathsf{E}}} G_{\widetilde{\Omega}_{\mathsf{E}}}(z, z_0)\,d\nu(T_n^{\mathsf{E}})(z) = 0.\end{aligned}$$

Through this chain of reasoning we have verified that the hypothesis of Theorem 25 is satisfied and therefore $\nu(T_n^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}}$ as $n \to \infty$. □

Remark. If a sequence of monic polynomials $\{P_n\}$, where $\deg P_n = n$, satisfies

$$\limsup_{n\to\infty} \|P_n\|_{\mathsf{E}}^{1/n} = \mathrm{Cap}(\mathsf{E})$$

then we say that P_n is *asymptotically extremal* on E, a terminology originating from[61, p. 169]. By Theorem 13, the sequence $\{T_n^{\mathsf{E}}\}$ is asymptotically extremal for any compact set E. Many results on weak-star limits of zero counting measures as in (46) can be phrased in terms of asymptotic extremality and therefore implicitly hold for Chebyshev polynomials. As an example, Theorems 25, 27, and 29 all hold in this extended setting. A generalization of Theorem 25 was shown by Grothmann in [62] and can be found in [58, Theorem 2.1.1].

In general it is not the case that the zeros of T_n^{E} approach the outer boundary of E. In particular this can never happen for closures of analytic Jordan domains.

Theorem 30 (Saff and Totik (1990) [63]). *Let $\mathsf{E} \subset \mathbb{C}$ be a compact set with connected interior and connected complement. There exists a neighborhood U of $\partial\mathsf{E}$ and $N \in \mathbb{N}$ such that*

$$\nu(T_n^{\mathsf{E}})(U) = 0, \quad n \geq N$$

if and only if $\partial\mathsf{E}$ is an analytic Jordan curve.

This entails that the zeros of T_n^{E} stay strictly away from a neighborhood of $\partial\mathsf{E}$ for large n precisely when $\partial\mathsf{E}$ is analytic. The proof in [63] is performed using a comparison with the Faber polynomials which exhibit this very property. A local result on asymptotic zero distributions of Chebyshev polynomials is given in the following.

Theorem 31 (Christiansen, Simon and Zinchenko (2020) [28]). *Let* E *be polynomially convex and* U *an open connected set with connected complement so that* $U \cap \partial\mathsf{E}$ *is a continuous arc that divides* U *into two pieces: one contained in the interior of* E *and one contained in* $\mathbb{C} \setminus \mathsf{E}$*. If*

$$\liminf_{n\to\infty} \nu(T_n^{\mathsf{E}})(U) = 0$$

then $U \cap \partial\mathsf{E}$ *is an analytic arc.*

These considerations conclude our discussion on the background of Chebyshev polynomials and we now advance toward novel results.

4 Summary of research papers

In this section we discuss the work from Papers I to VI as they constitute the main scientific contribution of this thesis. The aim of this section is to place the scientific novelties of the articles within a general framework. Papers I to IV are either published or submitted for peer-review. Paper V is more or less complete and is in the process of proof-reading. Paper VI contains several complete results, however, this is still work in progress.

Paper I – Extremal polynomials and sets of minimal capacity

In [14], Harold Widom performs a detailed study of the limiting behavior of Chebyshev polynomials corresponding to Jordan curves and arcs. As we discussed in Section 3 he completely determines the asymptotics of, what we now call the Widom-factors,

$$\mathcal{W}_n(\mathsf{E}, w) := \frac{t_n(\mathsf{E}, w)}{\mathrm{Cap}(\mathsf{E})^n} \tag{58}$$

in the case where E is a union of smooth Jordan curves and w is a nice enough weight. Based on the example

$$\mathcal{W}_n([a, b]) = 2, \quad a < b$$

he conjectured that the formula (28) should still hold if at least one of the components of E is an arc if one first multiplies the right-hand side by 2. In particular he writes

> Thus M_n (t_n, author's edit) is asymptotically twice as large for an interval as for a closed curve of the same capacity. We conjecture that this is true generally; that is, if at least one of the E_k is an arc then the asymptotic formula for $M_{n,\rho}$ ($t_n(\mathsf{E}, w)$, author's edit), given in Theorem 8.3 ((28) in Section 3, author's edit) must be multiplied by 2...
>
> ...Unfortunately we cannot prove these statements and so they are nothing but conjectures.

Widom shows in [14, Theorem 11.4] that the right-hand side of (28), multiplied with 2, provides an upper bound of any limit point of $\{\mathcal{W}_n(\mathsf{E}, w)\}$. However, he fails to show that the conjectured asymptotic formula holds. In the case of a single smooth arc E, Widom's conjecture would imply that asymptotically

$$\mathcal{W}_n(\mathsf{E}) \sim 2$$

as $n \to \infty$. As we already discussed previously, this conjecture is false as shown by Thiran and Detaille in [31]. In fact, Widom's conjecture fails for almost every arc as was recently shown by G. Alpan [64, Theorem 1.3]. In particular, the following holds.

Theorem 32 (Alpan (2022) [64]). *Let* E *denote a Jordan arc with* C^{2+} *regularity. If there exists an interior point* $z \in \mathsf{E}$ *such that*

$$\frac{\partial G_{\mathsf{E}}}{\partial_{n_+}}(z) \neq \frac{\partial G_{\mathsf{E}}}{\partial_{n_-}}(z) \tag{59}$$

where n_+ *and* n_- *denote the normal directions from each respective side of the arc, then*

$$\limsup_{n\to\infty} \mathcal{W}_n(\mathsf{E}) < 2. \tag{60}$$

If the normal derivatives of Green's function at infinity from both sides of a smooth arc are equal at all interior points, then we say that the arc possesses the *S-property*. This property was initially considered by Stahl in [65, 66]. We adopt a simplified definition, following [67, Definition 2].

Definition 1. *Let* $\mathsf{E} \subset \mathbb{C}$ *be a compact set with* $\mathrm{Cap}(\mathsf{E}) > 0$ *and suppose that* $\mathbb{C} \setminus \mathsf{E}$ *is connected. Assume further that there exists a compact subset* $\mathsf{E}_0 \subset \mathsf{E}$ *with* $\mathrm{Cap}(\mathsf{E}_0) = 0$ *such that*

$$\mathsf{E} \setminus \mathsf{E}_0 = \bigcup_{i\in I} \gamma_i \tag{61}$$

where the γ_i*'s are disjoint open analytic Jordan arcs and* $I \subset \mathbb{N}$*. Then* E *is said to possess the* S*-property if*

$$\frac{\partial G_{\mathsf{E}}}{\partial_{n_+}}(z) = \frac{\partial G_{\mathsf{E}}}{\partial_{n_-}}(z), \tag{62}$$

for all $z \in \bigcup_{i\in I} \gamma_i$.

While Theorem 32 is stated for arcs with C^{2+} regularity, it is not difficult to see that any arc for which (62) is satisfied at all interior points is, in fact, analytic.

The first result we show in Paper 1 is that the S-property of an arc can be rephrased. In particular, the only way that equality can hold in (60) is if E is a straight line segment.

Theorem 33. *Let* E *denote a Jordan arc with* C^{2+} *regularity. Then*

$$\limsup_{n\to\infty} \mathcal{W}_n(\mathsf{E}) \leq 2 \tag{63}$$

and equality holds if and only if E *is a straight line segment.*

Theorem 32 serves as a guiding principle in the further investigations undertaken in Paper 1. Our main inquiry concerns whether there are other examples of sets in the complex plane for which

$$\lim_{n\to\infty} \mathcal{W}_n(\mathsf{E}) = 2 \tag{64}$$

holds. If (62) fails even for one arc then (64) is impossible. We therefore restrict our study to sets satisfying the S-property. As it turns out, this property is intimately tied to minimal capacity conditions. To be precise, the Chebotarev problem – so-called since it was posed as a question from Chebotarev to Pólya [68] – concerns finding the following set.

Problem 4 (Chebotarev). *Given a finite number of points $a_1, \dots, a_m \in \mathbb{C}$, determine the compact connected set E containing these points with minimal logarithmic capacity.*

The non-trivial fact that such a set exists uniquely for any given point configuration $a_1, \dots, a_m$ was established by H. Grötzsch [69]. That such sets necessarily satisfy the S-property was shown by Stahl, see e.g. [70]. In our pursuit of examples of sets satisfying (64) we are thus led to investigate sets of minimal capacity. Typically, the solution to a Chebotarev problem is difficult to attain and one needs to rely on numerical approximations. A notable exception is due to Schiefermayr [71]. He showed that if P is a polynomial and $P^{-1}([-1,1])$ is connected, then this set is a solution to a Chebotarev problem. To be precise, if $a_1, \dots, a_m$ are the simple zeros of $P^2 - 1$ then $P^{-1}([-1,1])$ is the Chebotarev set corresponding to $\{a_1, \dots, a_m\}$. The capacity of this polynomial preimage is easily determined from (37).

