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On the spectrum of (non)magnetic Schrödinger operators

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On the spectrum of (non)magnetic Schrödinger operators

On the spectrum of (non)magnetic Schrödinger operators

by Germán Miranda



LUND
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Thesis for the degree of Doctor of Technology

Thesis advisor: Docent Mikael Persson Sundqvist

Co-advisor: Docent Magnus Goffeng

Faculty opponent: Docent Jonathan Rohleder

To be presented, with the permission of the Faculty of Engineering of Lund University, for public criticism in
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MADE IN SWEDEN 

A mi madre Carmen y a mi padre Pablo

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List of publications

This thesis is based on the following publications and work in progress, referred to by their Roman numerals:

- I **Non-monotonicity of the first eigenvalue for the 3D magnetic Robin Laplacian**
G. Miranda
Archiv der Mathematik (Basel), 120 (2023), no. 6, 643–649
- II **Discrete spectrum of the magnetic Laplacian on almost flat magnetic barriers**
G. Miranda
Journal of Mathematical Physics, 65 (2024), no. 7, Paper No. 072101, 28 pp.
- III **The magnetic Laplacian on the disc for strong magnetic fields**
A. Kachmar and G. Miranda
Journal of Mathematical Analysis and Applications, 546. (2025), no. 2, Paper No. 129261.
- IV **Reproving Friedlander’s inequality with the de Rham complex**
M. Fries, M. Goffeng and G. Miranda
arXiv, 2412.03369 (2024) (to appear in Journal of Spectral Theory).

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Popular summary in English

A physical theory describing phenomena in the real world is usually based on a physical law which captures the evolution of the system and the experimental results. Typically, such laws involve a mathematical description of the physical quantities and the relations between them. In this thesis, we are interested in the mathematical models coming from superconductivity and wave theory.

In 1908 Kamerlingh Onnes was able to obtain liquid helium at a temperature of 1.5 K, the coldest temperature reached until that moment. Later, in 1911, he discovered superconductivity when using liquid helium to cool down mercury which, below a critical temperature, exhibited a new thermodynamic state. Superconductors are metals and alloys which under a critical temperature T_C present properties such as no electrical resistance and expulsion of a weak enough applied magnetic field, known as the Meissner effect. Superconductivity has been intensively studied since its discovery due to its wide application range from maglev trains to MRI to quantum computing. However, the highest critical temperature of a superconductor to this day is 139 K, about $-134\text{ }^\circ\text{C}$, which is still far from room temperature. This produces a problem in the implementation of superconductors since cryogenic refrigeration is needed.

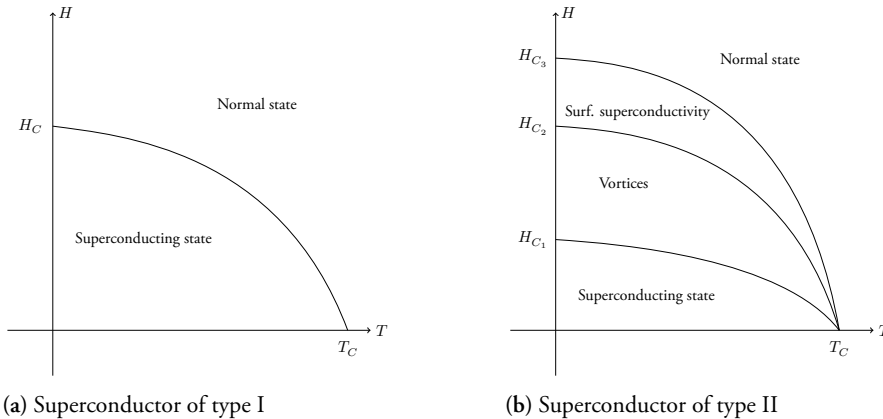


Figure 1: Sketch of the transitions between the superconducting and normal state for isolated superconductors of type I and II under a constant applied magnetic field. We denote the temperature by T and the applied magnetic field by H .

If we apply an external magnetic field to a superconductor sample, then a supercurrent is induced in the sample to compensate the applied magnetic field. Such current will flow without any voltage appearing. Because of this, it is natural to study superconductors under applied magnetic fields of varying strength. It is observed that under a strong enough applied magnetic field H_C , called critical field, the superconductor state is lost. This critical

field depends on the temperature, type of material and shape of the sample. Moreover, the transition between superconductor and normal state is global (all the sample passes from superconductor to normal state) in so-called Type I superconductors or partial in Type II superconductors.

The problems we study in this thesis are motivated by Type II superconductors under a constant or piecewise constant magnetic field. For example, if we consider a constant applied magnetic field and increase its strength then we reach a critical field H_{C_1} where the magnetic field penetrates the bulk creating vortices. If we keep increasing the magnetic field strength up to a critical field H_{C_2} , the superconducting state is lost in the bulk, but it remains in a very thin layer close to the surface of the sample. This state is called surface superconductivity. The localization of the superconducting state at the boundary can be modelled by spectral properties of one of the differential operators studied in this thesis: the magnetic Laplacian. Finally, if we reach an even higher critical magnetic field H_{C_3} , the superconductivity is lost in the sample and we return to a normal state.

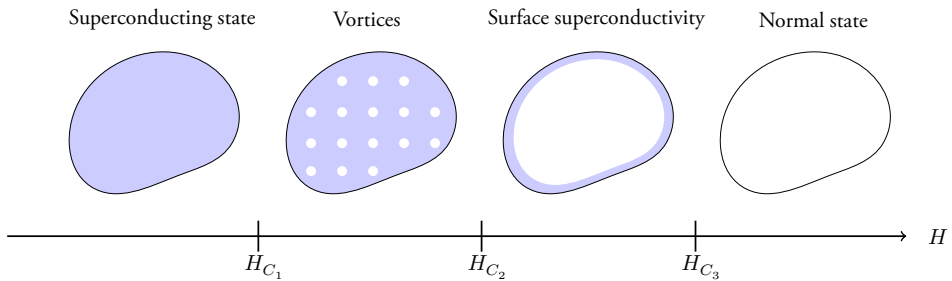


Figure 2: Representation of the different states a superconductor of type II goes through when increasing the strength of a constant applied magnetic field at constant temperature.

Considering different materials adjacent to the superconductor sample leads to treating spectral problems for the magnetic Laplacian with different boundary conditions. Moreover, the curvature of the boundary turns out to play a crucial role in surface superconductivity. Due to these reasons, we consider different boundary conditions of the magnetic Laplacian as well as different geometries of the samples.

Typically, the bottom of the spectrum of the magnetic Laplacian in the context of surface superconductivity is monotone with respect to the magnetic field strength, at least for strong fields. However, considering a particular boundary condition we were able to provide, to our knowledge, the first counterexample in three dimensions where this relation is not monotone.

Secondly, we are also interested in a problem in classical mechanics: a vibrating string or a vibrating drum. Both of these physical phenomena are governed by the wave equation. After a suitable change of variables one can see that the study reduces to the understanding

of the spectrum of another differential operator called the Laplacian. Fixing or leaving free the endpoints of our vibrating string or drum corresponds to considering a different boundary condition.

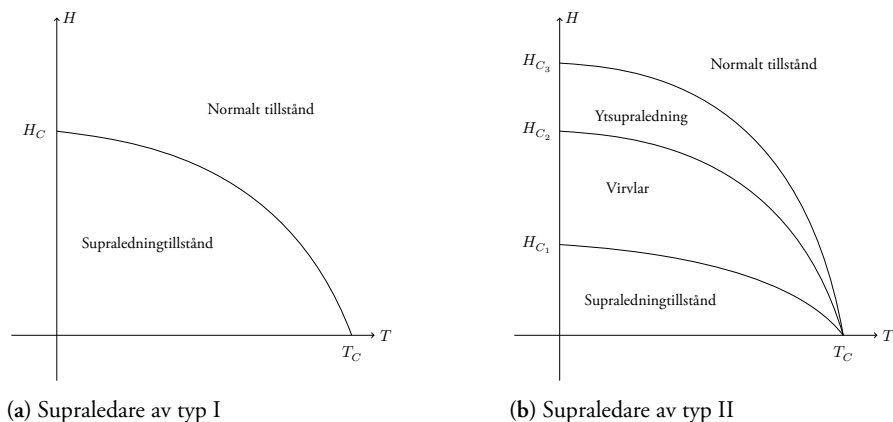
The relation between the spectrum of the Laplacian with different boundary condition has been intensively studied. This thesis provides additional insight on how to tackle these problems by using a different mathematical tool known as the de Rham complex.

Our study of these two models relies on getting a better understanding of the spectrum of differential operators which are, in particular, Schrödinger operators. Even though the two models differ from each other, most of the mathematical techniques used in this thesis are very similar.

Populärvetenskaplig sammanfattning på svenska

En fysikalisk teori som beskriver fenomen i den verkliga världen är oftast baserad på en fysikalisk lag som beskriver systemets utveckling och de experimentella resultaten. Sådana lagar involverar vanligen en matematisk beskrivning av fysikaliska storheter och förhållandet mellan dessa. I den här avhandlingen är vi intresserade av de matematiska modeller som härstammar från supraledning och teorierna för vågor.

År 1908 kunde Kamerlingh Onnes framställa flytande helium genom kondensering vid en temperatur på 1.5 K – den kallaste temperaturen som uppnåtts ditintills. Senare, år 1911 upptäckte han supraledning då han använde flytande helium för att kyla ner kvicksilver som, under en kritisk temperatur, uppvisade ett nytt termodynamiskt tillstånd. Supraledare är metaller och legeringar som under en kritisk temperatur T_C uppvisar egenskaper som ingen elektrisk resistens och utstrålning av ett svagt magnetiskt fält, känt som Meissner-effekten. Supraledning har studerats intensivt sedan dess upptäckt på grund av dess breda tillämpningar, till exempel maglevtåg, MRT och kvantumdatorer. Dock, den hittills högsta kritiska temperaturen för en supraledare är 139 K, eller ungefär $-134\text{ }^\circ\text{C}$, vilket är långt ifrån rumstemperatur. Detta medför ett problem i implementeringen av supraledare på grund av behovet av kryogen kylning.

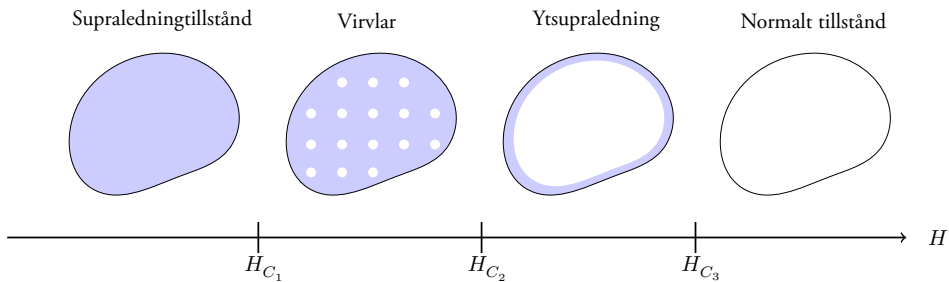


Figur 3: Skiss av övergångarna mellan supraledande och normalt tillstånd för isolerade supraledare av typ I och II under ett konstant yttre magnetfält. Vi betecknar temperaturen med T och det yttre magnetfältet med H .

Om vi applicerar ett externt magnetfält på ett supraledarprov, introduceras en superström i materialet för att kompensera för det applicerade magnetfältet. Denna ström kommer att flöda utan att någon spänning uppstår. På grund av detta är det naturligt att studera supraledare under applicerade magnetfält med varierande styrka. Det har påvisats att under ett tillräckligt starkt applicerat magnetfält, känt som kritiskt fält H_C , förloras supraledartill-

ståndet. Detta kritiska fält beror på temperaturen samt på provets materialtyp och form. Övergången mellan supraledartillståndet och det normala tillståndet global (hela provet går från supraledartillståndet till det normala tillståndet) i så kallade Typ I-supraledare, eller partiell i Typ II-supraledare.

De problem vi studerar i denna avhandling är motiverade av Typ II supraledare under ett konstant eller styckvis konstant magnetisk fält. Om vi till exempel betraktar ett konstant applicerat magnetfält och ökar dess styrka når vi ett kritiskt fält H_{C_1} , där magnetfältet tränger in i materialet och skapar virvlar. Om vi fortsätter att öka magnetfältets styrka upp till ett kritiskt fält H_{C_2} , förloras det supraledande tillståndet i bulkmaterialet, men kvarstår i ett mycket tunt lager nära provets yta. Detta tillstånd kallas ytsupraledning, och hur det supraledande tillståndet lokaliseras vid gränsen kan modelleras genom de spektrala egenskaperna hos en av de differentialoperatorer som studeras i denna avhandling – den magnetiska Laplaceoperatorn. Slutligen, om vi når ett ännu högre kritiskt magnetfält H_{C_3} , förloras supraledningen i provet och vi återgår till ett normalt tillstånd.



Figur 4: Representation av de olika tillstånd en supraledare av typ II genomgår när styrkan hos ett konstant yttre magnetfält ökas vid konstant temperatur.

Att beakta angränsande material till supraledarprovet leder till att man behandlar olika randvillkor för den magnetiska Laplaceoperatorn. Dessutom visar sig randens krökning spela en avgörande roll i ytsupraledningen. På grund av detta studerar vi olika randvillkor för den magnetiska Laplaceoperatorn samt olika geometrier för proverna.

Vanligtvis är botten av spektrumet för magnetiska Laplaceoperatorer, som uppstår i samband med ytsupraledning, monoton med avseende på styrkan hos magnetfältet. Genom att betrakta ett särskilt randvillkor kan vi emellertid, så vitt vi vet, ge ett av de första motexemplen i tre dimensioner där denna relation inte är monoton.

Utöver det, fokuserar vi på klassisk mekanik – en vibrerande sträng eller en vibrerande trumma. Båda dessa fysikaliska fenomen styrs av vågekvationen. Efter ett lämpligt byte av variabler reduceras problemet till att förstå spektrumet för en annan differentialoperator som kallas Laplaceoperatorn. I denna avhandling är vi också intresserade av hur de vibrerande lägena skiljer sig åt beroende på om vi betraktar fasta ändpunkter i vårt vibrerande system

eller om vi lämnar dem fria.

Relationen mellan spektrumet för Laplaceoperatorn med olika randvillkor har studerats ingående. Denna avhandling ger ytterligare insikter i hur dessa problem kan angripas med hjälp av ett matematiskt geometriskt verktyg känt som de Rhamkomplexet.

Vår studie av dessa två modeller syftar till att få en bättre förståelse för spektrumet av differentialoperatorer. Även om de två modellerna skiljer sig åt, är de flesta av de tekniker som används i denna avhandling väldigt lika.