This motivates an investigation of the Widom factors corresponding to the sets

$$\mathsf{E}_m = \{z : z^m \in [-2,2]\}. \tag{65}$$

For each fix $m \in \mathbb{N}$, the set E_m is the Chebotarev sets corresponding to $\{2^{1/m} e^{\pi i k/m} : k = 1, \dots, 2m\}$. Our main result in Paper I is the following.

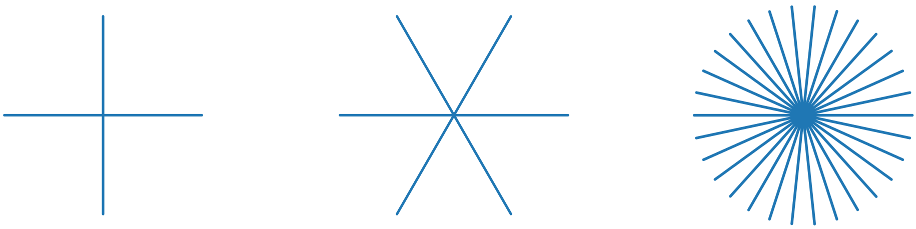


Figure 10: $\{z : z^m \in [-2,2]\}$ for $m = 2$, 3 and 15.

Theorem 34. *For $m \in \mathbb{N}$, let E_m be defined as in (65). Then*

$$\mathscr{W}_{nm}(\mathsf{E}_m) = 2, \quad n \geq 1$$

and

$$\lim_{n \to \infty} \mathscr{W}_n(\mathsf{E}_m) = 2. \tag{66}$$

In addition $n \mapsto \mathscr{W}_{2nm+l}(\mathsf{E}_m)$ is monotonically decreasing for $1 < l/m < 2$.

In short, the method we apply is transferring the domain using a change of variable to the Jacobi weighted Chebyshev polynomials on $[-1,1]$ corresponding to the weights (10). It turns out that Chebyshev polynomials corresponding to Jacobi weights exhibit different monotonicity structures depending on the values of the associated parameters. In fact, this was first observed numerically using a complex extension of the classical Remez algorithm which we discuss extensively in Paper III. The proof of such monotonicity structures eluded us for a long time. Our most comprehensive understanding is presented in Paper IV. A consequence of the main theorem proven there is that if $0 < l/m < 1$ then

$$\sup_n \mathcal{W}_{2nm+l}(\mathsf{E}_m) = 2.$$

It follows that the largest value is attained for $1 < l/m < 2$ and thus we have

$$\mathcal{W}_n(\mathsf{E}_m) \leq 2^{2-1/m}$$

for any n and this value is attained for $n = 2m-1$. By taking m large we can therefore produce Widom factors corresponding to connected sets which are arbitrarily close to 4. This observation led us to investigate which properties characterize sets with large Widom factors in Paper VI.

Determining the asymptotical values of the norms of Chebyshev polynomials corresponding to Jacobi weights on $[-1,1]$ can easily be attained from Theorem 4. However, we should note that, in our experiences, references to these results are hard to come by in the recently published literature. For this reason we provided a modification of the proof of Theorem 4 in the appendix of Paper I. To further emphasize the diverse applications of Theorem 4 we determined the asymptotics of Widom factors associated with quadratic preimages of $[-2,2]$. In particular we obtain the following result.

Theorem 35. *Let $P(z) = z^2 + az + b$ for $a, b \in \mathbb{C}$ and form $\mathsf{E}_P := \{z : P(z) \in [-2,2]\}$. Then*

$$\mathcal{W}_n(\mathsf{E}_P) = 2, \quad n \geq 1 \tag{67}$$

and

$$\lim_{n\to\infty} \mathcal{W}_{2n+1}(\mathsf{E}_P) = \sqrt{2|c+\sqrt{c^2-4}|}, \tag{68}$$

where $c = b - a^2/4$ and $z + \sqrt{z^2-4}$ maps the exterior of $[-2,2]$ to the exterior of the closed disk of radius 2 centered at 0. In particular, for $c \in [-2,2]$ we have

$$\lim_{n\to\infty} \mathcal{W}_n(\mathsf{E}_p) = 2. \tag{69}$$

It follows from [72, Theorem 40] that the preimage $P^{-1}([-2,2])$ is connected if and only if it contains all zeros of P', see also [73]. Theorem 35 thus shows that E_P is connected if and only if

$$\mathcal{W}_n(\mathsf{E}_P) \to 2, \quad n \to \infty.$$

In Paper I, our final investigation addresses a conjecture formulated in [74]. This conjecture builds upon an observation due to G. Alpan and M. Zinchenko from [75]. They showed that Widom factors corresponding to L^2 and L^∞ minimizers can be related on circular arcs. With the notation $\Gamma_\alpha = \{z : |\arg z| \leq \alpha\}$ where $\alpha \in (0, \pi)$ they showed that

$$\mathcal{W}_n(\Gamma_\alpha) \sim \min_{c_k} \int_{\Gamma_\alpha} \left| z^n + \sum_{k=0}^{n-1} c_k z^k \right|^2 d\mu_{\Gamma_\alpha}(z) \Big/ \operatorname{Cap}(\Gamma_\alpha)^{2n}$$

as $n \to \infty$. The conjecture in [74] proposes that this should hold for arcs in much greater generality. Indeed it is conjectured that the particular arc Γ_α could be replaced by any smooth arc Γ.

In Paper I, we verify that the conjecture is also valid for the sets E_m.

Theorem 36. *For $m \in \mathbb{N}$, let E_m be defined as in (65). Then as $n \to \infty$*

$$\mathcal{W}_n(\mathsf{E}_m) \sim \min_{c_k} \int_{\mathsf{E}_m} \left| z^n + \sum_{k=0}^{n-1} c_k z^k \right|^2 d\mu_{\mathsf{E}_m}(z) \Big/ \operatorname{Cap}(\mathsf{E}_m)^{2n} \sim 2.$$

We show this by explicitly computing both asymptotical values and then observing that they are equal.

Paper II – Chebyshev polynomials corresponding to a vanishing weight

We return to discuss weighted Chebyshev polynomials $T_n^{\mathsf{E},w}$, this time focusing on the case where the weight vanishes for some point of the set. Theorem 4 provides a method of determining the asymptotics of $t_n([-1,1], w)$ in the case where

$$w(x) = \prod_{k=1}^{m} |x - b_k|^{\alpha_k}.$$

Here we consider the generalized problem of vanishing weights on the boundary of a Jordan domain and how they affect the asymptotics of the corresponding Chebyshev polynomials. Theorem 8 provides a way of determining the asymptotics of $t_n(\mathsf{E}, w)$ for quite general weights, even those that vanish on boundaries. Lacking is however the asymptotical pointwise behavior on E. To exemplify the case of a vanishing weight we consider the Chebyshev polynomials on the unit circle and assume that the associated weight function is of the form

$$w(z) = \left| \prod_{k=1}^{m} (z - e^{i\alpha_k}) \right|, \tag{70}$$

where $\alpha_k \in [0, 2\pi)$. Initial interest in providing upper bounds for $t_n(\mathbb{T}, w)$ seems to originate with G. Halász. In [76, Lemma p.264] he showed, in relation to Turán's theory of power sums, that there exists, for each positive integer n, a polynomial P_n such that $P_n(0) = 1$, $P_n(1) = 0$ and

$$\|P_n\|_{\mathbb{T}} \le 1 + \frac{2}{n}. \tag{71}$$

Explicitly this polynomial is given by

$$P_n(z) := \sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right)\left(\frac{n-1}{n+1}\right)^j (1 - z^j) \Bigg/ \sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right)\left(\frac{n-1}{n+1}\right)^j. \tag{72}$$

It is clear by the maximum modulus theorem that any polynomial satisfying $Q(0) = 1$ also satisfies $\|Q\|_{\mathbb{T}} \ge 1$, so his results shows that there exists an asymptotically minimizing sequence of polynomials, which also vanishes at a point of $\mathbb{T}$. This is not quite the setting that we have been working in so far but we can easily transform it to a result on monic polynomials. If Q is an arbitrary polynomial of exact degree n then

$$Q^*(z) = \overline{Q(1/\overline{z})} z^n$$

is called the reciprocal of Q and possesses several interesting properties. It is a polynomial of degree at most n and if z_j is a zero of Q then $1/\overline{z_j}$ is a zero of Q^*. The zeros on $\mathbb{T}$ are preserved. Also preserved is the modulus on the unit circle, $|Q(z)| = |Q^*(z)|$ for all $z \in \mathbb{T}$. Finally, Q is a monic polynomial of degree n if and only if $Q^*(0) = 1$. By considering Halász' polynomial, denoted P_n, defined in (74), we conclude from the properties of reciprocals that

$$H_{n-1}(z) := P_n^*(z)/(z-1) \tag{73}$$

defines a monic polynomial of degree $n - 1$ satisfying

$$\max_{z\in\mathbb{T}} |H_{n-1}(z)(z-1)| \le 1 + \frac{2}{n}.$$

Introducing the notation $w(z) = (z - 1)$ we find that

$$t_n(\mathbb{T}, w) \le 1 + \frac{2}{n+1}.$$

As we will see, $O(1/n)$ decay to 1 is optimal in this case. Interestingly, the zeros of Halász polynomials $\{H_n\}$ exhibit very different behavior compared to z^n, the unweighted minimal polynomial on the unit circle. We show this using Theorem 29.

Proposition 1. *Let*

$$H_{n-1}(z)(z-1) = \sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right)\left(\frac{n-1}{n+1}\right)^j (z^j - 1) z^{n-j} \Bigg/ \sum_{j=1}^{n} \left(1 - \frac{j}{n+1}\right)\left(\frac{n-1}{n+1}\right)^j \tag{74}$$

then $\nu(H_{n-1}) \xrightarrow{*} \frac{d\theta}{2\pi}\Big|_{\mathbb{T}}$ *as* $n \to \infty$.

Proof. We write $P_n(z) = H_{n-1}(z)(z-1)$. From (71) we find that $\|P_n\|_{\mathbb{T}} \to 1$ implying that the sequence $\{P_n\}$ is asymptotically extremal. We determine a lower bound for $|P_n(0)|$ in order to show that (57) holds with $z_0 = 0$. Inserting 0 into (74) we find

$$P_n(0) = -\left(\frac{1}{n+1}\right)\left(\frac{n-1}{n+1}\right)^n \Bigg/ \sum_{j=1}^{n}\left(1-\frac{j}{n+1}\right)\left(\frac{n-1}{n+1}\right)^j.$$

Since

$$\sum_{j=1}^{n}\left(1-\frac{j}{n+1}\right)\left(\frac{n-1}{n+1}\right)^j \leq \sum_{j=1}^{n}\left(\frac{n-1}{n+1}\right)^j \leq \sum_{j=1}^{\infty}\left(\frac{n-1}{n+1}\right)^j = \frac{n-1}{2},$$

this gives us that

$$\liminf_{n\to\infty}|P_n(0)|^{1/n} \geq \liminf_{n\to\infty}\left(\frac{1}{n+1}\right)^{1/n}\left(\frac{n-1}{n+1}\right)\left(\frac{2}{n-1}\right)^{1/n} = 1.$$

However, $\mathrm{Cap}(\mathbb{T}) = 1$ and so we see that (57) is satisfied and the result follows from a suitable modification of Theorem 29 to asymptotically extremal sequences. □

As it turns out, something similar happens for the actual minimizer, T_n^w when $w(z) = |z-1|$. We will get back to this in the following discussion.

In [47, Problem 8.2], Halász had posed the related dual problem of determining the extremal value

$$\lambda_n := \max_P |P(0)|$$

among all polynomials of degree n which satisfy $P(1) = 1$ and $\|P\|_{\mathbb{T}} \leq 1$. It's easily seen that the solution is given by

$$\left(\frac{T_n^w w}{t_n(\mathbb{T}, w)}\right)^*$$

and hence

$$\lambda_n = t_n(\mathbb{T}, w)^{-1}.$$

The exact determination of λ_n (and $t_n(\mathbb{T}, w)$) was completely solved by Lachance, Saff and Varga [77]. They showed that

$$\lambda_n = \left(\cos\frac{\pi}{2(n+1)}\right)^{n+1}$$

but their considerations went much deeper. If we introduce the notation

$$w_s(z) = (z-1)^s, \quad s \in \mathbb{N} = \{1, 2, \dots\} \tag{75}$$

then $T_n^{w_s} w_s$ is the unique monic polynomial of degree $n+s$ that has a zero of order s at $z = 1$. Rather than having the perspective of weighted Chebyshev polynomials Lachance, Saff and Varga consider these polynomial as minimal monic polynomial with a prescribed zero of order s at the point 1. In connection to this they also consider polynomials

$$\mathring{T}_n^{w_s}(z) = \prod_{k=1}^{n}(z - e^{i\alpha_k^*}), \quad \alpha_k \in [0, 2\pi)$$

which are minimizers to the same problem however restricted to have all zeros on the unit circle

$$\|w_s\mathring{T}_n^{w_s}\|_{\mathbb{T}} = \min_{\alpha_1,\dots,\alpha_n\in\mathbb{R}} \max_{z\in\mathbb{T}} \left|(z-1)^s \prod_{k=1}^{n}(z - e^{i\alpha_k})\right|.$$

It is not apriori clear if such polynomials are unique. The following remarkable theorem shows how the seemingly unrelated minimization problems associated with $T_n^{w_s}$ and $\mathring{T}_n^{w_s}$ have strong links.

Theorem 37 (Lachance, Saff and Varga (1979) [77]). *Let $s, n \in \mathbb{N}$ then*

$$w_s(z)T_n^{w_s}(z) = \frac{d}{dz}\left(w_{s+1}(z)\mathring{T}_n^{w_{s+1}}\right)\Big/(n+s+1) \tag{76}$$

and

$$\|w_sT_n^{w_s}\|_{\mathbb{T}} = \|w_{s+1}\mathring{T}_n^{w_{s+1}}\|_{\mathbb{T}}\Big/2. \tag{77}$$

It is easy to show that all zeros of $T_n^{w_s}$ lie in $\mathbb{D}$. However, Theorem 37 shows that the primitive function of $w_sT_n^{w_s}$ normalised to have a zero of order $s+1$ at 1 has all its zeros on the unit circle. A consequence of (76) is that $\mathring{T}_n^{w_s}$ is unique for $s \geq 1$ while (77) implies that $\|w_{s+1}\mathring{T}_n^{w_{s+1}}\| \geq 2$. If we further recall the definition of the Jacobi weight

$$w^{(\alpha,\beta)}(x) = (1-x)^\alpha(1+x)^\beta$$

from (10) then there is an interesting relation between Chebyshev polynomials corresponding to the weight w_s on the unit circle and Chebyshev polynomials corresponding to $w^{(\alpha,\beta)}$ on $[-1, 1]$.

Theorem 38 (Lachance, Saff and Varga (1979)[77]). *Let $|z| \leq 1$ and set $x = (z + z^{-1})/2$. For $s, n \in \mathbb{N}$ and $l \in \{0, 1\}$*

$$w_s(z)\mathring{T}_{2n+l}^{w_s}(z) = (-1)^{s/2}(2z)^{n+(s+l)/2}w^{(s/2,l/2)}(x)T_n^{w^{(s/2,l/2)}}(x). \tag{78}$$

Consequently,

$$2t_{2n+l}(\mathbb{T}, w_s) = 2^{n+(s+l+1)/2}t_n([-1,1], w^{((s+1)/2,l/2)}). \tag{79}$$

Using Theorem 4 it is possible to determine the asymptotics of $t_n(\mathbb{T}, w_s)$. Since Lemma 3 implies that

$$\frac{1}{\pi}\int_{-1}^{1}\frac{w^{(\alpha,\beta)}(x)}{\sqrt{1-x^2}}dx = -(\alpha+\beta)\log 2$$

we conclude from (99) that

$$t_n([-1,1], w^{(\alpha,\beta)}(x)) \sim 2^{1-n-(\alpha+\beta)},$$

as $n \to \infty$. Inserting this into (79) one easily concludes that

$$t_n(\mathbb{T}, w_s) \sim 1$$

as $n \to \infty$. In the particular cases where the occurring weights are $w^{(1,0)}$ and $w^{(1,1/2)}$, the polynomials can be explicitly determined.

Using this explicit representation we prove the following result in Paper II.