On the spectrum of (non)magnetic Schrödinger operators

The aim of this summary is to present and motivate the physical and mathematical concepts contained in the papers constituting this thesis. In Chapter 1 we give an introduction to the two main physical phenomena involved in this thesis: superconductivity and wave theory. In Chapter 1.1 we describe how the study of Type II superconductors under strong applied magnetic fields relates to the spectral properties of the magnetic Laplacian. Similarly, in Chapter 1.2 we explain the connection between the vibrating properties of a string, or a drum, with the spectrum of the Laplacian.

In Chapter 2 we introduce the mathematical framework needed to study the spectral properties of the Laplacian and magnetic Laplacian which are two examples of linear unbounded operators. We start by giving an overview of the tools required to study the spectrum of linear unbounded operators. In particular, we introduce results such as the min-max principle, Temple's inequality and Persson's lemma. We finish this chapter by focusing on a particular type of linear differential operators known as Sturm–Liouville operators. These one-dimensional operators appear in this thesis after applying a fiber direct integral decomposition (introduced in Chapter 3) for the magnetic Laplacian introduced in Chapter 1. We describe in further detail two examples of Sturm–Liouville operators such as the harmonic oscillator on the half-line and band functions for magnetic steps. These two operators appear as modal operators in Paper II and III.

In Chapter 3 we present an important tool which allows us to work with the aforementioned modal operator known as constant fiber direct integral decomposition. We give a more detail overview of this concept since, to the author's knowledge, this has not been done in the literature. Moreover, we give two examples on how we can connect the magnetic Laplacian with different magnetic fields with two different modal operators. This is done using Fourier series and Fourier transform in the appropriate variable.

Lastly, in Chapter 4 we provide an introduction to the main tool used in Paper IV: the de Rham complex. We give a quick overview of the key concepts in smooth manifolds and multilinear algebra in order to introduce later differential forms and the Hodge–Laplacian. We finish introducing the de Rham complex for compact smooth manifolds.

I Introduction

1.1 Superconductivity and Ginzburg–Landau theory

There are materials which when cooled down under a critical temperature T_C pass from a normal state to a new thermodynamic state known as *superconducting state*. The materials in this new superconducting state are called *superconductors*. One of the striking properties of superconductors is that they have zero electrical resistance. As a consequence, one can

observe an electric current flowing in the superconductor without any voltage appearing. This means that the current can persist for long periods of time (experiments have shown such current to persist over a year). We call these permanent currents *supercurrents*.

The first superconductor was discovered by Kamerlingh Onnes in 1911 when cooling down mercury below 4.2 K, he observed that the electric resistance of the material dropped to zero. Many metals can become superconductors and adding impurities, which typically increase electric resistance in the normal state, does not affect the metal in becoming superconductor. However, alloying can affect the electrical properties of superconductors.

A way to generate a supercurrent is applying an external magnetic field¹ H to a superconductor. Therefore, it is very convenient to study superconductors under applied magnetic fields in order to understand properties of the supercurrent. When we consider superconductors under external applied magnetic fields, they exhibit another striking property known as the Meissner effect. For a weak enough applied magnetic field the superconductor is at an equilibrium state where there is an expulsion of the magnetic flux from the bulk of the superconductor. This phenomenon was shown by Meissner and Ochsenfeld in 1933 [MO33], where they also showed that the superconducting state is a true thermodynamic equilibrium state since the Meissner effect is independent of first applying the magnetic field or lowering the sample below the critical temperature.

Experimental results (see [Liv63] for lead and some of its alloys) show that when the applied magnetic field reaches a critical value, which we denote by H_C , then it penetrates in the bulk of the sample switching from the superconducting state to the normal state. The shape of the superconductor has a big influence in its magnetic behaviour. If we consider a sphere or a long ellipsoid the flux density in the different regions is very different in the two cases. In the sphere we have that the flux density is bigger at the equator than in other regions since the magnetic lines get closer there. Because of this, when the applied magnetic field is strong we have that some magnetic flux has to penetrate in the bulk of the sphere. This breaks up the sphere into alternating regions of superconducting and normal states, this new state is known as *intermediate state*. However, this phenomenon does not occur in long ellipsoids and cylinders where the flux is approximately the same as in the distant field. In this case, the behaviour differs if we have a Type I or Type II superconductor. In Type I superconductors the applied magnetic field fully penetrates the samples as soon as the critical field H_C is reached. Most of the pure metals (except vanadium and niobium) which can become superconductors are Type I superconductors. For Type II superconductors, the Meissner effect is only observed, in the case of a long ellipsoid or an infinite cylinder, for applied magnetic fields with strength $H < H_{C_1}$, where H_{C_1} is called the *first critical field*. For $H > H_{C_1}$, it is convenient to introduce the second and third critical fields H_{C_2} and H_{C_3} . For $H > H_{C_1}$ we have the following two situations:

¹In most cases we are considering a uniform applied magnetic field.

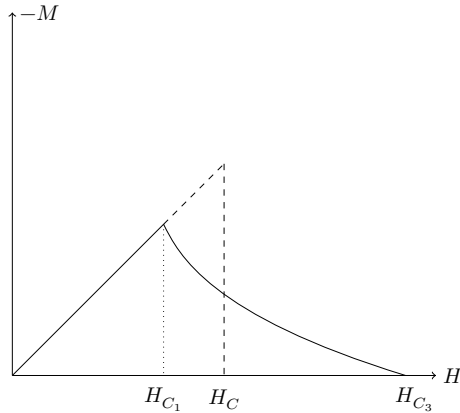


Figure 5: Magnetization (physical quantity measuring the strength of the induced magnetic field in the superconductor) with respect to the applied magnetic field H for a Type I superconductor (dashed curve) and a Type II superconductor (solid curve).

- $H_{C_1} < H < H_{C_2}$. The Vortex (or Schubnikov) phase starts when $H > H_{C_1}$, then the applied magnetic field is strong enough so the magnetic flux penetrates the bulk of the superconductor through vortices, but the normal state is not restored completely in the bulk of the sample. This mixed state was first observed by Schubnikov in 1937 in experiments with alloys. We also refer to the influential work of Abrikosov about this phase [Abr57].
- $H_{C_2} < H < H_{C_3}$. This phase is known as surface superconductivity. The superconducting state persists as a thin sheath (typical thickness $\sim 1000 \text{ \AA}$) on the surface, the rest of the bulk is in normal state. In most cases $H_{C_3} \sim 1.69H_{C_2}$ [DG18].

Most Type II superconductors are alloys and present higher critical temperatures and critical fields than Type I superconductors. These properties make them more interesting for applications. For example, strong superconducting magnets used in MRI machines typically are made with Type II superconductors.

The transition between normal and superconducting state is a virtually perfect second-order phase transition meaning that there is no latent heat in the phase transition and a sharp finite discontinuity in the specific heat [Til19]. In 1935 Fritz and Heinz London developed a theory trying to explain superconductivity [LL35]. This model was able to explain the Meissner effect, but it is not valid when considering strong magnetic fields or when the source of the applied magnetic field is a coil or if the order parameter is far from being constant in space [DG18]. In 1950 Ginzburg and Landau introduced a phenomenological model to describe superconductivity which can be found in the collection of articles [GL65]. Previously,

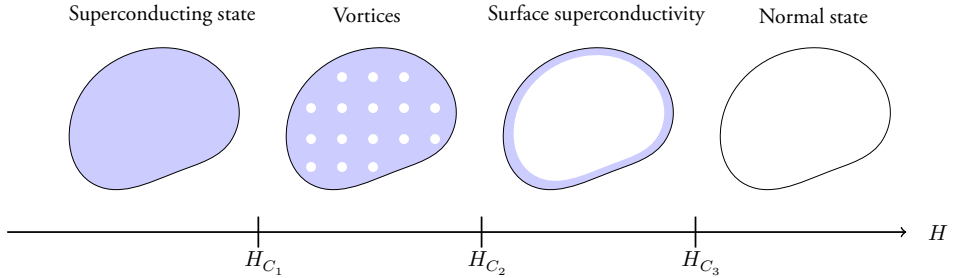


Figure 6: Distribution of superconductivity on the cross-section of a cylindrical sample of a Type II superconductor under a constant applied magnetic field parallel to the axis of the cylinder.

Landau had developed a general theory for second-order phase transitions characterizing the phase transition with an order parameter ψ . This approach does not always work, but it is valid for superconductors. The Ginzburg–Landau theory is a macroscopic theory which has proven to be very effective to understand and predict the behaviour of superconductors.

The microscopic theory of superconductivity appeared later than the Ginzburg–Landau theory and started in the fundamental paper [BCS57] by Bardeen, Cooper and Schrieffer in 1957. The microscopic theory of superconductivity is generally known as the BCS theory. The main idea of this theory is that superconductivity arises from an attractive interaction between the electrons via the positive ions of the metallic lattice. This interaction binds the electron in pairs which are called Cooper pairs. This binding energy is the one allowing the superconducting state to be energy favourable for the system. In 1959 Gorkov showed [Gor59] that in certain regimes of temperature and magnetic field strength, the Ginzburg–Landau theory can be derived from the BCS theory. Moreover, he gave a physical meaning to $|\psi|^2$ being the local density of Cooper pairs. We refer to [FHSS12] for a partial but rigorous derivation.

In this thesis, we will focus on the mathematical models coming from Ginzburg–Landau theory, which is an effective model to describe superconductors near its critical temperature. After a suitable rescaling similar as in [SS07, Equation 2.2] we have that the Gibbs energy of the superconductor near T_C is given by the Ginzburg–Landau functional

$$\mathcal{G}_{\kappa,H}(\psi, A) = \int_{\Omega} (|(-i\nabla - \kappa HA)\psi|^2 - \kappa^2|\psi|^2 + \frac{\kappa^2}{2}|\psi|^4) dx + (\kappa H)^2 \int_{\mathbb{R}^n} |\text{curl } A - \beta|^2 dx, \quad (1)$$

where $\Omega \subset \mathbb{R}^n$ with $n = 2, 3$ is an open and bounded domain, A is the magnetic potential, κ is a material parameter depending on the temperature and H denotes the applied magnetic field strength. Note that after the rescaling we denote by β the profile of the applied magnetic field in this setup.

We focus on the case $n = 2$ and refer to [FH10] for the case $n = 3$. Let $\Omega \subset \mathbb{R}^2$ be the cross-section of a cylindrical sample and consider an applied magnetic field which is parallel to the cylinder axis. If we assume that Ω is simply connected then the three dimensional setup can be reduced to study the Ginzburg–Landau functional in two dimensions [FH10, Section 10.5] given by

$$\mathcal{G}_{\kappa,H}(\psi, A) = \int_{\Omega} (|(-i\nabla - \kappa HA)\psi|^2 - \kappa^2 |\psi|^2 + \frac{\kappa^2}{2} |\psi|^4) dx + (\kappa H)^2 \int_{\Omega} |\operatorname{curl} A - \beta|^2 dx. \quad (2)$$

We are interested in finding minimizers $(\psi, A) \in H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ of this functional because they correspond to thermodynamic equilibrium states of the sample. The material parameter κ determines if the superconductor is Type I if $\kappa < 1/\sqrt{2}$ or Type II if $\kappa > 1/\sqrt{2}$ (see [Abr57]). In this thesis we are interested in extreme Type II superconductors corresponding to the London limit $\kappa \rightarrow +\infty$ which are in surface superconductivity phase. For the vortex phase please see [SS07] and references therein.

The quantities $|(-i\nabla - \kappa HA)\psi|$, $|\psi|$ and $\operatorname{curl} A$ in (2) are invariant under the Gauge transformation $(\psi, A) \mapsto (e^{i\varphi\kappa H}\psi, A + \nabla\varphi)$ for any $\varphi \in H^2(\Omega, \mathbb{R})$, then by [FH10, Section D.1] we can use this Gauge invariance to restrict the minimization problem from $H^1(\Omega, \mathbb{C}) \times H^1(\Omega, \mathbb{R}^2)$ to $H^1(\Omega, \mathbb{C}) \times H_{\operatorname{div}}^1(\Omega)$ with

$$H_{\operatorname{div}}^1(\Omega) := \{v = (v_1, v_2) \in H^1(\Omega, \mathbb{R}^2) : \operatorname{div} v = 0 \text{ in } \Omega, v \cdot \nu = 0 \text{ on } \partial\Omega\}, \quad (3)$$

where ν is the outward normal vector of $\partial\Omega$. Since Ω is bounded, the existence of a minimizer of $\mathcal{G}_{\kappa,H}(\psi, A)$ is rather standard (see [FH10, Theorem 10.2.1] for smooth domains). If the superconductor is surrounded by an insulator, a minimizer of $\mathcal{G}_{\kappa,H}(\psi, A)$ is a weak solution of the Euler–Lagrange equations given by

$$\begin{cases} (-i\nabla - \kappa HA)^2 \psi = \kappa^2 (1 - |\psi|^2) \psi & \text{in } \Omega, \\ j = -\frac{1}{\kappa H} \operatorname{Re}(\bar{\psi}(-i\nabla - \kappa HA)\psi) & \text{in } \Omega, \\ \nu \cdot (-i\nabla - \kappa HA)\psi = 0 & \text{on } \partial\Omega, \\ \operatorname{curl} A = \beta & \text{on } \partial\Omega, \end{cases} \quad (4)$$

where $j = \operatorname{curl}(\operatorname{curl} A - \beta)$ is the supercurrent. If we consider a superconductor–normal metal junction at the boundary [DG18, Section 6.3] then the first boundary condition, known in the literature as magnetic Neumann boundary condition, is replaced by a magnetic Robin boundary condition of the form

$$\nu \cdot (-i\nabla - \kappa HA)\psi = i\gamma\psi, \quad (5)$$

where $\gamma \in \mathbb{R}$ is the Robin parameter. The normal state $(0, F)$, where $F \in H_{\operatorname{div}}^1(\Omega)$ with $\operatorname{curl} F = \beta$, solves (4), so it is a trivial critical point of the Ginzburg–Landau functional. It

is a natural question if the normal state is a global or local minimizer. Due to the results of Giorgi and Phillips [GP02] (see also [FH10, Theorem 10.4.1]) we know that for a constant applied magnetic field the normal state is a global minimizer for $H > C \max\{\kappa, 1\}$, where C is a constant depending on Ω and the applied magnetic field. Hence, we can define, as in [LP99b], the third critical field

$$H_C(\kappa) = \inf \{H > 0 : (0, F) \text{ is the unique minimizer of } \mathcal{G}_{\kappa, H}\}. \quad (6)$$

To determine $H_C(\kappa)$ and the existence of non-trivial minimizers of the Ginzburg–Landau functional, it turns out to be useful to investigate the eigenvalue problem of the magnetic Laplacian given by

$$\begin{cases} (-i\nabla - \kappa HF)^2 \psi = \lambda \psi & \text{in } \Omega, \\ \nu \cdot (-i\nabla - \kappa HF) \psi = i\gamma \psi & \text{on } \partial\Omega, \end{cases} \quad (7)$$

where $\lambda, \gamma \in \mathbb{R}$. For example, if we consider $(\mu u, F)$ where $\mu > 0$ is a small parameter and u is a non-zero solution of (7), then, for μ small enough, the non-linear term in (1) is small, so we can see that if $\lambda < \kappa^2$ then $G_{\kappa, H}$ has a non-trivial minimizer [FH10, Lemma 13.1.1]. For more details we refer to [FH10] for a mathematical approach and to [DG18, Til19] for the physical background about superconductivity. In this thesis, we will also consider applied magnetic fields which are piecewise constant. The reader can find in [Ass19] a detailed exposition of this case.