Theorem 39. *Let $w_1(z) = (z-1)$, then*

$$\nu(T_n^{w_1}) \xrightarrow{*} \frac{d\theta}{2\pi}\bigg|_{\mathbb{T}}$$

as $n \to \infty$.

Consequently the zeros of $T_n^{w_1}$ distribute according to equilibrium measure on the unit circle. The way to prove this is by applying Theorem 29 together with Theorems 37 and 38 in the case where $s = 1$. In essence, this behavior mirrors that of the polynomials constructed by Halász.

A natural question arises: what are the effects when the prescribed zeros are raised to some non-integer power? In other words, what if

$$w(z) = \prod_{k=1}^{m} |z - e^{i\alpha_k}|^{s_k} \tag{80}$$

for $s_k \in (0, \infty)$? Our motivation for this originates with a study of Chebyshev polynomials corresponding to the lemniscatic sets

$$\mathsf{L}_m := \{z : |z^m - 1| = 1\}. \tag{81}$$

These sets represent examples of unions of curves with a self-intersection. In other words, they are not closures of Jordan domains and thus the machinery developed by Widom in [14] does not apply. It is easily seen using symmetry that if $l \in \{0, 1, \dots, m-1\}$ then

$$T_{nm+l}^{\mathsf{L}_m}(z) = z^l Q_n^{\mathsf{L}_m}(z^m) \tag{82}$$

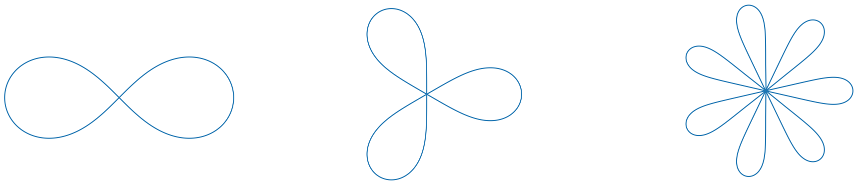


Figure 11: $\{z : |z^m - 1| = 1\}$ for $m = 2, 3, 7$.

for some monic polynomial $Q_n^{\llcorner_m}$ of degree n. By changing the variable to $\zeta = z^m - 1 \in \mathbb{T}$ in (82) we find that

$$T_{nm+l}^{\llcorner_m}(z) = w_{l/m}(\zeta) T_n^{w_{l/m}}(\zeta)$$

which brings us to the case illustrated in (80). In the particular case where $l = 0$ we obtain

$$T_{nm}^{\llcorner_m}(z) = (z^m - 1)^n.$$

We see that all zeros corresponding to such degrees are located at the mth roots of unity. Our interest centers around better understanding the behavior of the Chebyshev polynomials for the remaining degrees, when $0 < l < m$. This leads us to extend Theorems 37 and 38 to the case where s is not necessarily an integer.

The proof of Theorem 37 relies on the Erdős–Lax inequality.

Theorem 40 (Lax (1944) [78]). *If P is a polynomial of degree n which is zero-free on $\mathbb{D}$ then*

$$\|P'\|_{\mathbb{T}} \leq \frac{n}{2}\|P\|_{\mathbb{T}}. \tag{83}$$

Equality holds in (83) *if and only if P has n zeros, counting multiplicity, all situated on $\mathbb{T}$.*

The case of equality in Theorem 40 establishes the connection between Chebyshev problems with unrestricted zeros and those restricted to the unit circle, as discussed in [77]. Initially conjectured by Erdős, the first proof of (83) which was built upon techniques of Szegő and Pólya was given by P. Lax [78], see also [79]. In order to generalize Theorem 37 to the case of weight functions w_s with $s \geq 0$ we need to extend Theorem 40 to the case of non-integer powers. In Paper II we established the following.

Theorem 41. *Let $s_k \geq 1$ and $\theta_k \in [0, 2\pi)$ for $k = 1, \ldots, n$ then*

$$\max_{|z|=1} \left| \frac{d}{dz} \left\{ \prod_{k=1}^{n} (z - e^{i\theta_k})^{s_k} \right\} \right| = \frac{\sum_{k=1}^{n} s_k}{2} \max_{|z|=1} \left| \prod_{k=1}^{n} (z - e^{i\theta_k})^{s_k} \right|.$$

This in turn allows us to establish extensions of Theorems 37 and 38 to the case where the parameter s is any positive real number.

Our initial aim with determining these extensions was the prospect towards determining properties of Chebyshev polynomials corresponding to the lemniscates L_m. One of the consequences is that we can easily determine that

$$\mathscr{W}_n(\mathsf{L}_m) \to 1 \tag{84}$$

as $n \to \infty$ using Theorem 4.

However, we lack a precise description of the zeros. Our initial hope was to establish a generalization of Theorem 39 showing that

$$\nu(T_n^{w_s}) \xrightarrow{*} \frac{d\theta}{2\pi}\bigg|_{\mathbb{T}}$$

as $n \to \infty$ for any $s \geq 0$. One of the implications such a result would have is that if $0 < l < m$ then

$$\nu(T_{nm+l}^{\mathsf{L}_m}) \xrightarrow{*} \mu_{\mathsf{L}_m} \tag{85}$$

as $n \to \infty$. As of yet, we have unfortunately been unable to do so. We find this an intriguing question. If true, this would illustrate radically different behaviors among subsequences of Chebyshev polynomials associated with the same compact set in terms of their zeros. We remark that this is known for the corresponding Faber polynomials, see [80]. To investigate this further, we aim to gain a deeper understanding of Jacobi-weighted Chebyshev polynomials with general parameters. This is the topic of Paper IV.

We end this section by considering several recent results concerning polynomials with prescribed zeros on the boundary of domains. In particular both upper and lower bounds have been provided for $t_n(\mathbb{T}, w)$ where w is as in (70).

Theorem 42 (Totik and Varjú (2007) [81]). *Let w be as in* (70) *with m zeros. There exists a constant $c > 0$ such that for $n > m$*

$$t_{n-m}(\mathbb{T}, w) \geq 1 + c \cdot \frac{m}{n}. \tag{86}$$

Under the assumption that

$$0 \leq \alpha_1 < \ldots \alpha_m < 2\pi \tag{87}$$

and

$$\alpha_{j+1} - \alpha_j \geq (1+\delta)\frac{2\pi}{n} \tag{88}$$

for some $\delta > 0$ there exists a constant D_δ (only depending on δ) such that

$$t_{n-m}(\mathbb{T}, w) \leq 1 + D_\delta\sqrt{\frac{m}{n}} \tag{89}$$

where the constant D_δ only depends on δ.

The upper bound (89) was improved in [82] to show that (86) is sharp.

Theorem 43 (Andrievskii, Blatt (2010) [82]). *Let w be as in* (70) *and assume that $\alpha_1, \ldots, \alpha_m$ satisfy* (87) *and* (88) *for some $\delta > 0$. Then for $n > m$*

$$t_{n-m}(\mathbb{T}, w) \leq 1 + \widetilde{D}_\delta \frac{m}{n} \tag{90}$$

where $\tilde{D}_\delta$ is a constant that only depends on δ.

Andrievskii and Blatt went much further by considering the same problem of prescribing zeros on any analytic Jordan curve. Totik in [83] extended their results to the case of C^2 curves. These results highlight the fact that by prescribing a zero on the boundary of a curve in the form of a vanishing weight, the corresponding Widom factors may decay much slower.

In Paper IV, we returned to the problem of studying Chebyshev polynomials corresponding to Jacobi weights on $[-1, 1]$. Our motivation for this was different than in Paper II since we wanted to discern general monotonicity patters for the corresponding Widom factors. To this end, Theorem 41 proved useful.

Paper III – Computing Chebyshev polynomials using the complex Remez algorithm

Orthogonal polynomials may be efficiently computed using the Arnoldi iteration, a modified version of the classical Gram-Schmidt procedure, see e.g. [84, 85, 86, 87]. In theory this provides an explicit iterative formula for computing minimizing polynomials relative to the L^2 norm. There is no corresponding explicit method to compute L^∞ minimizers such as Chebyshev polynomials except for certain special cases where they can be determined explicitly.

In [88, 89], E. Remez describes an iterative algorithm to compute best approximations in the real setting. As such, his algorithm can be used to compute weighted Chebyshev polynomials relative to real sets. Several proposed generalization exists with the goal of computing best approximations in the complex setting. One robust algorithm which can be shown to converge quadratically given that certain regularity conditions are fulfilled has been developed by P. T. P Tang, B. Fischer and J. Modersitzki, see [90, 91, 92]. The aim of Paper III is to employ this algorithm and illustrate how it can be used to compute Chebyshev polynomials corresponding to a variety of sets. While there are only a handful of theoretical results in Paper III, the focus lies in illustrating connections to other classes of polynomials using numerical experiments. We believe that Tang's algorithm is a useful tool in discerning patterns which can later be turned into proven mathematical statements.