1.2 Strings, drums and the wave equation

When playing a string instrument such a violin or a guitar, one can notice that thicker strings produce lower sounds and pressing down the strings results in a higher pitch sound. These physical phenomena can be explained using the model of a *vibrating string*. We assume that the string has constant density. In two dimensions, we can consider a *vibrating drum or membrane* instead [LMP23]. If we let $u : \mathbb{R}_+ \times [0, L] \rightarrow \mathbb{R}$ be the function $u(t, x)$ giving the deviation of the string at the point x and the time t , one can see that the *wave equation*

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \quad (8)$$

is the differential equation governing this system where c is the wave propagation speed (see [Bla60, CH53]). This model can also be generalized to higher dimensions. The wave equation in n dimensions is given by

$$\frac{1}{c^2} \frac{\partial^2 u}{\partial t^2} = \Delta u \quad (9)$$

where $u : \Omega \subset \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}$ is the deflection with respect to the equilibrium position, $\Omega \subset \mathbb{R}^n$ is a bounded domain and c is the wave propagation speed. For simplicity, we set

$c = 1$. If the system has fixed boundary, then the boundary condition is $u(x, t) = f(x)$ for $x \in \partial\Omega$ and f being a function with the boundary values. In particular, we are interested in the case where $u(x, t) = 0$ known as the *Dirichlet boundary condition*. This boundary condition corresponds to having our string or membrane fixed in equilibrium position on its boundary. We are also interested in having the free boundary condition which is given by

$$\nu \cdot \nabla_x u(x, t) = 0 \quad \text{on } \partial\Omega \quad (10)$$

where ν is the outward normal vector. This boundary condition is also known as the *Neumann boundary condition* [CH53]. Using the method of separation of variables, we look for solutions of (8) of the form $u(x, t) = v(x)w(t)$. Then the differential equation (8) can be written in the form

$$\frac{w''(t)}{w(t)} = \frac{\Delta v(x)}{v(x)}. \quad (11)$$

The left-hand side is independent of x and the right-hand side of t , hence both sides must be equal to the same constant $-\lambda$. Thus, $v(x)$ can be determined by the boundary value problem given by

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega \\ v = 0 & \text{on } \partial\Omega \end{cases} \quad (12)$$

when considering the Dirichlet boundary condition, and

$$\begin{cases} -\Delta v = \lambda v & \text{in } \Omega \\ \nu \cdot \nabla v = 0 & \text{on } \partial\Omega \end{cases} \quad (13)$$

if we consider the Neumann boundary condition. It can be seen that (12) and (13) can be solved only for certain $\lambda \in \mathbb{R}$. Those λ are called *eigenfrequencies* or natural frequencies. Due to the *principle of superposition* [CH53, Chapter V] we know that we can use the $v(x)$ associated to the eigenfrequencies as building blocks of the general solution of (8). This means that a better understanding of the eigenfrequencies will help in understanding vibrating phenomena. We are interested in the relation between the eigenfrequencies corresponding to Dirichlet and Neumann boundary conditions. For a given $C > 0$ it is expected to have more $\lambda \leq C$ fulfilling (13) than (12). We are interested in quantifying how many more eigenfrequencies we can have fulfilling the Neumann boundary condition with respect to the Dirichlet boundary condition for a general domain $\Omega \subset \mathbb{R}^n$.

2 Spectral theory of linear unbounded operators

In this section, we present the most important mathematical concepts used in this thesis. Most of the results from this section can be found in [Wei80], [Sch12] and [BS87]. Recall that a Hilbert space \mathcal{H} is a complex or real vector space together with an inner product $\langle \cdot, \cdot \rangle$

such that it is complete with respect to the induced norm $\|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle}$. A Hilbert space is *separable* if it has a countable dense subset; we will only work in separable Hilbert spaces.

A *linear operator* in \mathcal{H} is a mapping $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ such that

$$T(\lambda u + \mu v) = \lambda T u + \mu T v \quad \text{for all } u, v \in \mathcal{D}(T) \text{ and } \lambda, \mu \in \mathbb{C}, \quad (14)$$

where $\mathcal{D}(T)$ is a linear subspace of \mathcal{H} called the *domain* of T . The set $\text{Ran}(T) := \{Tx : x \in \mathcal{D}(T)\}$ of \mathcal{H} is called the *range* of T and the subspace $\ker(T) := \{x \in \mathcal{D}(T) : Tx = 0\}$ is called the *kernel* of T .

Let $M \subset \mathcal{H}$, the set $M^\perp = \{u \in \mathcal{H} : \langle u, v \rangle = 0 \forall v \in M\}$ is called the *orthogonal complement* of M . If M is closed then each $u \in \mathcal{H}$ can be uniquely decomposed as $u = v + w$ with $v \in M$ and $w \in M^\perp$ [Wei80, Theorem 3.2]. The mapping $P_M : \mathcal{H} \rightarrow \mathcal{H}$ with $P_M(u) = v$ is a linear operator called the *orthogonal projection onto* M .

A family $\{e_i : i \in I\}$ is called an *orthonormal system* if $\|e_n\| = 1$ and $\langle e_n, e_m \rangle = \delta_{nm}$ for all $n, m \in I$. An orthonormal system $\{e_i : i \in I\}$ is called an *orthonormal basis (ONB)* of \mathcal{H} if $\overline{\text{Span}\{e_i : i \in I\}} = \mathcal{H}$.

2.1 Basic notions of spectral theory of self-adjoint operators

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle)$ be a Hilbert space over \mathbb{C} and $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ a linear operator. The number $\lambda \in \mathbb{C}$ is called an *eigenvalue* of T if there exists a $u \in \mathcal{D}(T) \setminus \{0\}$ such that $Tu = \lambda u$, i.e. if $\lambda - T$ is not injective. If λ is not an eigenvalue then the operator $R(\lambda, T) = (\lambda - T)^{-1}$ is well-defined. Denote by $\mathcal{L}(\mathcal{H})$ the set of bounded linear operators in \mathcal{H} . The set

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is injective and } R(\lambda, T) \in \mathcal{L}(\mathcal{H})\} \quad (15)$$

is called the *resolvent set* of T and

$$\sigma(T) = \mathbb{C} \setminus \rho(T) \quad (16)$$

is called the *spectrum* of T . The map

$$R(\cdot, T) : \rho(T) \rightarrow \mathcal{L}(\mathcal{H}), \quad \lambda \mapsto (\lambda - T)^{-1} \quad (17)$$

is called the *resolvent operator* of T .

We say that T is a *closed operator* if its graph

$$\mathcal{G}(T) : \{(u, Tu) : u \in \mathcal{D}(T)\} \quad (18)$$

is closed in $\mathcal{H} \times \mathcal{H}$. An operator $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ is said to be *closable* if $\overline{\mathcal{G}(T)}$ is the graph of a linear operator $\bar{T} : \mathcal{D}(\bar{T}) \subset \mathcal{H} \rightarrow \mathcal{H}$. We call the operator \bar{T} the *closure* of T . Let T be a closed operator, a subspace $\mathcal{D} \subset \mathcal{D}(T)$ is called a *core* of T if $T = \overline{T|_{\mathcal{D}}}$.

If T is not closed then $\rho(T) = \emptyset$, and the characterization of the spectrum is trivial. If T is a closed operator, then by the closed graph theorem [Wei80, Theorem 5.6] we have

$$\rho(T) = \{\lambda \in \mathbb{C} : \lambda - T \text{ is bijective}\}. \quad (19)$$

We are interested in the spectral properties of a particular type of closed operators called *self-adjoint operators*. An operator $S : \mathcal{D}(S) \subset \mathcal{H} \rightarrow \mathcal{H}$ is called a *formal adjoint* of T if

$$\langle v, Tu \rangle = \langle Sv, u \rangle, \text{ for all } u \in \mathcal{D}(T), v \in \mathcal{D}(S). \quad (20)$$

Let T be a densely defined operator, i.e. $\mathcal{D}(T)$ is dense in \mathcal{H} , and let

$$\mathcal{D}(T^*) := \{v \in \mathcal{H} : \exists w_v \in \mathcal{H} \text{ such that } \langle v, Tu \rangle = \langle w_v, u \rangle \forall u \in \mathcal{D}(T)\}, \quad (21)$$

we define the *adjoint operator* T^* of T as the formal adjoint operator of T with domain $\mathcal{D}(T^*)$. Since $\mathcal{D}(T)$ is dense, each $w_v \in \mathcal{H}$ in (21) is uniquely determined (see [Wei80, Section 4.4]) by v and T via

$$\langle v, Tu \rangle = \langle w_v, u \rangle \quad \forall u \in \mathcal{D}(T),$$

which means that T^* is a well-defined operator. Moreover, T^* is an extension of all formal adjoints of T and is always a closed operator [Wei80, Theorem 5.3]. An operator T is *self-adjoint* if T is densely defined and $T = T^*$. If T is self-adjoint then $\sigma(T) \subset \mathbb{R}$.

One can study the fine structure of the spectrum of a self-adjoint operator T evaluating for which $\lambda \in \mathbb{C}$, $\lambda - T$ is not bijective. In particular, we are interested in the set of all eigenvalues of T with finite multiplicity, i.e. $\dim \ker(\lambda - T)$ is finite, that are isolated points of $\sigma(T)$, we call this set *discrete spectrum* of T and denote it by $\sigma_d(T)$. The set $\sigma_{\text{ess}}(T) := \sigma(T) \setminus \sigma_d(T)$ is called the *essential spectrum* of T .

An operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is *compact* if every bounded sequence $(u_n)_{n \in \mathbb{N}} \subset \mathcal{D}(T)$ contains a subsequence (u_{n_k}) for which (Tu_{n_k}) is convergent in \mathcal{H} . In other words, T maps bounded sets onto relatively compact sets. Let $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a closed operator, we say that T has *compact resolvent* if there exists $\lambda_0 \in \rho(T)$ such that $R(\lambda_0, T)$ is compact.

Theorem 1 ([Wei80, Theorem 6.7]). *Let $K : \mathcal{H} \rightarrow \mathcal{H}$ be a compact operator. We have that any $\lambda \in \sigma(K) \cap (\mathbb{C} \setminus \{0\})$ is an eigenvalue. If \mathcal{H} is infinite dimensional, then $\sigma(K) = \sigma_d(K) \cup \{0\}$. The operator K has at most countably many eigenvalues, and they can cluster only at 0.*

Corollary 1. *Let \mathcal{H} be infinite dimensional, and $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a self-adjoint operator with compact resolvent, then $\sigma(T) = \sigma_d(T)$, i.e. T has purely discrete spectrum. Equivalently, the spectrum consists of a real sequence $(\lambda_n)_{n \in \mathbb{N}}$ with an associated orthonormal basis $\{e_n : n \in \mathbb{N}\}$ of \mathcal{H} such that $\lim_{n \rightarrow \infty} |\lambda_n| = +\infty$ and $Te_n = \lambda_n e_n$ for $n \in \mathbb{N}$.*

The corollary is a consequence of Theorem 1 with $K = R(\lambda_0, T)$ and the identity

$$\begin{aligned} T - \lambda &= -(\lambda - \lambda_0) + T - \lambda_0 \\ &= -(R(\lambda_0, T) + (\lambda - \lambda_0)^{-1})(T - \lambda_0)(\lambda_0 - \lambda). \end{aligned}$$

2.2 Self-adjoint operators defined by sesquilinear forms

A mapping $s : \mathcal{D}(s) \times \mathcal{D}(s) \rightarrow \mathbb{C}$, where $\mathcal{D}(s)$ is a linear subspace of \mathcal{H} , is called a *sesquilinear form* on \mathcal{H} if for all $u, v, w \in \mathcal{D}(s)$ and $\lambda, \mu \in \mathbb{C}$ we have

$$s(u, \lambda v + \mu w) = \lambda s(u, v) + \mu s(u, w), \quad (22)$$

$$s(\lambda u + \mu v, w) = \bar{\lambda} s(u, w) + \bar{\mu} s(v, w). \quad (23)$$

A sesquilinear form s is symmetric if $s(u, v) = \overline{s(v, u)}$ for all $u, v \in \mathcal{D}(s)$ and is *bounded from below* if there exists a $\gamma \in \mathbb{R}$ such that $s(u, u) \geq \gamma \|u\|^2$ for all $u \in \mathcal{D}(s)$. Note that if s is bounded below for some $\gamma \in \mathbb{R}$, then s is symmetric (one can use polarization identity to see this) and

$$\langle u, v \rangle_s := s(u, v) + (1 - \gamma) \langle u, v \rangle \quad \text{for } u, v \in \mathcal{D}(s) \quad (24)$$

defines a scalar product in $\mathcal{D}(s)$. A sesquilinear form is *densely defined* if $\mathcal{D}(s)$ is dense in \mathcal{H} and *closed* if $(\mathcal{D}(s), \langle \cdot, \cdot \rangle_s)$ is a Hilbert space. The mapping $q : \mathcal{D}(s) \subset \mathcal{H} \rightarrow \mathbb{C}$ defined by $q(u) := s(u, u)$ is called the *quadratic form* induced by s . We say that a quadratic form is *closed* if the inducing sesquilinear form is closed. The following theorem is an adaptation of [Wei80, Theorem 5.36] and it is the key result which will allow us to define self-adjoint operators from closed sesquilinear (or quadratic) forms.

Theorem 2. *Let $s : \mathcal{D}(s) \times \mathcal{D}(s) \rightarrow \mathbb{C}$ be a densely defined, bounded from below with lower bound $\gamma \in \mathbb{R}$ and closed sesquilinear form. Then there exists exactly one self-adjoint operator T on \mathcal{H} such that*

$$\mathcal{D}(T) \subset \mathcal{D}(s) \text{ and } \langle v, Tu \rangle = s(v, u) \text{ for } u \in \mathcal{D}(T), v \in \mathcal{D}(s). \quad (25)$$

We have that $\langle u, Tu \rangle \geq \gamma \|u\|^2$ for all $u \in \mathcal{D}(T)$ and T can be defined by the equalities

$$\left. \begin{aligned} \mathcal{D}(T) &= \{u \in \mathcal{D}(s) : \text{there exists a } w \in \mathcal{H} \text{ such that } s(v, u) = \langle v, w \rangle \forall v \in \mathcal{D}(s)\}, \\ Tu &= w. \end{aligned} \right\} \quad (26)$$

$\mathcal{D}(T)$ is dense in $(\mathcal{D}(s), \|\cdot\|_s)$ where $\|\cdot\|_s$ is the norm induced by $\langle \cdot, \cdot \rangle_s$.

This theorem allows the identification of closed quadratic forms, which are bounded from below, with self-adjoint operators which are also bounded from below. In this case we say that T is the self-adjoint operator associated with the closed quadratic form q , or the sesquilinear form s .

An operator $S : \mathcal{D}(S) \subset \mathcal{H} \rightarrow \mathcal{H}$ is *symmetric* if it is densely defined and a formal adjoint of itself, i.e.

$$\langle v, Su \rangle = \langle Sv, u \rangle \quad \forall u, v \in \mathcal{D}(S). \quad (27)$$

It is clear that every self-adjoint operator is symmetric, but the converse is not true. A symmetric operator is *bounded from below* if there exists a $\gamma \in \mathbb{R}$ such that $\langle u, Su \rangle \geq \gamma \|u\|^2$ for all $u \in \mathcal{D}(S)$. Given a symmetric operator S we are interested in finding a “large enough” self-adjoint extension T , so that

$$S \subset T = T^* \subset S^*.$$

A symmetric operator is called *essentially self-adjoint* if it has precisely one self-adjoint extension. Next we will see how the use of sesquilinear forms allows us to prove that every symmetric operator that is bounded from below has at least one self-adjoint extension with the same bound from below; it is known as the *Friedrichs extension*.

Let $s : \mathcal{D}(s) \times \mathcal{D}(s) \rightarrow \mathcal{H}$ be a densely defined sesquilinear form bounded from below by $\gamma \in \mathbb{R}$. Then we denote by $\|\cdot\|_s$ the norm induced by the scalar product $\langle \cdot, \cdot \rangle_s$ defined in (24) and by $(\mathcal{H}_s, \|\cdot\|_s)$ the Hilbert space which has a dense subspace isomorphic to $(\mathcal{D}(s), \|\cdot\|_s)$. The Hilbert space \mathcal{H}_s is called the completion of $\mathcal{D}(s)$, and we refer to [Wei80, Theorem 4.11] for a proof of the existence and uniqueness (up to isomorphism) of the completion and to [Wei80, Section 5.5] for a discussion on why \mathcal{H}_s can be considered as a subspace of \mathcal{H} .

Theorem 3. *Let $S : \mathcal{D}(S) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a symmetric operator which is bounded from below with lower bound γ . If we define the sesquilinear form $s(u, v) = \langle u, Sv \rangle$ for $u, v \in \mathcal{D}(S)$ and \mathcal{H}_s as above, then the operator S_F defined by*

$$\mathcal{D}(S_F) = \mathcal{D}(S^*) \cap \mathcal{H}_s \text{ and } S_F u = S^* u \text{ for } u \in \mathcal{D}(S)$$

is a self-adjoint extension with lower bound γ and it is the only self-adjoint extension such that $\mathcal{D}(S_F) \subset \mathcal{H}_s$.

2.3 Variational theorems

An important property of self-adjoint operators is that they admit an integral representation in terms of spectral integrals called *spectral resolutions*. Next, we present the spectral theorem for unbounded self-adjoint operators. A *resolution of the identity*, or spectral family, on \mathcal{H} is a one-parameter family $\{E(\lambda) : \lambda \in \mathbb{R}\}$ of orthogonal projections on \mathcal{H} such that

- (i) $\langle u, E(\lambda_1)u \rangle \leq \langle u, E(\lambda_2)u \rangle$ for all $u \in \mathcal{H}$ if $\lambda_1 \leq \lambda_2$ (monotonicity),
- (ii) $\lim_{\epsilon \rightarrow 0^+} E(\lambda + \epsilon)u = E(\lambda)u$ for $u \in \mathcal{H}$ and $\lambda \in \mathbb{R}$ (strong right continuity),
- (iii) $\lim_{\lambda \rightarrow -\infty} E(\lambda)x = 0$ and $\lim_{\lambda \rightarrow +\infty} E(\lambda)x = x$ for $x \in \mathcal{H}$ (completeness)

The condition (i) guarantees that $E(\lambda_0^-) := \lim_{\lambda \rightarrow \lambda_0^-} E(\lambda)$ exists for all $\lambda_0 \in \mathbb{R}$ and that it is again a spectral projection (see [Wei80, Theorem 4.32] for more details). We define the spectral projections for intervals as

$$\begin{aligned} E([a, b]) &:= E(b) - E(a^-), & E([a, b)) &:= E(b^-) - E(a^-) \\ E((a, b]) &:= E(b) - E(a), & E((c, d)) &:= E(d^-) - E(c), \end{aligned} \tag{28}$$

for $a, b, c, d \in \mathbb{R} \cup \{-\infty\} \cup \{+\infty\}$, $a \leq b$, $c < d$ and setting $E(-\infty) = 0$ and $E(+\infty) = I$. A mapping μ from the Borel σ -algebra $\mathcal{B}(\mathbb{R})$ into the orthogonal projections on \mathcal{H} is a *spectral measure* if

1. $\mu(\mathbb{R}) = I$,
2. $\mu(\bigcup_{n=1}^{\infty} M_n) = \sum_{n=1}^{\infty} \mu(M_n)$ for any sequence $(M_n)_{n \in \mathbb{N}}$ of pairwise disjoint sets from $\mathcal{B}(\mathbb{R})$.

There is a one-to-one correspondence (see [Sch12, Theorem 4.6]) between resolutions of the identity $\{E(\lambda) : \lambda \in \mathbb{R}\}$ and spectral measures μ on $\mathcal{B}(\mathbb{R})$ via

$$E(\lambda) = \mu((-\infty, \lambda]), \quad \lambda \in \mathbb{R}. \tag{29}$$

Let $M \in \mathcal{B}(\mathbb{R})$ and μ a spectral measure, then we can define the integral on the step function

$$\chi_M(\lambda) := \begin{cases} 1 & \text{if } \lambda \in M \\ 0 & \text{otherwise,} \end{cases}$$

by

$$\int_{\mathbb{R}} \chi_M(\lambda) \, dE(\lambda) := \mu(M).$$

This definition can be extended to Borel-measurable functions $f : \mathbb{R} \rightarrow \mathbb{C} \cup \{\infty\}$ for which $E(\{t \in \mathbb{R} : f(t) = \infty\}) = 0$, and we denote it by $\int_{\mathbb{R}} f(\lambda) \, dE(\lambda)$. More details of this construction can be found in [Sch12, Chapter 4].

Theorem 4 (Spectral Theorem). *Let T be a self-adjoint operator on \mathcal{H} . Then there exists a unique spectral measure, which we denote by E_T , on the Borel σ -algebra $\mathcal{B}(\mathbb{R})$ such that*

$$T = \int_{\mathbb{R}} \lambda \, dE_T(\lambda).$$

The support of the spectral measure E_T is equal to the spectrum of T . One of the most important consequence of the spectral theorem is the *functional calculus* of the self-adjoint operator T which is given by

$$f \mapsto f(T) = \int_{\mathbb{R}} f(\lambda) \, dE_T(\lambda) \tag{30}$$

where f is a Borel-measurable function with $E(\{t \in \mathbb{R} : f(t) = \infty\}) = 0$. Some important properties, which we will use later, are $\langle u, f(T)v \rangle = \int_{\mathbb{R}} f(\lambda) \langle u, dE_T(\lambda)v \rangle$ for $u, v \in \mathcal{D}(f(T))$ and $(fg)(T) = f(T)g(T)$ where g is a Borel-measurable function fulfilling the same property as f .

Moreover, λ_0 is an eigenvalue if and only if $\chi_{\{\lambda_0\}}(T) = E_T(\{\lambda_0\}) \neq 0$, and in this case $E_T(\{\lambda_0\})$ is the projection of \mathcal{H} on the eigenspace of T at λ_0 . If in addition $\lambda_0 \in \sigma_d(T)$ then

$$E_T(\{\lambda_0\}) = \sum_{i=1}^m \langle u_i, \cdot \rangle u_i \quad (31)$$

where $m = \dim \ker(T - \lambda_0)$ and $\{u_i\}_{i=1}^m$ is an orthonormal basis of the eigenspace. The next result is a characterization of the essential spectrum of a self-adjoint operator. We refer to [GS20, Theorem 25.57] for a proof.

Proposition 1 (Weyl's criterion). *Let $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a self-adjoint operator. Then $\lambda \in \sigma_{\text{ess}}(T)$ if, and only if, there exists a sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(T)$ such that $\|u_n\| = 1$ for all $n \in \mathbb{N}$, $\lim_{n \rightarrow \infty} \|(T - \lambda)u_n\| = 0$ and $\lim_{n \rightarrow \infty} \langle v, u_n \rangle = 0$ for all $v \in \mathcal{H}$.*

A sequence $\{u_n\}_{n \in \mathbb{N}}$ with the same properties as the one given in Proposition 1 is called a *singular or Weyl sequence* for T and λ . We prove a lemma which is a consequence of Weyl's criterion following the proof in [CR21, Lemma 6.19].

Lemma 1. *Let $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a self-adjoint operator and $\lambda \in \sigma_{\text{ess}}(T)$. Then, for all $N \in \mathbb{N}$ and $\epsilon > 0$ there exists an orthonormal system $\{u_n^\epsilon\}_{1 \leq n \leq N} \subset \mathcal{D}(T)$ such that for all $n \in \{1, \dots, N\}$*

$$\|(T - \lambda)u_n^\epsilon\| \leq \epsilon. \quad (32)$$

Proof. If λ is an isolated eigenvalue then it has infinite multiplicity and we can take any set of N distinct eigenfunctions and the results follows. If λ is not isolated, then for any $\epsilon > 0$ we can find a sequence $\{\lambda_n\}_{n \in \mathbb{N}} \subset \sigma_{\text{ess}}(T)$ with $\lambda_i \neq \lambda_j$ if $i \neq j$ such that $\lim_{n \rightarrow \infty} \lambda_n = \lambda$ and $|\lambda_i - \lambda_j| \leq \frac{\epsilon}{2}$ for all $i, j \in \mathbb{N}$. Let $N \in \mathbb{N}$ be fixed and assume $\epsilon \in (0, \frac{1}{2})$. Then by Weyl's criterion, we get the existence of $\{v_i^\epsilon\}_{1 \leq i \leq N} \subset \mathcal{D}(T)$ with $\|v_i^\epsilon\| = 1$ for all $1 \leq i \leq N$ such that

$$\|(T - \lambda_i)v_i^\epsilon\| \leq \frac{\epsilon}{2N!} \min_{i \neq j} |\lambda_i - \lambda_j|,$$

for all $i \in \{1, \dots, N\}$. Since T is self-adjoint we have for $i \neq j$

$$\langle (\lambda_i - \lambda_j)v_i^\epsilon, v_j^\epsilon \rangle = \langle v_i^\epsilon, (T - \lambda_j)v_j^\epsilon \rangle + \langle (\lambda_i - T)v_i^\epsilon, v_j^\epsilon \rangle,$$

which implies that $|\langle v_i^\epsilon, v_j^\epsilon \rangle| \leq \frac{\epsilon}{N!}$. In other words, we have that $\{v_1^\epsilon, \dots, v_N^\epsilon\}$ is an almost orthogonal set. Applying the Gram-Schmidt process to $\{v_i^\epsilon\}_{1 \leq i \leq N}$ we can get an orthonormal family $\{u_i^\epsilon\}_{1 \leq i \leq N}$ given by $u_i^\epsilon = \frac{\tilde{u}_i^\epsilon}{\|\tilde{u}_i^\epsilon\|}$ where $\tilde{u}_i^\epsilon = v_i^\epsilon - \sum_{j=1}^{i-1} \langle v_i^\epsilon, v_j^\epsilon \rangle v_j^\epsilon$ for all $i \in \{1, \dots, N\}$.

This family is indeed orthonormal since $\|\tilde{u}_i^\epsilon\| \geq 1 - \sum_{j=1}^{k-1} |\langle v_i^\epsilon, v_j^\epsilon \rangle| \geq 1 - \epsilon > 0$ where recall that we assumed that $\epsilon \in (0, \frac{1}{2})$. Such family fulfils (32) because

$$\|(T - \lambda)u_i^\epsilon\| \leq |\lambda - \lambda_i| + \|(T - \lambda_i)u_i^\epsilon\| \leq \frac{\epsilon}{2} + \|(T - \lambda_i)u_i^\epsilon\|.$$

Since $|\lambda - \lambda_i| \leq |\lambda - \lambda_n| + \frac{\epsilon}{2} \rightarrow \frac{\epsilon}{2}$ as $n \rightarrow \infty$, and by the previous estimates, we have

$$\begin{aligned} \|(T - \lambda_i)u_i^\epsilon\| &\leq \frac{1}{\|\tilde{u}_i^\epsilon\|} \left(\|(T - \lambda_i)v_i^\epsilon\| + \sum_{j=1}^{i-1} |\langle v_i^\epsilon, v_j^\epsilon \rangle| (\|(T - \lambda_j)v_j^\epsilon\| + |\lambda_j - \lambda_i|) \right) \\ &\leq \frac{\epsilon^2}{1 - \epsilon} \leq \epsilon, \end{aligned}$$

since $\|\tilde{u}_i^\epsilon\| \geq 1 - \epsilon$ and $\epsilon \in (0, \frac{1}{2})$. \square

We are now ready to present one of the main tools to estimate the discrete spectrum and the bottom of the essential spectrum of a self-adjoint operator that is bounded from below: the *min-max principle*, or the *Courant–Fischer–Weyl min-max principle* [Cou20, Fiso5]. We follow [CR21, Section 6.4], [Dav95, Section 4.5] and [Lew24, Theorem 5.4].