In this regard, use of the algorithm has been tremendously useful in our studies as many of our results have first been suggested by numerical experiments.

With Paper III we mainly investigate three properties related with Chebyshev polynomials in the complex plane.

- The Widom factors,
- A relation between Chebyshev polynomials on growing equipotential lines and Faber polynomials,
- Zero distributions of Chebyshev polynomials.

Widom Factors

Existing results concerning the actual limit values of Widom factors related to closures of Jordan domains all require at least $C^{1+\alpha}$ regularity of the bounding curve. Theorem 9 implies that if E is a $C^{1+\alpha}$ Jordan curve then $\mathcal{W}_n(\mathsf{E}) \to 1$ as $n \to \infty$. In our premier experimental analysis, we wished to investigate what happens when the regularity conditions are loosened. The examples we investigated are the regular polygons, hypocycloids and circular lunes, all of which are examples of piecewise analytic Jordan curves with singularities on their boundaries. Our findings appear to point in the direction that if the corner points are sufficiently mild then the minimal possible limit still seems to hold. That is $\mathcal{W}_n(\mathsf{E}) \to 1$ as $n \to \infty$. With a "mild" corner point we mean a singularity on the boundary curve other than a cusp. Sets such as regular polygons and circular lunes all seem to satisfy minimization of the Widom factors in the limit, with the apparent emergence of monotonicity properties. We therefore conjectured the following.

Conjecture 1. *Let* E *denote the closure of a Jordan domain with piecewise analytic boundary where none of the singularities of* $\partial\mathsf{E}$ *are cusp points. Then*

$$\lim_{n\to\infty} \mathcal{W}_n(\mathsf{E}) = 1. \tag{91}$$

While this result seems to be true based on plenty of numerical experiments there is no clear way in how to show this. The results of Faber [9], Widom [14] and Suetin [32] have all compared the norm of the Chebyshev polynomials to those of the related Faber polynomials. If a corner point appears on the boundary of E, then it is known from [93, Theorem II.2.1] that for the corresponding Faber polynomials $\{F_n^{\mathsf{E}}\}$ defined in (24), it holds that

$$\liminf_{n\to\infty} \|F_n^{\mathsf{E}}\| \Big/ \operatorname{Cap}(\mathsf{E})^n > 1.$$

If we choose to believe that Conjecture 1 is valid then we need to consider some other class of polynomials to show this conjecture.

The reason for not including sets with cusp points in the conjecture comes from the numerical experiments related to the hypocycloids which have an outward pointing cusp. It appears that the decay of $\{\mathcal{W}_n\}$ is much slower in this case and it is not clear if it converges to 1.

A relation to Faber polynomials

In our study of Chebyshev polynomials related to the family of lemniscates

$$\mathsf{L}_m(r) = \{z : |z^m - 1| = r^m\} \tag{92}$$

an interesting pattern emerged upon letting the equipotential value r grow. Letting $F_n^{\mathsf{L}_m}$ denote the associated Faber polynomials (which do not depend on the parameter r) we found, to our surprise, that

$$\lim_{r\to\infty} T_n^{\mathsf{L}_m(r)} = F_n^{\mathsf{L}_m}$$

seemed to hold. It is classically known that

$$T_{nm}^{\mathsf{L}_m(r)}(z) = (z^m - 1)^n = F_{nm}^{\mathsf{L}_m(r)}(z).$$

To the best of our knowledge, in all known cases where the Chebyshev polynomials for a specific set can be explicitly determined, they always coincide with the corresponding Faber polynomials. Using an explicit computation, we can show that

$$T_3^{\mathsf{L}_2(r)} \to F_3^{\mathsf{L}_2}$$

as $r \to \infty$.

We investigated this potential link between Chebyshev polynomials and Faber polynomials on several other domains. For the closure of a Jordan domain E we let $\Phi : \overline{\mathbb{C}}\setminus\mathsf{E} \to \overline{\mathbb{C}}\setminus\overline{\mathbb{D}}$ be the canonical conformal map as in (22). By letting $\mathsf{E}(r) = \{z : |\Phi(z)| = r\}$ we considered the limit points of $T_n^{\mathsf{E}_m(r)}$ as r increases for fixed degrees n. To our surprise, the pattern appeared consistent insofar as

$$\lim_{r\to\infty} T_n^{\mathsf{E}_m(r)} = F_n^{\mathsf{E}}$$

was valid for every set we considered. We suspect that the regularity of the boundary is of lesser importance since $\mathsf{E}(r)$ will always be an analytic curve for $r > 1$ and the corresponding F_n^{E} are independent of the value of r. In Paper III we therefore conjecture the following.

Conjecture 2. *Let* E *denote a connected compact set with simply connected complement and let* $\Phi : \overline{\mathbb{C}} \setminus \mathsf{E} \to \overline{\mathbb{C}} \setminus \overline{\mathbb{D}}$ *denote the conformal map as in* (23). *If* $\mathsf{E}(r) = \{z : |\Phi(z)| = r\}$ *and* $n \in \mathbb{N}$ *then*

$$\lim_{r\to\infty} T_n^{\mathsf{E}_m(r)} = F_n^{\mathsf{E}}. \tag{93}$$

One reason to expect this to be true is that the norms of F_n^{E} relative to the equipotential lines $\mathsf{E}(r)$ approach the minimal value of $\mathrm{Cap}(\mathsf{E}(r))$ as r grows. To be precise,

$$\lim_{r\to\infty} \frac{\|F_n^{\mathsf{E}}\|_{\mathsf{E}(r)}}{\mathrm{Cap}(\mathsf{E}(r))^n} = 1.$$

Presumably this is enough to ensure that for large values of r the polynomials $T_n^{\mathsf{E}(r)}$ and F_n^{E} are measurably close.

Zero distributions

Our final numerical experiments concerns determining the zero placements of Chebyshev polynomials corresponding to compact sets in the complex plane. Motivated by Conjecture 2 we considered sets for which the asymptotic zero distributions of the corresponding Faber polynomials were known. From [94, Theorem 1.5] we gather that if E is the closure of a piecewise analytic Jordan domain with a corner other than an outward cusp then there is a subsequence of degrees n_k such that

$$\nu(F_{n_k}^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}} \tag{94}$$

as $n_k \to \infty$. An outward cusp truly provides an exceptional case as shown in [95], see also [96, 97]. They show that for the m-cusped hypocycloid, all zeros of the corresponding Faber polynomials are situated on the lines connecting the vertices with the origin. For an inward cusp the situation is quite different, and the asymptotic distribution of the zeros Chebyshev polynomials is known. Indeed it was shown in [98] that $\nu(T_n^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}}$ as $n \to \infty$ in this case. Our numerical experiments suggest that if E is the closure of a piecewise analytic Jordan domain with a singularity other than an outward cusp then it appears as though

$$\nu(T_n^{\mathsf{E}})(A) \to 0$$

as $n \to \infty$ for any closed set A contained in the interior of E. Assuming that this is true, Theorem 25 implies that $\nu(T_n^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}}$ as $n \to \infty$ holds. Based on this, we conjecture the following in Paper III.

Conjecture 3. *Let $\mathsf{E} \subset \mathbb{C}$ denote the closure of a Jordan domain with piecewise analytic boundary such that $\partial\mathsf{E}$ has a singularity other than an outward cusp. Then there is a subsequence $\{T_{n_k}^{\mathsf{E}}\}$ such that*

$$\nu(T_{n_k}^{\mathsf{E}}) \xrightarrow{*} \mu_{\mathsf{E}} \tag{95}$$

as $n_k \to \infty$.

Again our main motivation comes from the result due to Kuijlaars and Saff from [94] where they showed that the result holds for Faber polynomials. Our conjectured relation between

Chebyshev polynomials and Faber polynomials serves to motivate why one could expect similar behavior for the corresponding Chebyshev polynomials.

For a set with an outward cusp it appears as though the result is not valid, just as for Faber polynomials. The zeros of Chebyshev polynomials corresponding to m-cusped hypocycloids seem to lie on the lines connecting the vertices with the origin and we conjecture that the same is true for the corresponding Chebyshev polynomials.

We conclude our experimental study by investigating the zeros of Chebyshev polynomials associated with lemniscates of the form $\mathsf{E}(r) = \{z : |P(z)| = r\}$ where P is a monic polynomial of degree m. It is known from Theorem 16 that

$$T_{nm}^{\mathsf{E}(r)} = P(z)^n$$

and so $\nu(T_{nm}^{\mathsf{E}(r)})$ is a fixed discrete measure for all values of n. What is interesting is that $\nu(T_{nm+l}^{\mathsf{E}(r)})$ seem to behave quite different when $0 < l < m$. Indeed, for large r, the zeros of the Chebyshev polynomials of degrees $nm + l$ seem to distribute along some curve in the complex plane as n grows. Numerical evidence from Paper III seem to suggest that this curve is given by $\mathsf{E}(r_0)$ where $r_0 \geq 0$ is the smallest value for which $\mathsf{E}(r)$ is connected. The value r_0 is the largest modulus of any critical value of P. In the case of the family $\{z : |z^m - 1| = r^m\}$, we have $r_0 = 1$. In this case, it is known that the $\nu(F_{nm+l}^{\mathsf{L}_m})$ converges weak-star to equilibrium measure on $\{z : |z^m - 1| = 1\}$ for $0 < l < m$, see [80, 96].

Conjecture 4. *Let P be a polynomial of degree m with largest critical value in terms of absolute value given by c. For any $r \geq |c|$ let*

$$\mathsf{E}(r) = \{z : |P(z)| = r\}.$$

For a fixed $l \in \{1, 2, \dots, m-1\}$

$$\nu(T_{nm+l}^{\mathsf{E}(r)}) \xrightarrow{*} \mu_{\mathsf{E}(|c|)} \tag{96}$$

as $n \to \infty$.

Paper IV – Chebyshev polynomials related to Jacobi weights

Paper IV is dedicated to the analysis of Chebyshev polynomials relative to Jacobi weights. Recall from (10) that we use the notation $w^{(\alpha,\beta)}$ for the weight function

$$w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta}.$$

Such considerations are classical and in the case where $\alpha = \beta \in \{0, 1/2\}$ the minimizers of (3) are given by the Chebyshev polynomials of the 1st to 4th kind. In [12], Bernstein considers the minimizing properties of orthogonal polynomials in terms of the maximum norm. Given a weight function $w_0 : [-1,1] \to [1/M, M]$ where $M \geq 1$ he shows that the minimizing coefficients $c_0^*, \dots, c_{n-1}^*$ satisfying

$$\int_{-1}^{1} \left| x^n + \sum_{k=0}^{n-1} c_k^* x^k \right|^2 \frac{w_0(x)^2}{\sqrt{1-x^2}} dx = \min_{c_k} \int_{-1}^{1} \left| x^n + \sum_{k=0}^{n-1} c_k x^k \right|^2 \frac{w_0(x)^2}{\sqrt{1-x^2}} dx \tag{97}$$

also satisfy

$$\max_{x \in [-1,1]} \left| w_0(x) \left(x^n + \sum_{k=0}^{n-1} c_k^* x^k \right) \right| \sim t_n([-1,1], w_0) \tag{98}$$

as $n \to \infty$. In other words the orthogonal polynomials are also minimal in terms of the maximum norm with respect to a related weight function. These investigations are extended in [24]. There, uniform estimates of T_n^w in terms of the orthogonal polynomials corresponding to the weight $w^2/\sqrt{1-x^2}$ are established under the additional assumption that the weight possesses certain smoothness and does not vanish on $[-1,1]$, see [24, Corollary 2.6].

In [13], Bernstein proceeds to investigate what happens when a zero is added to the weight. To provide an example, he investigates Jacobi weights (10). Following the notation from [99, Chapter IV], we use $P_n^{(\alpha,\beta)}$ to denote the classical Jacobi polynomials. The associated monic Jacobi polynomial is given by

$$\hat{P}_n^{(\alpha,\beta)} := 2^n P_n^{(\alpha,\beta)} \Big/ \binom{2n+\alpha+\beta}{n}.$$

These are the monic polynomials which are orthogonal relative to $w^{(\alpha,\beta)}$. Based on (97) and (98) it is suggested that we should consider $\hat{P}_n^{(2\alpha-1/2,2\beta-1/2)}$ in relation to the weighted maximum norm corresponding to the weight $w^{(\alpha,\beta)}$. Bernstein determines that if $0 \leq \alpha, \beta \leq 1/2$ then

$$\|w^{(\alpha,\beta)} \hat{P}_n^{(2\alpha-1/2,2\beta-1/2)}\|_{[-1,1]} \sim t_n([-1,1], w^{(\alpha,\beta)}) \tag{99}$$

as $n \to \infty$. However, if one of the conditions fails, i.e. $\max\{\alpha, \beta\} > 1/2$, then (99) fails.

Our aim in Paper IV is to provide a more detailed description of the convergence determined in Theorem 4. Namely, in what manner does the convergence

$$\mathcal{W}_n(w^{(\alpha,\beta)}, [-1,1]) \sim 2^{1-\alpha-\beta}$$

as $n \to \infty$ occur. One part of our analysis consists of showing that as $n \to \infty$

$$2^n \|w^{(\alpha,\beta)} \hat{P}_n^{(2\alpha-1/2,2\beta-1/2)}\|_{[-1,1]} \to 2^{1-\alpha-\beta}$$

from below if $0 \leq \alpha, \beta \leq 1/2$. As a consequence, the same follows for $\mathcal{W}_n(w^{(\alpha,\beta)}, [-1,1])$. This is shown using an estimate from [100] on the Jacobi polynomials. In a completely different manner we show convergence from above for the case where $\alpha, \beta \in \{0\} \cup [1/2, \infty)$. Our main result in Paper IV is the following.

Theorem 44. *For any parameters $\alpha, \beta \geq 0$ it holds that*

$$\mathcal{W}_n(w^{(\alpha,\beta)}, [-1,1]) \sim 2^{1-\alpha-\beta}$$

as $n \to \infty$. Furthermore:

1. *If $\alpha, \beta \in \{0, 1/2\}$ the quantity $\mathcal{W}_n(w^{(\alpha,\beta)}, [-1,1])$ is constant.*
2. *If $\alpha, \beta \in [0, 1/2]$ then*

$$\sup_n \mathcal{W}_n(w^{(\alpha,\beta)}, [-1,1]) = 2^{1-\alpha-\beta}.$$

3. *If $\alpha, \beta \in \{0\} \cup [1/2, \infty)$ then*

$$\inf_n \mathcal{W}_n(w^{(\alpha,\beta)}, [-1,1]) = 2^{1-\alpha-\beta}$$

$$\sup_n \mathcal{W}_n(w^{(\alpha,\beta)}, [-1,1]) = \left(\frac{2\alpha}{\alpha+\beta}\right)^{\alpha} \left(\frac{2\beta}{\alpha+\beta}\right)^{\beta}$$

and $\mathcal{W}_n(w^{(\alpha,\beta)}, [-1,1])$ decreases monotonically with n.

The way to prove the different cases uses completely different methods. In the case where $\alpha, \beta \in \{0, 1/2\}$ the Chebyshev polynomials are classically known. If $\alpha, \beta \in [0, 1/2]$ we carefully manipulate an estimate from [100]. If $\alpha, \beta \in \{0\} \cup [1/2, \infty)$ we apply a similar technique used in Paper I to relate the Chebyshev polynomials relative to Jacobi weights with weighted polynomials on the unit circle. The difference now is that we allow for the case of asymmetric weight functions, i.e. $\alpha \neq \beta$.

It would be interesting to consider uniform estimates between $\hat{P}_n^{(2\alpha-1/2, 2\beta-1/2)}$ and $T_n^{w^{(\alpha,\beta)}}$ when $0 \leq \alpha, \beta \leq 1/2$ in the style of [24].

Paper V – Chebyshev polynomials and circular arcs (work in progress)

We turn now to describe work in progress concerning Chebyshev polynomials on circular arcs of the form

$$\Gamma_{\alpha,\beta} := \{z \in \partial\mathbb{D} : \beta \leq |\arg z| \leq \alpha\} \tag{100}$$

for $0 \leq \beta < \alpha \leq \pi$. Note that if $\beta > 0$ the set contains two components which are symmetrically placed with respect to the real line while the case $\beta = 0$ degenerates to a single arc. To ease with notation we let

$$\Gamma_\alpha = \Gamma_{\alpha,0}. \tag{101}$$

Chebyshev polynomials corresponding to circular arcs have been previously studied for the one arc case corresponding to $\beta = 0$ by several authors. The problem originated with [31] and was followed up by considerations in [101, 26, 102]. The case of two arcs, i.e. when $\beta > 0$ has been studied in [46]. For $\beta = 0$ the norm asymptotics of the corresponding Chebyshev polynomials have been completely determined.