Theorem 5 (Min-max principle). *Let $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a self-adjoint operator that is bounded from below by γ associated with the closed sesquilinear form s with domain $\mathcal{D}(s) \subset \mathcal{H}$. Let*

$$\mu_k(T) := \inf_{\substack{M \subset \mathcal{D}(T) \\ \dim(M)=k}} \max_{u \in M} \frac{\langle u, Tu \rangle}{\|u\|_{\mathcal{H}}^2} = \inf_{\substack{M \subset \mathcal{D}(s) \\ \dim(M)=k}} \max_{u \in M} \frac{s(u, u)}{\|u\|_{\mathcal{H}}^2}. \quad (33)$$

Then for every $k \geq 1$ one of the following holds

- (a) *There are k eigenvalues of finite multiplicity below $\sigma_{\text{ess}}(T)$ and $\mu_k(T)$ is the k -th eigenvalue, where each eigenvalue is repeated a number of times equal to its multiplicity.*
- (b) *$\mu_k(T) = \inf \sigma_{\text{ess}}(T) = \mu_{k+1}(T) = \mu_{k+2}(T) = \dots$ and there are at most $(k - 1)$ eigenvalues of finite multiplicity below $\mu_k(T)$, where each eigenvalue is repeated a number of times equal to its multiplicity.*

Proof. The second equality in (33) follows from the fact that $\mathcal{D}(T)$ is dense in $(\mathcal{D}(s), \|\cdot\|_s)$ as stated in Theorem 2. We refer to [Dav95, Theorem 4.5.3] for more details. For the first equality, we start by proving by contradiction that $\mu_k(T) \leq \inf \sigma_{\text{ess}}(T)$ for all $k \in \mathbb{N}$. By definition, the sequence $\{\mu_k(T)\}_{k \in \mathbb{N}}$ is non-decreasing. Let $c < \mu_k(T)$ and assume that we can find a $\lambda \in (-\infty, c) \cap \sigma_{\text{ess}}(T)$. Then by Lemma 1, for any given $N \geq k$ and $\epsilon > 0$, we can find an orthonormal system $\{u_k\}_{1 \leq k \leq N} \subset \mathcal{D}(T)$ such that

$$\|(T - \lambda)u_k\| \leq \frac{\epsilon}{\sqrt{N}}.$$

Let $u \in U := \text{Span}(u_1, \dots, u_N)$, then $u = \sum_{i=1}^N \alpha_i u_i \in \mathcal{D}(T)$ with $\alpha_i \in \mathbb{C}$ for all $i \in \{1, \dots, N\}$, $\dim U = N \geq n$ and

$$\frac{\langle u, Tu \rangle}{\|u\|^2} \leq \lambda + \frac{\|(T - \lambda)u\|}{\|u\|} \leq \lambda + \left(\sum_{i=1}^N \|(T - \lambda)u_i\|^2 \right)^{\frac{1}{2}} \leq \lambda + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, $\mu_k(T) \leq c$ which is a contradiction. The previous reasoning allows us to show that $\mu_k(T) \leq \inf \sigma_{\text{ess}}(T)$. Moreover, if $\mu_k(T) = +\infty$ for some $k \in \mathbb{N}$ this means that the essential spectrum is empty.

T can only have isolated eigenvalues with finite multiplicity in the interval $(-\infty, \inf \sigma_{\text{ess}}(T))$. If $\sigma(T) \cap (-\infty, \inf \sigma_{\text{ess}}(T)) = \emptyset$, then we are in the case (b) for every $k \geq 1$ and the proof is done. If this is not the case, then we need to prove that (a) holds. We can list the isolated eigenvalues with finite multiplicity in the interval $(-\infty, \inf \sigma_{\text{ess}}(T))$ in increasing order counting multiplicities by $\{\lambda_k(T)\}_{k=1}^N$ where $N \in \mathbb{N} \cup \{\infty\}$. Denote by $\{u_i\}_{i=1}^N$ the corresponding orthonormal system of eigenfunctions. Since T is bounded below, if $N = \infty$ the eigenvalues $\{\lambda_k(T)\}_{k=1}^N$ converge monotonically to $\inf \sigma_{\text{ess}}(T)$.

1. $\mu_k(T) \leq \lambda_k(T)$. Let $L_k := \text{Span}\{u_1, \dots, u_k\}$ and $u \in L_k \setminus \{0\}$ then $u \in \mathcal{D}(T)$ and

$$\langle u, Tu \rangle = \sum_{i=1}^k \lambda_i(T) |\langle u, u_i \rangle|^2 \leq \lambda_k(T) \sum_{i=1}^k |\langle u, u_i \rangle|^2 = \lambda_k(T) \|u\|^2.$$

It follows that $\mu_k(T) \leq \lambda_k(T)$.

2. $\mu_k(T) \geq \lambda_k(T)$. We proceed by induction. Let $k = 1$ then for any $u \in \mathcal{D}(T)$ since $\sigma(T) = \text{supp } E_T$ we have

$$\begin{aligned} \langle u, Tu \rangle &= \int_{\mathbb{R}} \lambda \langle u, dE_T(\lambda)u \rangle \\ &\geq \lambda_1(T) \int_{\mathbb{R}} \langle u, dE_T(\lambda)u \rangle \\ &= \lambda_1(T) \|u\|^2, \end{aligned}$$

for all $u \in \mathcal{D}(T)$. Then $\mu_1(T) \geq \lambda_1(T)$ follows. Assume that $\mu_{k-1}(T) \geq \lambda_{k-1}(T)$ for some $k \in \{2, \dots, N\}$, we want to show $\mu_k(T) \geq \lambda_k(T)$. If $\lambda_k(T) = \lambda_{k-1}(T)$ (this can happen if the eigenvalue is not simple) then $\mu_k(T) \geq \mu_{k-1}(T) \geq \lambda_{k-1}(T) = \lambda_k(T)$, so we assume that $\lambda_k(T) > \lambda_{k-1}(T)$. Let L_n be any k -dimensional linear subspace of $\mathcal{D}(T)$ and consider the orthogonal projection $P_{U_{k-1}}$ onto $U_{k-1} = \{u_1, \dots, u_{k-1}\}$ given by

$$P_{U_{k-1}} u = \sum_{i=1}^{k-1} \langle u, u_i \rangle u_i.$$

Since $\text{Ran } P_{U_{k-1}} \subset \mathcal{D}(T)$ and $\dim(\text{Ran } P_{U_{k-1}}) = k - 1 < k = \dim L_k$, there must exist a $u \in L_k$ with $\|u\| = 1$ such that $P_{U_{k-1}} u = 0$. Because of this, and using the properties of the functional calculus,

$$\begin{aligned} \langle u, Tu \rangle &= \langle u, TP_{U_{k-1}} u \rangle + \langle u, T(I - P_{U_{k-1}})u \rangle \\ &= \int_{\mathbb{R}} \lambda \chi_{(\lambda_{k-1}(T), +\infty)}(\lambda) \langle u, dE_T(\lambda)u \rangle \\ &\geq \lambda_k(T) \|u\|^2, \end{aligned}$$

where $I \in \mathcal{L}(\mathcal{H})$ is the identity operator. This implies that $\max_{u \in L_k \setminus \{0\}} \langle u, Tu \rangle \geq \lambda_k(T)$ and since L_k was chosen arbitrarily we get $\mu_k(T) \geq \lambda_k(T)$. \square

Remark. In applications, it is often more convenient to use the min-max principle given in terms of the sesquilinear, or quadratic, form since $\mathcal{D}(T) \subset \mathcal{D}(s)$. There is an alternative formulation of the min-max principle given by a max-min version where

$$\begin{aligned} \mu_n(T) &= \sup_{u_1, \dots, u_{n-1}} \inf_{\substack{u \in (\text{Span}(u_1, \dots, u_{n-1}))^\perp \\ u \in \mathcal{D}(T)}} \frac{\langle u, Tu \rangle}{\|u\|_{\mathcal{H}}^2} \\ &= \sup_{u_1, \dots, u_{n-1}} \inf_{\substack{u \in (\text{Span}(u_1, \dots, u_{n-1}))^\perp \\ u \in \mathcal{D}(s)}} \frac{s(u, u)}{\|u\|_{\mathcal{H}}^2}. \end{aligned} \tag{34}$$

We refer to [Hel13, Chapter 11] for a proof of the max-min principle since it will not be used in this thesis.

The min-max principle is useful when we want to find upper bounds of the eigenvalues of a bounded from below self-adjoint operator. To get more precise lower bounds we will use *Temple's inequality* which was introduced in [Tem28] when studying the eigenfrequencies of a vibrating string with not constant density. We present the generalization of Temple's inequality to self-adjoint operators given by Kato in [Kat49].

Theorem 6 (Temple's inequality). *Let $T : \mathcal{D}(T) \subset \mathcal{H} \rightarrow \mathcal{H}$ be a self-adjoint operator and for $w \in \mathcal{D}(T)$ with $\|w\|_{\mathcal{H}} = 1$ define $\eta = \langle w, Tw \rangle$ and $\epsilon = \|(T - \eta)w\|$. Assume that the interval (α, β) contains no point of the spectrum except at most a simple eigenvalue and that $\epsilon^2 < (\eta - \alpha)(\beta - \eta)$. Then there is a unique eigenvalue $\mu \in (\alpha, \beta)$ and*

$$\eta - \frac{\epsilon^2}{\beta - \eta} \leq \mu \leq \eta + \frac{\epsilon^2}{\eta - \alpha}. \tag{35}$$

Proof. The two bounds rely on the following equality

$$(\lambda - \xi)(\lambda - \mu) = \lambda^2 - (\xi + \mu)\lambda + \xi\mu. \tag{36}$$

Let $\xi = \alpha \in \mathbb{R}$ and assume that $\sigma(T) \cap (\alpha, \mu)$ is empty, then (36) is non-negative when we evaluate λ over the spectrum of T . Using the properties of the functional calculus we have

$$\int_{\mathbb{R}} \lambda^2 \langle w, dE_T(\lambda)w \rangle = \langle Tw, Tw \rangle = \eta^2 + \epsilon^2,$$

and

$$\int_{\mathbb{R}} \lambda \langle w, dE_T(\lambda)w \rangle = \langle w, Tw \rangle = \eta.$$

Then (36) is still non-negative after multiplying by $\langle w, dE_T(\lambda)w \rangle$, and integrating it over \mathbb{R} we get

$$\epsilon^2 + \eta^2 - (\alpha + \mu)\eta + \alpha\mu \geq 0 \Rightarrow \mu \leq \eta + \frac{\epsilon^2}{\eta - \alpha}.$$

Note that if $\alpha = -\infty$ we can use $\lambda = \mu$ instead of (36). We have seen that we have a point in the spectrum in (α, μ) if $\mu > \eta + \epsilon^2/(\eta - \alpha)$. But we know that the spectrum is a closed set (in the region we are looking we know it is an isolated point) so we have a point in the spectrum in the interval $(\alpha, \eta + \epsilon^2/(\eta - \alpha)]$.

Similarly, if we let $\xi = \beta \in \mathbb{R}$ and assume that $\sigma(T) \cap (\mu, \beta)$ is empty, we have that (36) is non-negative, so multiplying again by $\langle w, dE_T(\lambda)w \rangle$ and integrating over \mathbb{R} we get

$$\epsilon^2 + \eta^2 - (\beta + \mu)\eta + \beta\mu \geq 0 \Rightarrow \mu \geq \eta - \frac{\epsilon^2}{\beta - \eta},$$

and this means that we have a point in the spectrum if $\mu < \eta - \frac{\epsilon^2}{\beta - \eta}$ and, as before, we can deduce that we have a point in the spectrum in $[\eta - \epsilon^2/(\beta - \eta), \beta)$. We can set $\lambda = \mu$ instead of (36) if $\beta = +\infty$. \square

The last main theorem of this section is Persson's lemma. This result was first proven by Arne Persson in Lund [Per60] in 1960. This lemma gives a "variational" formulation of the bottom of the essential spectrum of Schrödinger operators and can be interpreted as the fact that the bottom of the essential spectrum is not affected by "what happens" in compact sets [CFKS87].

We follow the ideas in [Per60, Bono3] to give a version of Persson's lemma for magnetic Schrödinger operators acting on $L^2(\mathbb{R}^n)$ which we will denote by H . As already stated in [Per60, Remark 3], the result can be proven for the Friedrichs extension of more general elliptic operators with compactly supported smooth functions as a core. We introduce some notation

$$\mathcal{M}_R := \{u \in \mathcal{D}(H) : \text{supp } u \subset \{x \in \mathbb{R}^n : |x| > R\}\}, \quad (37)$$

$$\Sigma_R := \inf\{\langle u, Hu \rangle : u \in \mathcal{M}_R, \|u\|_{L^2(\mathbb{R}^n)} = 1\}, \quad (38)$$

and

$$\Sigma := \sup_{R>0} \Sigma_R. \quad (39)$$

Σ is called the ionization threshold when we deal with Schrödinger operators for atoms and molecules. Σ_R is an increasing function since $\mathcal{M}_{R_2} \subset \mathcal{M}_{R_1}$ if $R_1 < R_2$. This means that if H is bounded from below by γ , then $\Sigma \in (\gamma, +\infty]$ and $\Sigma = \lim_{R \rightarrow \infty} \Sigma_R$. Consider the differential operator

$$H_0 = (-i\nabla - A)^2 + V(x) \quad (40)$$

with domain $\mathcal{D}(H_0) = C_c^\infty(\mathbb{R}^n)$, $A = (A_1, \dots, A_n) \in L_{\text{loc}}^4(\mathbb{R}^n)^n$ with $\nabla \cdot A \in L_{\text{loc}}^2(\mathbb{R}^n)$ and $0 \leq V \in L_{\text{loc}}^2(\mathbb{R}^n)$. Note that V could just be bounded from below since we can consider $\tilde{V} = V + C$ where C is a suitable constant so that $\tilde{V} \geq 0$. Under these assumptions H_0 is a symmetric and bounded from below operator. Moreover, by [LS81, Theorem 2] we know that H_0 is essentially self-adjoint. Let H be the self-adjoint extension of H_0 , we have

$$\mathcal{D}(H) = \{u \in \mathcal{D}(h) : \exists w \in L^2(\mathbb{R}^n) \text{ such that } h(v, u) = \langle v, w \rangle_{L^2(\mathbb{R}^n)} \forall v \in \mathcal{D}(h)\}, \quad (41)$$

where $h : \mathcal{D}(h) \times \mathcal{D}(h) \rightarrow \mathbb{C}$ is the symmetric and closed sesquilinear form (see [LS81, Lemma 1])

$$h(u, v) = \int_{\mathbb{R}^n} \overline{(-i\nabla - A)u} (-i\nabla - A)v \, dx + \int_{\mathbb{R}^n} V(x) \bar{u}v \, dx, \quad (42)$$

with $\mathcal{D}(h) = \{u \in L^2(\mathbb{R}^n) : (-i\nabla - A)u \in L^2(\mathbb{R}^n) \text{ and } \sqrt{V}u \in L^2(\mathbb{R}^n)\}$. This follows from Theorem 2 and the fact that $C_c^\infty(\mathbb{R}^n)$ is dense in $(\mathcal{D}(h), \|\cdot\|_h)$ where $\|\cdot\|_h$ is just the induced norm by the scalar product $\langle \cdot, \cdot \rangle_h$ with $\gamma = 0$ in (24).