Thiran and Detaille [31] established that

$$\mathscr{W}_n(\Gamma_\alpha) \to 2\cos^2\left(\frac{\alpha}{4}\right) \in (1,2) \tag{102}$$

if $\alpha \in (0,\pi)$. The significance of this result lies in the fact that it constitutes a counter example to the conjecture of Widom from [14] stating that if Γ is a sufficiently smooth arc then

$$\mathscr{W}_n(\Gamma) \to 2$$

as $n \to \infty$.

The way that (102) is shown in [31] comes from relating Chebyshev polynomials on the circular arc with weighted Chebyshev polynomials relative to two disjoint intervals.

Theorem 45 (Thiran & Detaille 1991 [31]). *Let* $\Gamma_\alpha = \{z \in \partial\mathbb{D} : -\alpha \leq \arg z \leq \alpha\}$ *and* $\mathsf{E}(a) = [-1,-a] \cup [a,1]$ *with* $a = \cos(\alpha/2)$ *then*

$$\|T_{2n}^{\Gamma_\alpha}\|_{\Gamma_\alpha} = 2^{2n}\|T_{2n+1}^{\mathsf{E}(a)}\|_{\mathsf{E}(a)} \tag{103}$$

$$\|T_{2n+1}^{\Gamma_\alpha}\|_{\Gamma_\alpha} = 2^{2n+1}\|T_{2n+1}^{\mathsf{E}(a),w}w\|_{\mathsf{E}(a)} \tag{104}$$

where $w(x) = \sqrt{1-x^2}$.

The even Chebyshev polynomials $T_{2n}^{\mathsf{E}(a)}$ and $T_{2n}^{\mathsf{E}(a),w}$ are simply rescaled variants of classical Chebyshev polynomials on an interval. The odd Chebyshev polynomials $T_{2n+1}^{\mathsf{E}(a)}$ and $T_{2n+1}^{\mathsf{E}(a),w}$ have representations in terms of Jacobi elliptic functions and theta functions as shown in [10, 103]. For a proof of the representations (103) and (104) we refer the reader to [26]. It is immediate that Theorem 5 can be used to compute the asymptotical values of (103). To compute the asymptotic behavior of (104) an analogous result can be shown using Theorem 4.

Recently, Schiefermayr and Zinchenko [102] showed that $\mathscr{W}_n(\Gamma_\alpha)$ is strictly monotonically increasing. In [46], the same authors related the odd sequence of Chebyshev polynomials on two symmetric disjoint arcs to weighted Chebyshev polynomials relative to disjoint intervals.

Theorem 46 (Schiefermayr & Zinchenko 2022 [46]). *Let* $\Gamma_{\alpha,\beta} = \{z \in \partial\mathbb{D} : \beta \leq |\arg z| \leq \alpha\}$, $0 \leq \beta < \alpha \leq \pi$ *and* $\mathsf{E}(a,b) = [-b-a] \cup [a,b]$ *with* $a = \cos(\alpha/2)$ *and* $b = \cos(\beta/2)$ *then*

$$\|T_{2n+1}^{\Gamma_{\alpha,\beta}}\|_{\Gamma_{\alpha,\beta}} = 2^{2n}\|T_{2n+1,w}^{\mathsf{E}(a,b)}w\|_{\mathsf{E}(a,b)}$$

where $w(x) = \sqrt{1-x^2}$.

They also prove upper bounds on the Widom factors related to $\Gamma_{\alpha,\beta}$.

Theorem 47 (Schiefermayr & Zinchenko 2022 [46]). *Let* $\Gamma_{\alpha,\beta} = \{z \in \partial\mathbb{D} : \beta \leq |\arg z| \leq \alpha\}$, $0 \leq \beta < \alpha \leq \pi$ *and* $n \in \mathbb{N}$ *then*

$$\mathcal{W}_{2n}(\Gamma_{\alpha,\beta}) \leq 2\cos\frac{\alpha-\beta}{4}, \tag{105}$$

and

$$\mathcal{W}_{2n+1}(\Gamma_{\alpha,\beta}) \leq \sqrt{\frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}}\left(1+\cos\frac{\alpha-\beta}{2}\right). \tag{106}$$

They conjecture that (105) is not sharp but should be replaced with

$$\mathcal{W}_{2n}(\Gamma_{\alpha,\beta}) \leq 1+\cos\frac{\alpha-\beta}{2}. \tag{107}$$

Using Theorems 4 and 46, it is actually possible to show that

$$\lim_{n\to\infty}\mathcal{W}_{2n+1}(\Gamma_{\alpha,\beta}) = \sqrt{\frac{\sin\frac{\alpha+\beta}{2}}{\sin\frac{\alpha-\beta}{2}}}\left(1+\cos\frac{\alpha-\beta}{2}\right). \tag{108}$$

The proof is completely analogous to the proof of Theorem 5 so we refrain from providing it. Furthermore, numerical experiments suggest that (107) is valid and that

$$\lim_{n\to\infty}\mathcal{W}_{2n}(\Gamma_{\alpha,\beta}) = 1+\cos\frac{\alpha-\beta}{2}. \tag{109}$$

should hold. However, we have been unable of providing a proof of this.

The main content of Paper v concerns establishing extensions of Theorem 4 to the setting of one circular arc. In particular, we determine the following.

Theorem 48. *Let* $w_0 : \Gamma_\alpha \to [0,\infty)$ *denote a Riemann integrable function with respect to arc-length on* Γ_α, *satisfying* $1/M < w_0(x) < M$ *for all* $x \in \Gamma_\alpha$ *where* $M \geq 1$. *If* $w : \Gamma_\alpha \to [0,\infty)$ *is a weight function of the form*

$$w(x) = w_0(x)\prod_{j=1}^{k}|x-x_j|^{s_j}$$

where $x_j \in \Gamma_\alpha$ *and* $s_j \in \mathbb{R}$, *then*

$$t_n(\Gamma_\alpha, w) \sim 2\cos(\alpha/4)^2\,\mathrm{Cap}(\Gamma_\alpha)^n \exp\left\{\int_{\Gamma_\alpha}\log w(x)\,d\mu_{\Gamma_\alpha}(x)\right\} \tag{110}$$

as $n \to \infty$.

This result should be compared with those in [14] for weighted Chebyshev polynomials in relation to Jordan curves, here presented as Theorem 8. Widom's approach to proving Theorem 8 begins by considering smooth, positive weights. To relax the regularity conditions and allow for zeros in the weight function, he approximates the weight from above. However, this method fails in the case of arcs. Specifically, for an arc Γ with an associated weight function w, the expression

$$\|T_n^w w\|_\Gamma \Big/ \operatorname{Cap}(\Gamma)^n$$

does not asymptotically saturate the lower bound

$$\exp\left\{\int_\Gamma \log w(x) d\mu_\Gamma(x)\right\}.$$

In fact, the results in [44] illustrate that this fails even when w is a constant function. In order to determine (110) a careful analysis is needed. The way we approach this is by first considering residual polynomials relative to the circular arcs Γ_α and weight functions w of the form

$$w(u) = \prod_{j=1}^{k} \frac{1}{|u - u_j|}$$

where $|u_j| > 1$. The residual polynomials are the solutions to the extremal problems

$$\sup\left\{|P(u_0)| : \deg P \le n,\ \|wP\|_{\Gamma_\alpha} \le 1\right\} \tag{111}$$

if $u_0 \in \mathbb{C} \setminus \Gamma_\alpha$ and

$$\sup\left\{\lim_{z\to\infty}\left|\frac{P(z)}{z^n}\right| : \deg P \le n,\ \|wP\|_{\Gamma_\alpha} \le 1\right\} \tag{112}$$

at infinity. It is clear that the solution to (112) is given by $e^{i\phi}T_n^{\Gamma_\alpha,w}/\|wT_n^{\Gamma_\alpha,w}\|_{\Gamma_\alpha}$ for any $\phi \in [0, 2\pi)$. In Paper v we build upon work in [101] where the unweighted case is considered. A detailed account of residual polynomials, detailing existence and uniqueness results can be found in [104].

Similar to the approach in [101] we establish so-called Szegő–Widom asymptotics of the residual polynomials on the domain $\overline{\mathbb{C}} \setminus \Gamma_\alpha$. The main difference between our results and those in [101] is that our formula include the outer function F_w which is analytic in $\overline{\mathbb{C}} \setminus \Gamma_\alpha$ and satisfies $|F_w(u)| = w(u)$ for $u \in \Gamma_\alpha$. Similar to [101] this can be concluded from results in [105] where residual polynomials on $[-1, 1]$ are studied.