We give an alternative version of condition (B) given by [Per60, Equation (1.1)]: For any $R > 0$ there exists $C_R > 0$, such that

$$\int_{B(0,R)} |Au|^2 \, dx \leq C_R \int_{B(0,R)} |u|^2 \, dx, \quad u \in \mathcal{D}(h). \quad (43)$$

Remark. If $A \in C(\mathbb{R}^n)^n$ then (43) is satisfied since A is bounded in $B(0, R)$, in particular

$$\int_{B(0,R)} |Au|^2 \, dx \leq \max_{x \in B(0,R)} |A(x)|^2 \int_{B(0,R)} |u|^2 \, dx.$$

Let

$$\|u\|_{H_A^1(B(0,R))} := \left(\int_{B(0,R)} (|(-i\nabla - A)u(x)|^2 + |u(x)|^2) \, dx \right)^{\frac{1}{2}},$$

and

$$\|u\|_{H^1(B(0,R))} := \left(\int_{B(0,R)} (|\nabla u(x)|^2 + |u(x)|^2) \, dx \right)^{\frac{1}{2}}.$$

Proposition 2. *If (43) holds, then for any $R > 0$ the norms $\|\cdot\|_{H^1(B(0,R))}$ and $\|\cdot\|_{H_A^1(B(0,R))}$ are equivalent on $\mathcal{D}(h)$.*

Proof. Note that

$$\begin{aligned} \|u\|_{H^1(B(0,R))}^2 &= \|(-i\nabla - A)u + Au\|_{L^2(B(0,R))}^2 + \|u\|_{L^2(B(0,R))}^2 \\ &\leq 2\|(-i\nabla - A)u\|_{L^2(B(0,R))}^2 + 2\|Au\|_{L^2(B(0,R))}^2 + \|u\|_{L^2(B(0,R))}^2 \\ &\leq 2\|(-i\nabla - A)u\|_{L^2(B(0,R))}^2 + (1 + 2C_R)\|u\|_{L^2(B(0,R))}^2 \\ &\leq C^2(\|(-i\nabla - A)u\|_{L^2(B(0,R))}^2 + \|u\|_{L^2(B(0,R))}^2), \end{aligned}$$

where in the last inequality we have used (43) and $C^2 := \max\{2, 1 + 2C_R\}$. Taking square roots we get that $\|u\|_{H^1(B(0,R))} \leq C\|u\|_{H_A^1(B(0,R))}$. Similarly,

$$\begin{aligned} \|u\|_{H_A^1(B(0,R))}^2 &= \|(-i\nabla - A)u\|_{L^2(B(0,R))}^2 + \|u\|_{L^2(B(0,R))}^2 \\ &\leq 2\|\nabla u\|_{L^2(B(0,R))}^2 + 2\|Au\|_{L^2(B(0,R))}^2 + \|u\|_{L^2(B(0,R))}^2 \\ &\leq 2\|\nabla u\|_{L^2(B(0,R))}^2 + (1 + 2C_R)\|u\|_{L^2(B(0,R))}^2 \\ &\leq \tilde{C}^2(\|\nabla u\|_{L^2(B(0,R))}^2 + \|u\|_{L^2(B(0,R))}^2), \end{aligned}$$

where again we have used (43) and $\tilde{C}^2 := \max\{2, 1 + 2C_R\}$. Taking square roots we get $\|u\|_{H_A^1(B(0,R))} \leq \tilde{C}\|u\|_{H^1(B(0,R))}$. \square

Theorem 7 (Persson's lemma). *Let Σ be defined as in (39). Then $\Sigma = \inf \sigma_{\text{ess}}(H)$.*

We divide the proof in two lemmas.

Lemma 2. *Let Σ be defined as in (39). Then $\inf \sigma_{\text{ess}}(H) \leq \Sigma$.*

Proof. Let $0 < \varepsilon < 1$, if $M \subset L^2(\mathbb{R}^n)$ is compact, then we can find a finite subset $\{u_n\}_{n=1}^N \subset C_c^\infty(\mathbb{R}^n)$ such for any $u \in M$ we have $\|u - u_i\|_{L^2(\mathbb{R}^n)}^2 < \varepsilon$ for at least one $i \in \{1, \dots, N\}$. We can find an $R_\varepsilon > 0$ such that all u_i vanish outside the ball $B(0, R_\varepsilon)$. Hence, for every $\varepsilon > 0$ we can find a $R_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^n \setminus B(0, R_\varepsilon)} |u(x)|^2 dx < \varepsilon \quad (44)$$

for all $u \in M$. Let $\mu < \inf \sigma_{\text{ess}}(H)$, similarly as in (31) we can find an orthonormal system of eigenfunctions of H such that

$$E_H((-\infty, \mu]) = \sum_{i=1}^m \langle \cdot, u_i \rangle u_i. \quad (45)$$

This implies that $\text{Ran}(E_H)$ is finite dimensional and the closed unit sphere $M_\mu := \{u \in \text{Ran}(E_H(-\infty, \mu]) : \|u\|_{L^2(\mathbb{R}^n)} \leq 1\}$ is compact (this follows from Heine-Borel theorem [Rud76, Theorem 2.41] since $\text{Ran}(E_H(-\infty, \mu])$ is homeomorphic to \mathbb{R}^{2m} and M_μ is closed and bounded). Then, by (44) for every $\varepsilon > 0$ we have a $R_\varepsilon > 0$ such that

$$\int_{\mathbb{R}^n \setminus B(0, R_\varepsilon)} |u(x)|^2 dx < \varepsilon \quad (46)$$

for all $u \in M_\mu$. Let $u \in C_c^\infty(\mathbb{R}^n)$ with $\text{supp } u \subset \mathbb{R}^n \setminus B(0, R_\varepsilon)$ and $\|u\|_{L^2(\mathbb{R}^n)} = 1$, then $u \in \mathcal{D}(H)$ (recall that $C_c^\infty(\mathbb{R}^n) \subset \mathcal{D}(H)$) and we can expand $u = v + w$ with $v \in \text{Ran}(E_H(-\infty, \mu])$ and $w \in \text{Ran}(E_H(-\infty, \mu])^\perp$. Since v and w are orthogonal $\|u\|_{L^2(\mathbb{R}^n)}^2 = \|v\|_{L^2(\mathbb{R}^n)}^2 + \|w\|_{L^2(\mathbb{R}^n)}^2$ implying that both v and w have norm less than one and, in particular, that $v \in M_\mu$.

Recall that $\text{supp } u \subset \mathbb{R}^n \setminus B(0, R_\varepsilon)$, so we have $v(x) + w(x) = 0$ almost everywhere in $B(0, R_\varepsilon)$. Taking the complex conjugate and multiplying by $v(x)$ we also have

$$|v(x)|^2 + \overline{w(x)}v(x) = 0 \quad \text{almost everywhere in } B(0, R_\varepsilon). \quad (47)$$

Integrating (47) over $B(0, R_\varepsilon)$

$$\int_{B(0, R_\varepsilon)} |v(x)|^2 dx = - \int_{B(0, R_\varepsilon)} \overline{w(x)}v(x) dx = \int_{\mathbb{R}^n \setminus B(0, R_\varepsilon)} \overline{w(x)}v(x) dx,$$

since $\langle w, v \rangle_{L^2(\mathbb{R}^n)} = 0$. By Cauchy–Schwarz inequality,

$$\begin{aligned} \int_{B(0, R_\varepsilon)} |v(x)|^2 dx &\leq \|v\|_{L^2(\mathbb{R}^n \setminus B(0, R_\varepsilon))} \|w\|_{L^2(\mathbb{R}^n \setminus B(0, R_\varepsilon))} \\ &\stackrel{(46)}{\leq} \varepsilon \|w\|_{L^2(\mathbb{R}^n \setminus B(0, R_\varepsilon))} \leq \varepsilon, \end{aligned}$$

where in the last inequality we used the fact that w has norm less than one. Then we have

$$\|v\|_{L^2(\mathbb{R}^n)}^2 = \int_{B(0, R_\varepsilon)} |v(x)|^2 dx + \int_{\mathbb{R}^n \setminus B(0, R_\varepsilon)} |v(x)|^2 dx < 2\varepsilon,$$

and $\|w\|_{L^2(\mathbb{R}^n)}^2 = \|u\|_{L^2(\mathbb{R}^n)}^2 - \|v\|_{L^2(\mathbb{R}^n)}^2 > 1 - 2\varepsilon$. Finally, it is clear that $Hv \in \text{Ran}(E_H(-\infty, \mu])$ and $Hw \in \text{Ran}(E_H(-\infty, \mu])^\perp$, so it follows that

$$\langle u, Hu \rangle = \langle v, Hv \rangle + \langle w, Hw \rangle \geq \mu \langle w, w \rangle > \mu(1 - 2\varepsilon),$$

where we have used $\langle v, Hv \rangle \geq 0$ because H is bounded from below with bound $\gamma = 0$. We have seen that $\langle u, Hu \rangle > \mu(1 - 2\varepsilon)$ for all $u \in C_c^\infty(\mathbb{R}^n)$ with support outside $B(0, R_\varepsilon)$, so, using the fact that Σ_R is an increasing function, we have

$$\Sigma > \mu(1 - 2\varepsilon). \quad (48)$$

Since the choice of $\varepsilon > 0$ was arbitrary, and we can take μ as close to $\inf \sigma_{\text{ess}}(H)$ as we want, we deduce that $\inf \sigma_{\text{ess}}(H) \leq \Sigma$, which finishes the proof. \square

Lemma 3. *Let Σ be defined as in (39). Then $\inf \sigma_{\text{ess}}(H) \geq \Sigma$.*

Proof. Let $\varepsilon > 0$ be arbitrary and let $\lambda \in \sigma_{\text{ess}}(H)$, by Proposition 1 we can find a Weyl sequence $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(H)$ for H and λ . Next, we want to prove that given any bounded measurable set $K \subset \mathbb{R}^n$ we have that $\lim_{n \rightarrow \infty} \|u_n\|_{L^2(K)} = 0$. Given such a set $K \subset \mathbb{R}^n$ we can find an $R > 0$ big enough such that $K \subset B(0, R)$, so it is enough to prove that $\lim_{n \rightarrow \infty} \|u_n\|_{L^2(B(0, R))} = 0$ for all $R > 0$. To prove this we will show first that $\{u_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $(H^1(B(0, R)), \|\cdot\|_{H^1(B(0, R))})$. Since $\lim_{n \rightarrow \infty} \|(H - \lambda)u_n\|_{L^2(\mathbb{R}^n)} = 0$, we know that for n sufficiently large we have

$$\|(H - \lambda)u_n\|_{L^2(\mathbb{R}^n)} \leq 1.$$

Recall that $\|u_n\|_{L^2(\mathbb{R}^n)} = 1$, so by Cauchy-Schwarz inequality,

$$\langle Hu_n, u_n \rangle - \lambda = \langle (H - \lambda)u_n, u_n \rangle \leq \|(H - \lambda)u_n\|_{L^2(\mathbb{R}^n)}.$$

Since $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(T)$ we have, for all $n \in \mathbb{N}$,

$$\langle Hu_n, u_n \rangle = h(u_n, u_n) = \int_{\mathbb{R}^n} |(-i\nabla - A)u_n|^2 dx + \int_{\mathbb{R}^n} V(x)|u_n(x)|^2 dx,$$

where h was defined in (42). Combining the three previous expressions we have, for n large enough,

$$\int_{\mathbb{R}^n} |(-i\nabla - A)u_n|^2 dx \leq 1 + \lambda - \int_{\mathbb{R}^n} V(x)|u_n(x)|^2 dx \leq 1 + \lambda, \quad (49)$$

where we have used that $V(x) \geq 0$. This means that for all $R > 0$ and large enough n

$$\|(-i\nabla - A)u_n\|_{L^2(B(0, R))}^2 + \|u_n\|_{L^2(B(0, R))}^2 \leq \lambda + 2. \quad (50)$$

By Proposition 2 we get

$$\|\nabla u_n\|_{L^2(B(0, R))}^2 + \|u_n\|_{L^2(B(0, R))}^2 \leq C(\lambda + 2). \quad (51)$$

This means that $\{u_n\}_{n \in \mathbb{N}}$ is a bounded sequence in $(H^1(B(0, R)), \|\cdot\|_{H^1(B(0, R))})$. By Rellich-Kondrachov theorem [BreII, Theorem 9.16] we know that the inclusion

$$H^1(B(0, R)) \hookrightarrow L^2(B(0, R))$$

is compact, which implies that $\{u_n\}_{n \in \mathbb{N}}$ has a convergent subsequence in $L^2(B(0, R))$. Since $\lim_{n \rightarrow \infty} \langle v, u_n \rangle = 0$ for all $v \in L^2(\mathbb{R}^n)$, then the convergent subsequence must converge also to zero in $L^2(B(0, R))$. This forces the original sequence $\{u_n\}_{n \in \mathbb{N}}$ to converge to zero in $L^2(B(0, R))$. This means that $\lim_{n \rightarrow \infty} \|u_n\|_{L^2(K)} = 0$ for any bounded set $K \subset \mathbb{R}^n$.

Let $R' > R > 0$ and consider a function $\chi \in C^\infty(\mathbb{R}^n, [0, 1])$ such that $|\nabla \chi(x)| \leq C$ for some constant $C > 0$ and

$$\chi(x) = \begin{cases} 1 & \text{if } x \in \mathbb{R}^n \setminus B(0, R') \\ 0 & \text{if } x \in B(0, R) \end{cases}. \quad (52)$$

For all $n \in \mathbb{N}$ it holds that $\chi u_n \in \mathcal{D}(h)$. Indeed, $(-i\nabla - A)(\chi u_n) = \chi(-i\nabla - A)u_n + u_n \nabla \chi$, so

$$\begin{aligned} \|(-i\nabla - A)\chi u_n\|_{L^2(\mathbb{R}^n)} &\leq \|\chi(-i\nabla - A)u_n + u_n \nabla \chi\|_{L^2(\mathbb{R}^n)} \\ &\leq \|(-i\nabla - A)u_n\|_{L^2(\mathbb{R}^n)} + C\|u_n\|_{L^2(\mathbb{R}^n)}. \end{aligned}$$

Moreover, $\|\sqrt{V}\chi u_n\|_{L^2(\mathbb{R}^n)} \leq \|\sqrt{V}u_n\|_{L^2(\mathbb{R}^n)}$.

We are going to prove that

$$\|(-i\nabla - A)(\chi u_n)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{V}\chi u_n\|_{L^2(\mathbb{R}^n)}^2 \leq \lambda \| \chi u_n \|_{L^2(\mathbb{R}^n)}^2 + o(1), \quad (53)$$

where $o(1) \rightarrow 0$ as $n \rightarrow \infty$. By the triangle inequality we have

$$\|(-i\nabla - A)(\chi u_n)\|_{L^2(\mathbb{R}^n)} \leq \|\chi(-i\nabla - A)u_n\|_{L^2(\mathbb{R}^n)} + \|u_n \nabla \chi\|_{L^2(\mathbb{R}^n)}. \quad (54)$$

The second term can be bounded as $\|u_n \nabla \chi\|_{L^2(\mathbb{R}^n)} \leq C\|u_n\|_{L^2(B(0, R'))}$ since $|\nabla \chi| \leq C$ and $\text{supp}(\nabla \chi) \subset B(0, R') \setminus B(0, R)$. Before we have shown that $\lim_{n \rightarrow \infty} \|u_n\|_{L^2(B(0, R'))} = 0$, then there exists an $N_1 \in \mathbb{N}$ such that for all $n \geq N_1$

$$\|u_n \nabla \chi\|_{L^2(\mathbb{R}^n)}^2 \leq \varepsilon. \quad (55)$$

On the other hand, since $0 \leq \chi(x) \leq 1$

$$\begin{aligned} \|\chi(-i\nabla - A)u_n\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{V}\chi u_n\|_{L^2(\mathbb{R}^n)}^2 &\leq \|(-i\nabla - A)u_n\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{V}u_n\|_{L^2(\mathbb{R}^n)}^2 \\ &\leq \langle u_n, H u_n \rangle_{L^2(\mathbb{R}^n)} \\ &\leq \|u_n\|_{L^2(\mathbb{R}^n)} \|H u_n\|_{L^2(\mathbb{R}^n)}, \end{aligned}$$

where we recall that $\{u_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(H)$. Since $\lim_{n \rightarrow \infty} \|(H - \lambda)u_n\|_{L^2(\mathbb{R}^n)} = 0$ then we can find $N_2 \in \mathbb{N}$ such that for all $n \geq N_2$

$$\|\chi(-i\nabla - A)u_n\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{V}\chi u_n\|_{L^2(\mathbb{R}^n)}^2 \leq (\lambda + \varepsilon) \|u_n\|_{L^2(\mathbb{R}^n)}^2,$$

and using the fact that $\lim_{n \rightarrow \infty} \|u_n\|_{L^2(B(0, R'))} = 0$ and the assumptions on χ , we can choose N_3 large enough so that for $n \geq N_3$

$$\|u_n\|_{L^2(\mathbb{R}^n)}^2 \leq \|u_n\|_{L^2(B(0, R'))}^2 + \|\chi u_n\|_{L^2(\mathbb{R}^n)}^2 \leq \|\chi u_n\|_{L^2(\mathbb{R}^n)}^2 + \varepsilon. \quad (56)$$

Thus, for $n \geq \max\{N_2, N_3\}$, we have

$$\|\chi(-i\nabla - A)u_n\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{V}\chi u_n\|_{L^2(\mathbb{R}^n)}^2 \leq (\lambda + \epsilon)\|\chi u_n\|_{L^2(\mathbb{R}^n)}^2 + \epsilon(\lambda + \epsilon). \quad (57)$$

Using this inequality in (54) after squaring, we get for $n \geq \max\{N_1, N_2, N_3\}$

$$\begin{aligned} \|(-i\nabla - A)(\chi u_n)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{V}\chi u_n\|_{L^2(\mathbb{R}^n)}^2 &\leq \|\chi(-i\nabla - A)u_n\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad + \|\sqrt{V}\chi u_n\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad + \|u_n \nabla \chi\|_{L^2(\mathbb{R}^n)}^2 \\ &\quad + 2 \operatorname{Re}\langle \chi(-i\nabla - A)u_n, u_n \nabla \chi \rangle_{L^2(\mathbb{R}^n)} \\ &\leq \lambda \|\chi u_n\|_{L^2(\mathbb{R}^n)}^2 + o(1), \end{aligned}$$

because by (55)

$$2|\langle \chi(-i\nabla - A)u_n, u_n \nabla \chi \rangle_{L^2(\mathbb{R}^n)}| \leq 2\|\chi(-i\nabla - A)u_n\|_{L^2(\mathbb{R}^n)}\|u_n \nabla \chi\|_{L^2(\mathbb{R}^n)} = o(1).$$

Because of this, for $n \geq \max\{N_1, N_2, N_3\}$ we obtain

$$\frac{\|(-i\nabla - A)(\chi u_n)\|_{L^2(\mathbb{R}^n)}^2 + \|\sqrt{V}\chi u_n\|_{L^2(\mathbb{R}^n)}^2}{\|\chi u_n\|_{L^2(\mathbb{R}^n)}^2} \leq \lambda + o(1), \quad (58)$$

where $1/\|\chi u_n\|_{L^2(\mathbb{R}^n)}^2 \leq 1/(1 - \epsilon)$ because of (56) and $\|u_n\|_{L^2(\mathbb{R}^n)} = 1$. Since $\Sigma = \lim_{R \rightarrow \infty} \Sigma_R$, where Σ_R is an increasing function, we can find a $R_\epsilon > 0$ large enough such that

$$\Sigma \leq \Sigma_{R_\epsilon} + \epsilon. \quad (59)$$

Note that we can choose $R > R_\epsilon$ so that $\chi u_n \in \mathcal{M}_{R_\epsilon}$. For large enough n we have, by (58),

$$\Sigma_{R_\epsilon} \leq \lambda + o(1), \quad (60)$$

where $o(1) \rightarrow 0$ as $n \rightarrow +\infty$. Since $\epsilon > 0$ is arbitrary, the proof is complete. \square

Remark. We refer to [FH10] for a generalization of Persson's lemma following the ideas in [Agm82] and to [Bono3, Ass21] for results concerning magnetic Schrödinger operators in unbounded domains with non-empty boundary and magnetic Neumann boundary conditions.

2.4 Sturm–Liouville operators

In this section, we introduce second order ordinary differential operators and, in particular, we focus on Sturm–Liouville operators. We refer to [Wei87, Zeto5] for a complete survey

on the subject. Let (a, b) be a bounded or unbounded interval in \mathbb{R} and $w : (a, b) \rightarrow \mathbb{R}$ be a continuous and positive function, then the weighted space $L^2((a, b), w(x) dx)$ given by measurable functions u defined on (a, b) such that $\int_a^b |u(x)|^2 w(x) dx < \infty$ together with the scalar product

$$\langle u, v \rangle_{L^2((a,b),w(x) dx)} := \int_a^b \overline{u(x)}v(x) w(x) dx \quad (61)$$

is a Hilbert space. This can be seen from the fact that $U_w : L^2((a, b), w(x) dx) \rightarrow L^2(a, b)$ given by $U_w u = \sqrt{w}u$ is an isomorphism and $L^2(a, b)$ is a Hilbert space (see [BREI1, Theorem 4.8]). Consider the differential expression L defined by

$$Lu := \frac{1}{w}(- (pu')' + qu), \quad (62)$$

where $p > 0$ and q are real-valued continuous functions defined on (a, b) . We are interested in eigenvalue problems of the form $Lu = \lambda u$ with $\lambda \in \mathbb{R}$. The endpoint a (respectively b) is said to be *regular* if $1/p, q \in L^1(a, d)$ (respectively $1/p, q \in L^1(d, b)$) for some $d \in (a, b)$. Next, we define operators acting on some $L^2((a, b), w(x) dx)$ using the differential expression L .

2.4.1 Harmonic oscillator on the half-line and band functions for magnetic steps

There are two spectral constants, Θ_0 and β_0 , appearing in different contexts in this thesis. These constants come from minimizing the bottom of the spectrum of Sturm–Liouville operator families with respect to a parameter. We follow [BONO3, Chapter 2] and [FH10, Chapter 3]. For $k \in \mathbb{N}$ consider

$$B^k(\mathbb{R}^+) := \{u \in L^2(\mathbb{R}^+) : t^p u^{(q)} \in L^2(\mathbb{R}^+), \forall p, q \in \mathbb{Z}_{\geq 0} \text{ s.t. } p + q \leq k\}, \quad (63)$$

and for $u \in B^k(\mathbb{R}^+)$ define

$$\|u\|_{B^k(\mathbb{R}^+)} := \sum_{p+q \leq k} \|t^p u^{(q)}\|_{L^2(\mathbb{R}^+)}. \quad (64)$$

We define similarly $(B^k(\mathbb{R}), \|\cdot\|_{B^k(\mathbb{R})})$. Consider the closed quadratic form

$$q^\xi(u) = \int_0^\infty (|u'(t)|^2 + (t - \xi)^2 |u(t)|^2) dt \quad (65)$$

with domain $\mathcal{D}(q^\xi) = B^1(\mathbb{R}^+)$. The inducing sesquilinear form coming from q^ξ is densely defined and also bounded from below since $q^\xi(u) \geq 0$ for any $u \in B^1(\mathbb{R}^+)$. Then, by Theorem 2 and integration by parts, we have that the self-adjoint operator associated with q^ξ is the Neumann realization of the harmonic oscillator on the half-line given by

$$\mathfrak{h}^\xi = -\frac{d^2}{dt^2} + (t - \xi)^2 \quad (66)$$

with domain $\mathcal{D}(\mathfrak{h}^\xi) = \{u \in B^1(\mathbb{R}^+) : \mathfrak{h}^\xi u \in L^2(\mathbb{R}^+) \text{ and } u'(0) = 0\}$. Using the difference quotients method [BreII, Section 9.5], one can show, similarly as done for the harmonic oscillator in [Ray17, Proposition 2.2], that $\mathcal{D}(\mathfrak{h}^\xi) = \{u \in B^2(\mathbb{R}^+) : u'(0) = 0\}$. \mathfrak{h}^ξ has compact resolvent because $(B^1(\mathbb{R}^+, \|\cdot\|_{B^1(\mathbb{R}^+)})$ is compactly embedded in $L^2(\mathbb{R}^+)$. We show this in the following proposition which is an adaption of [BreII, Corollary 4.27].

Proposition 3. $(B^1(\mathbb{R}^+, \|\cdot\|_{B^1(\mathbb{R}^+)})$ is compactly embedded in $L^2(\mathbb{R}^+)$.

Proof. The proof is divided in two stages. Let $R > 0$ be fixed and consider the bounded set $\mathcal{F}_R := \overline{B(0, R)}$ in $(B^1(\mathbb{R}, \|\cdot\|_{B^1(\mathbb{R})})$. The first stage is proving that $\lim_{|h| \rightarrow 0} \|\tau_h u - u\|_{L^2(\mathbb{R})} = 0$ uniformly in \mathcal{F}_R , where $\tau_h u(t) := u(t + h)$.

Let $u \in \mathcal{F}_R$, then $u \in B^1(\mathbb{R}) \subset H^1(\mathbb{R})$. By [BreII, Proposition 8.5] we know that there exists a constant C , in fact $C = \|u'\|_{L^2(\mathbb{R})}$, such that for all $h \in \mathbb{R}$

$$\|\tau_h u - u\|_{L^2(\mathbb{R})} \leq \|u'\|_{L^2(\mathbb{R})} |h| \leq R|h|,$$

where in the last inequality we used $u \in \mathcal{F}_R$, so $\|u\|_{B^1(\mathbb{R})} \leq R$. Therefore,

$$\lim_{|h| \rightarrow 0} \|\tau_h u - u\|_{L^2(\mathbb{R})} = 0$$

uniformly in \mathcal{F}_R . Using [BreII, Theorem 4.26] we know that for any measurable set $\Omega \subset \mathbb{R}$ of finite measure, the closure of $\mathcal{F}_R|_\Omega$ in $L^2(\Omega)$ is compact.

For a given $\varepsilon > 0$, we choose the bounded interval $\Omega_\varepsilon := (0, \frac{\sqrt{2}}{\varepsilon}) \subset \mathbb{R}^+$, then for any $u \in B^1(\mathbb{R}^+)$

$$\int_{\mathbb{R}^+ \setminus \Omega_\varepsilon} |u(t)|^2 dt \leq \frac{\varepsilon^2}{2} \int_{\mathbb{R}^+} t^2 |u(t)|^2 dt < \varepsilon^2 \|u\|_{B^1(\mathbb{R}^+)}^2. \quad (67)$$

Let $\mathcal{G} := \overline{B(0, 1)}$ in $(B^1(\mathbb{R}^+, \|\cdot\|_{B^1(\mathbb{R}^+)})$ and for any $u \in B^1(\mathbb{R}^+)$ define the reflection

$$T(u)(t) := \begin{cases} u(t) & t \geq 0 \\ u(-t) & t < 0 \end{cases}. \quad (68)$$

Note that $\|T(u)\|_{B^1(\mathbb{R})} = \sqrt{2} \|u\|_{B^1(\mathbb{R}^+)}$ which means that $T(\mathcal{G}) \subseteq \mathcal{F}_R$ with $R = \sqrt{2}$. Let Ω_ε be defined as before, by [BreII, Theorem 4.26] we know that $\mathcal{F}_{\sqrt{2}}|_{\Omega_\varepsilon}$ has compact closure in $L^2(\Omega_\varepsilon)$, so we can find a finite cover such that

$$T(\mathcal{G})|_{\Omega_\varepsilon} \subset \bigcup_{n=1}^N B(g_n, \varepsilon)$$

with $g_n \in L^2(\Omega_\varepsilon)$ and some $N \in \mathbb{N}$. Set

$$h_n(t) = \begin{cases} g_n(t) & \text{if } t \in \Omega_\varepsilon \\ 0 & \text{if } t \in \mathbb{R}^+ \setminus \Omega_\varepsilon \end{cases}, \quad (69)$$

then using that $T(\mathcal{G})|_{\Omega_\varepsilon} = \mathcal{G}|_{\Omega_\varepsilon}$ and (67) we have that \mathcal{G} is covered by the finite cover $\{B(h_n, 2\varepsilon)\}_{1 \leq n \leq N}$. \square

Remark. One can show that q^ξ is closed using the fact that, for a given $\xi \in \mathbb{R}$, the norm induced by the quadratic form is equivalent to the norm

$$\|u\|_{B^1(\mathbb{R}^+)} = \|u'\|_{L^2(\mathbb{R}^+)} + \|tu\|_{L^2(\mathbb{R}^+)} + \|u\|_{L^2(\mathbb{R}^+)},$$

for $u \in B^1(\mathbb{R}^+)$. Then by reflection we can relate a sequence in $B^1(\mathbb{R}^+)$ with a sequence in $B^1(\mathbb{R}) = \{u \in L^2(\mathbb{R}) : u \in H^1(\mathbb{R}), \hat{u} \in H^1(\mathbb{R})\}$ with the norm

$$\|u\|_{B^1(\mathbb{R})} = \|u'\|_{L^2(\mathbb{R})} + \|tu\|_{L^2(\mathbb{R})} + \|u\|_{L^2(\mathbb{R})}.$$

Finally, we can prove that $(B^1(\mathbb{R}), \|\cdot\|_{B^1(\mathbb{R})})$ is a Hilbert space using Plancherel theorem and the fact that $(H^1(\mathbb{R}), \|\cdot\|_{H^1(\mathbb{R})})$ is a Hilbert space [BREI1, Proposition 8.1].