Having obtained the limiting behavior of $\{t_n(\Gamma_\alpha, w)\}$ for weights which are given as reciprocals of polynomials we obtain the non-vanishing case of Theorem 48 through an approximation argument using the Stone-Weierstraß Theorem. To allow for zeros of the weight we use a technique similar to Bernstein's in [13]. We emphasize that this is still work in progress and the manuscript may change before submission to reviewers.

Paper VI – Large Widom factors (work in progress)

In Paper I we showed that the minimal capacity sets

$$\mathsf{E}_m = \{z : z^m \in [-2, 2]\}$$

satisfy $\mathcal{W}_n(\mathsf{E}_m) \to 2$ as $n \to \infty$. We also established the upper bound

$$\mathcal{W}_n(\mathsf{E}_m) \le 2^{2-1/m}.$$

This demonstrates that for this particular family of sets, the inequality $\mathcal{W}_n(\mathsf{E}_m) \le 4$ holds. Interestingly, it is possible to approach saturation of the bound 4 by letting $n = m - 1$ and letting m grow. This led us to consider if the Widom factors corresponding to compact connected sets could be upper bounded. The formula (43) defining $\mathcal{W}_n(\mathsf{E})$ suggests that there are two perspectives to take into account if one wants to obtain large Widom factors. One is to make $\|T_n^{\mathsf{E}}\|_{\mathsf{E}}$ large and the other is to make $\mathrm{Cap}(\mathsf{E})$ small. The latter exemplifies the possible relation between large Widom factors and minimal capacity sets. As we already discussed in relation to Paper I, E_m are minimal capacity sets. Their symmetry further "forces" certain zeros of the corresponding Chebyshev polynomials to be placed at the origin. This combination of forced zero placement together with minimal capacity serves as our motivation that they could provide large Widom factors. In Paper VI we show that the sets E_m are extremal in providing large Widom factors certain degrees for symmetric sets. Based on this we conjecture the following.

Conjecture 5. *Let $\mathsf{E} \subset \mathbb{C}$ be an arbitrary non-degenerate continuum. Then*

$$\mathcal{W}_n(\mathsf{E}) \le 4^{1-1/(n+1)}$$

for all $n \ge 1$. In particular, the number 4 serves as a universal upper bound for all Widom factors corresponding to such sets.

This result would show that [47, Problem 4.4] is impossible and provide an explicit upper bound of 4. To investigate this further we consider the case $n = 1$ which states that

$$\mathcal{W}_1(\mathsf{E}) \le 2$$

for any compact connected set with positive capacity. It turns out that this problem is intimately connected to minimal capacity sets. Recalling Problem 4 we let $\mathsf{E}(a_1, \dots, a_m)$ denote the Chebotarev set corresponding to the points $a_1, \dots, a_m$. Then the assertion that $\mathcal{W}_1(\mathsf{E}) \le 2$ for any set compact connected set E with positive capacity is equivalent to the statement that

$$\mathrm{Cap}(\mathsf{E}(-1, e^{i\phi}, e^{-i\psi})) > 1/2, \quad 0 < \phi, \psi \le \pi/2. \tag{113}$$

While this remains a work in progress, numerical evidence strongly suggests the validity of the claim. The Chebotarev problem for two points is trivially solved by the straight line segment connecting them. However, the case of three points already demands significantly more advanced techniques. The solution, involving Jacobi theta functions, is presented in [106, 107]. A thorough analysis of these formulas is crucial for establishing (113).

In Paper VI we also present work concerning Widom factors on Julia sets. In particular, we consider a quasidisk E_c such that $\limsup_{n\to\infty} \mathscr{W}_n(\mathsf{E}_c) > 1$. Let E_c be the Julia set associated with

$$z^2 - c$$

where $-1/4 < c < 3/4$. It follows from [108, Theorem 12.1] that E_c is the closure of a Jordan domain and from [109, §VIII] that this closure is in fact a quasicircle. As such, Theorem 21 implies that $\{\mathscr{W}_n(\mathsf{E}_c)\}$ is a bounded sequence in n. Since

$$\{z : z^2 - c \in \mathsf{E}_c\} = \mathsf{E}_c$$

we conclude from Theorem 16 that

$$T_2^{\mathsf{E}_c}(z) = z^2 - c.$$

Iterating, we can further extend this to

$$T_{2^n}^{\mathsf{E}_c}(z) = \underbrace{T_2^{\mathsf{E}_c} \circ \cdots \circ T_2^{\mathsf{E}_c}}_{n \text{ times}}(z).$$

This result, originating from [42], was extended by Stawiska in [110] to show that if $r \geq 1$ and

$$\mathsf{E}_c(r) := \{z : G_{\mathsf{E}_c}(z) = \log r\}$$

then $T_{2^n}^{\mathsf{E}_c(r)} = T_{2^n}^{\mathsf{E}_c}$, see also [111]. This broadens Faber's result in [9] stating that the Chebyshev polynomials on ellipses with focii at ± 1 coincide with the Chebyshev polynomials of the first kind. It follows from [30, Theorem 6.5.1] that $\mathrm{Cap}(\mathsf{E}_c) = 1$ and that

$$\frac{1}{2} + \sqrt{\frac{1}{4} + c} \in \mathsf{E}_c.$$

If we additionally assume that $c > 0$ then this value also coincides with $t_{2^n}(\mathsf{E}_c)$. Combining the specific form of the norm with the fact that $\mathrm{Cap}(\mathsf{E}_c) = 1$ we find that

$$\mathscr{W}_{2^n}(\mathsf{E}_c) = \frac{1}{2} + \sqrt{\frac{1}{4} + c} > 1$$

for any n. Interestingly, this example shows the existence of a compact set E which is the closure of a Jordan domain such that

$$\limsup_{n\to\infty} \mathscr{W}_n(\mathsf{E}_c) \geq \frac{1}{2} + \sqrt{\frac{1}{4} + c} > 1. \tag{114}$$

It is an open problem whether

$$\mathscr{W}_n(\mathsf{E}_c) \geq \frac{1}{2} + \sqrt{\frac{1}{4} + c}$$

can be shown for every n. In a different phrasing, we question if the Julia set E_c can be mapped by a monic polynomial inside the disk of radius $\frac{1}{2} + \sqrt{\frac{1}{4} + c}$. In Paper VI we show that this is impossible for degrees of the form $n = k2^m$ where $k = 3$ and $m \in \mathbb{N}$. Similar proofs work for $k = 5, 7$.

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Scientific publications

Author contributions

Co-authors are abbreviated as follows:
Alex Bergman (AB), Jacob Stordal Christiansen (JSC), Benjamin Eichinger (BE) and Maxim Zinchenko (MZ).

Paper I: Extremal polynomials and sets of minimal capacity

I took part in the formulation of the research problem and in its solution together with JSC and BE. I also took part in deducing consequences our solution had. I wrote the first draft of the paper which was later modified by BE and JSC. BE related the problem to Chebyshev polynomials corresponding to Jacobi weights and JSC pointed out how Bernstein's results in [12, 13] could be applied in the computation of the asymptotic formulas of the corresponding Widom factors.

Paper II: Chebyshev polynomials corresponding to a vanishing weight

I suggested the problem to AB and provided a major part of the contributions made in Section 3 in relation to Chebyshev polynomials. AB did the majority of the work in proving Theorem 4 which was instrumental in the generalizations we considered. JSC suggested that Theorem 3 could be solved using explicit representations of classical Chebyshev polynomials.

Paper III: Computing Chebyshev polynomials using the complex Remez algorithm

I was the sole author of this paper. JSC offered valuable suggestions, and Frank Wikström provided insightful comments on the presentation of the numerical experiments.

Paper iv: Chebyshev polynomials related to Jacobi weights

I took part in proposing the problem and contributed to the majority of its solution as well as writing of the manuscript. JSC contributed with supervision and additional writing. The method used is an extension of the ideas used in Paper ii.

Paper v: Chebyshev polynomials and circular arcs

Work in progress. The specifics of the mathematical solution was carried out by BE and myself. JSC and MZ provided key insight into how the result could be obtained and took part in the formulation of the problem. I did most of the writing of the paper.

Paper vi: Large Widom factors

Work in progress. JSC wrote the paper based on our joint analysis. I further contributed with the numerical computations used in the paper. These serve as the experimental evidence toward the propsed conjectures.

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