Since \mathfrak{h}^ξ has compact resolvent and is bounded from below we have that $\sigma(\mathfrak{h}^\xi) = \sigma_d(\mathfrak{h}^\xi)$ and we can order its eigenvalues in increasing order as $\lambda_1(\mathfrak{h}^\xi) < \lambda_2(\mathfrak{h}^\xi) < \lambda_3(\mathfrak{h}^\xi) < \dots$. Note that the eigenvalues are simple due to Cauchy uniqueness.

The family $(\mathfrak{h}^\xi)_{\xi \in \mathbb{R}}$ is a holomorphic family of type (A) in the sense of Kato [Kat76, Chapter VII.2] which implies that $\xi \mapsto \lambda_1(\mathfrak{h}^\xi)$ is a holomorphic function. A more elementary proof of this fact can be found in [Ray17, Lemma 2.14]. It is known that $\xi \mapsto \lambda_1(\mathfrak{h}^\xi)$ has a unique minimum denoted by $\xi_0 > 0$. We define $\Theta_0 := \lambda_1(\mathfrak{h}^{\xi_0})$. We know that $\Theta_0 \approx 0.590106125$ up to an error of size 10^{-9} and $\xi_0^2 = \Theta_0$ (see [BN12]).

For the second family of operators, known in the literature as the ‘‘Trapping magnetic case’’ [HPRS16], consider $\mathfrak{b} = (b, 1)$ with $b \in [-1, 0)$ and

$$V_{\mathfrak{b}}(\xi, t) = \begin{cases} (bt + \xi)^2, & t < 0 \\ (t + \xi)^2, & t > 0 \end{cases}.$$

Then we can define the closed quadratic form

$$q_{\mathfrak{b}}^\xi(u) = \int_{\mathbb{R}} (|u'(t)|^2 + V_{\mathfrak{b}}(\xi, t)|u(t)|^2) dt, \quad (70)$$

with domain $\mathcal{D}(q_{\mathfrak{b}}^\xi) = B^1(\mathbb{R})$. Closedness of $q_{\mathfrak{b}}^\xi$ can be shown similarly as in Remark 2.4.1 for q^ξ . Then, as we did for \mathfrak{h}^ξ , we have that the self-adjoint operator associated with $q_{\mathfrak{b}}^\xi$ is given by

$$\mathfrak{h}_{\mathfrak{b}}^\xi = -\frac{d}{dt^2} + V_{\mathfrak{b}}(\xi, t), \quad (71)$$

with domain $\mathcal{D}(\mathfrak{h}_{\mathfrak{b}}^\xi) = \{u \in B^1(\mathbb{R}) : \mathfrak{h}_{\mathfrak{b}}^\xi u \in L^2(\mathbb{R})\}$. We have that $\mathfrak{h}_{\mathfrak{b}}^\xi$ has compact resolvent and that $\lambda_1(\mathfrak{h}_{\mathfrak{b}}^\xi)$ is a simple eigenvalue and a holomorphic function with respect to ξ . Let

$\beta_b := \inf_{\xi \in \mathbb{R}} \lambda_1(\mathfrak{h}_b^\xi)$, then we know (see [HPRS16]) that there exists a unique $\xi_b \in \mathbb{R}$ such that $\beta_b = \lambda_1(\mathfrak{h}_b^{\xi_b})$. Moreover, it was proven in [AKPS19, AK22] that

$$|b| \Theta_0 < \beta_b < \min\{|b|, \Theta_0\}. \quad (72)$$

2.4.2 Sturm-Liouville operators with singular potentials

In Paper III we study operators acting on $L^2((0, R), r \, dr)$ which are associated with the differential expression

$$H_{m,b} := -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \left(\frac{m}{r} - \frac{br}{2} \right)^2 \quad (73)$$

acting on $L^2((0, R), r \, dr)$ with $m \in \mathbb{Z}$. Note that these operators have a singularity at the origin and are closely related to Bessel-type operators. We adapt [KL22, GPS24, KL10] to our setting to describe in detail the domains of these operators.

We consider the family of non-negative closed quadratic forms $q_{m,R}^b$ with $R \in (0, +\infty)$

$$q_{m,R}^b(u) := \int_0^R \left(|u'(r)|^2 + \left(\frac{m}{r} - \frac{br}{2} \right)^2 |u(r)|^2 \right) r \, dr \quad (74)$$

acting on $L^2((0, R); r \, dr)$ with domain

$$\mathcal{D}(q_{m,R}^b) = \left\{ u \in L^2((0, R), r \, dr) : u', \frac{mu}{r} \in L^2((0, R), r \, dr) \right\}. \quad (75)$$

By Theorem 2 we know that there exists a unique self-adjoint operator $S_{m,b}$ on $L^2((0, R), r \, dr)$ such that $\mathcal{D}(S_{m,R}^b) \subset \mathcal{D}(q_{m,R}^b)$ and

$$\langle v, S_{m,R}^b u \rangle_{L^2((0,R),r \, dr)} = s_{m,R}^b(v, u) \quad \text{for } u \in \mathcal{D}(S_{m,R}^b), v \in \mathcal{D}(q_{m,R}^b),$$

where $s_{m,R}^b$ is the sesquilinear form inducing $q_{m,R}^b$. Next, we show that $S_{m,R}^b$ is actually given by the differential expression given in (73) with domain

$$\mathcal{D}(H_{m,R}^b) = \left\{ u : u, u/r, u', H_{m,b}u \in L^2((0, R), r \, dr), u'(R) = 0 \right\},$$

for $m \neq 0$ and

$$\mathcal{D}(H_{0,R}^b) = \left\{ u : u, H_{0,b}u \in L^2((0, R), r \, dr), u'(R) = 0 \text{ and } \lim_{r \rightarrow 0^+} \frac{u(r)}{\ln r} = 0 \right\}.$$

To this end, we need the following result which is an adaptation of [KL10, Theorem 2.2] (see also [LV11, Theorem 3.1]).

Proposition 4. Let $u \in \{u \in L^2((0, R), r \, dr) : H_{m,R}^b u \in L^2((0, R), r \, dr)\}$ if $m = 0$ and $u \in \{u \in L^2((0, R), r \, dr) : H_{m,R}^b u, u/r \in L^2((0, R), r \, dr)\}$ if $m \geq 1$. Then the following holds:

(a) If $m = 0$, for $r \leq R/2$ we can write u as

$$u(r) = c_1(u) + c_2(u) \ln r + \tilde{u},$$

where $c_1(u), c_2(u)$ are constants and $\tilde{u} \in C^1([0, R/2])$ such that $\tilde{u}(r) = \mathcal{O}(r)$, $\tilde{u}'(r) = \mathcal{O}(1)$.

(b) If $m \geq 1$, for $r \leq R/2$ we can write

$$u(r) = c_1(u, R)r^m + \tilde{u},$$

where $c_1(u, R)$ is a constant and $\tilde{u} \in C^1([0, R/2])$ such that $\tilde{u}(r) = \mathcal{O}(r)$, $\tilde{u}'(r) = \mathcal{O}(1)$.

Proof. (a) $m = 0$. Since $H_{m,R}^b u \in L^2((0, R), r \, dr)$ we can find a $f \in L^2((0, R), r \, dr)$ such that $H_{m,R}^b u = f$, and we can rearrange terms to get

$$(ru'(r))' = -rf(r) + \frac{b^2 r^3}{4}u(r),$$

then integrating we get

$$ru'(r) = c_1 + \int_0^r (-f(s) + \frac{b^2 s^2}{4}u(s)) s \, ds,$$

where c_1 is an integration constant. Let $\psi(r) := -f(r) + \frac{b^2 r^2}{4}u(r)$, since $f, u \in L^2((0, R), r \, dr)$ we have that $\psi \in L^2((0, R), r \, dr)$. By Cauchy–Schwarz we have

$$\left| \int_0^r \psi(s) s \, ds \right| \leq \left(\int_0^r s \, ds \right)^{\frac{1}{2}} \left(\int_0^R |\psi(r)|^2 s \, ds \right)^{\frac{1}{2}} = cr, \quad (76)$$

where $c = \frac{1}{\sqrt{2}} \|\psi\|_{L^2((0,R), r \, dr)}$. Integrating once more we obtain

$$u(r) = c_1 \ln r + c_2 + \int_0^r \frac{1}{t} \int_0^t \psi(s) s \, ds \, dt,$$

where again c_2 is an integration constant. By (76) we have

$$\left| \int_0^r \frac{1}{t} \int_0^t \psi(s) s \, ds \, dt \right| \leq \int_0^r c \, dt = cr.$$

This implies that $\tilde{u}(r) = \mathcal{O}(r)$ and

$$|\tilde{u}'(r)| = \left| \frac{1}{r} \int_0^r \psi(s) s \, ds \right| \leq c,$$

using again (76).

(b) $m \geq 1$. Let $u(r) = r^m v(r)$, then again we can find an $f \in L^2((0, R), r \, dr)$ such that $H_{m,R}^b u = f$ to get

$$\frac{1}{r^{m+1}} (r^{2m+1} v'(r))' = -f(r) + \left(\frac{m}{r} - \frac{br}{2} \right)^2 r^m v(r).$$

Integrating, we obtain

$$r^{2m+1} v' = c_1 + \int_0^r \left(-f(s) + \left(\frac{m}{s} - \frac{bs}{2} \right)^2 s^m v(s) \right) s^{m+1} \, ds.$$

Let $\psi(r) := -f(r) + \left(\frac{m}{r} - \frac{br}{2} \right)^2 r^m v(r)$ then

$$\left| \int_0^r s^m \psi(s) \, ds \right| \leq \left(\int_0^r s^{2m+1} \, ds \right)^{\frac{1}{2}} \left(\int_0^R |\psi(r)|^2 s \, ds \right)^{\frac{1}{2}} = cr^{m+1} \quad (77)$$

where $c = \frac{1}{\sqrt{2(m+1)}} \|\psi\|_{L^2((0,R), r \, dr)}$. Integrating again we get

$$v(r) = \frac{1}{(2m+1)r^{2m}} c_1 + c_2 + \int_r^{\frac{R}{2}} \frac{1}{t^{2m+1}} \int_0^t s^m \psi(s) \, ds \, dt,$$

where c_1 and c_2 are integration constants. Then we have that

$$u(r) = \frac{1}{(2m+1)r^m} c_1 + c_2 r^m + r^m \int_r^{\frac{R}{2}} \frac{1}{t^{2m+1}} \int_0^t s^m \psi(s) \, ds \, dt.$$

Since $u \in L^2((0, R), r \, dr)$ and $m \geq 1$ then $c_1 = 0$. By (77) we have

$$\begin{aligned} \left| r^m \int_r^{\frac{R}{2}} \frac{1}{t^{2m+1}} \int_0^t s^m \psi(s) \, ds \, dt \right| &\leq \left| cr^m \int_r^{\frac{R}{2}} t^{-m} \, dt \right| \\ &\leq \tilde{c}(r^m + r). \end{aligned}$$

This implies that $\tilde{u}(r) = \mathcal{O}(r)$ since $m \geq 1$ and

$$|\tilde{u}'(r)| \leq \left| mr^{m-1} \int_r^{\frac{R}{2}} \frac{1}{t^{2m+1}} \int_0^t s^m \psi(s) \, ds \, dt \right| + \left| r^{-m+1} \int_0^t s^m \psi(s) \, ds \, dt \right|$$

which gives $\mathcal{O}(1)$ using (77) once more. \square

Consider the differential expression

$$s_m(v) := v'(r) - \frac{m^2 - \frac{1}{2}}{r^2} v(r). \quad (78)$$

Using the transformation $u(r) = r^{-\frac{1}{2}}v(r)$ can see that if $v \in \mathcal{D}(q_{m,R}^b)$ then $\|s_m u\|_{L^2(0,R)} < \infty$. Then applying [GPS24, Lemma B.1] we have that

$$\lim_{r \rightarrow 0^+} \frac{|v(r)|}{\ln(\bar{R}/r)^{\frac{1}{2}}} = 0 \text{ if } m = 0 \quad \text{and} \quad \lim_{r \rightarrow 0^+} |v(r)| = 0 \text{ if } m \geq 1, \quad (79)$$

for $v \in \mathcal{D}(q_{m,R}^b)$ and $R < \bar{R} < \infty$. Let $u \in \mathcal{D}(H_{m,R}^b)$ and $v \in \mathcal{D}(q_{m,R}^b)$, integrating by parts we have

$$\begin{aligned} s_{m,R}^b(v, u) &= \langle v, H_{m,b}^R u \rangle_{L^2((0,R), r \, dr)} - \lim_{r \rightarrow 0^+} \overline{v(r)} r u'(r) + \overline{v(\bar{R})} R u'(R) \\ &= \langle v, H_{m,b}^R u \rangle_{L^2((0,R), r \, dr)}, \end{aligned}$$

where $\lim_{r \rightarrow 0^+} \overline{v(r)} r u'(r) = 0$ by the expansions in Proposition 4 and (79). This means that $H_{m,R}^b \subset S_{m,R}^b$, but $(H_{m,R}^b, \mathcal{D}(H_{m,R}^b))$ is self-adjoint (see [Zet05, Section 10.4]) which means that $H_{m,R}^b = S_{m,R}^b$.

3 The magnetic Laplacian and the constant fiber direct integral decomposition

An important part of this thesis is understanding the spectral properties of self-adjoint realizations of the *magnetic Laplacian*. This operator is given by the differential expression

$$(-i\nabla - A)^2 = -\Delta + i\nabla \cdot A + 2iA \cdot \nabla + |A|^2 \quad (80)$$

acting on $L^2(\Omega)$ with $\Omega \subset \mathbb{R}^n$ and $A = (A_1, \dots, A_n) \in H^1(\Omega; \mathbb{R}^n)$. We only consider $n \in \{2, 3\}$. In this section we present an important technique which allows us to reduce the study of the spectrum of the magnetic Laplacian to the study of the spectrum of a family of operators, called *fiber operators*.

3.1 Constant fiber direct integrals

Most of the operators we are considering possess either a translation or a rotation invariance. This means that one can find a unitary² operator U such that the operator commutes with U , allowing it to have a direct integral decomposition which reduces the study of the original operator to the study of the so-called *fiber operators*. We follow [RS78, Section XIII.16] and [RS80, Section II.1] to give an introduction to *constant fiber direct integrals* with concrete examples where this decomposition can be used. Some ideas come from [BS87] as well. This

²An operator $U : \mathcal{H}_1 \rightarrow \mathcal{H}_2$ is *unitary* if $\|U(u)\|_{\mathcal{H}_2} = \|u\|_{\mathcal{H}_1}$ for all $u \in \mathcal{H}_1$ and $\text{Ran}(U) = \mathcal{H}_2$.

decomposition was introduced and developed by John von Neumann [vN49] in a more general framework, but some key concepts can be found already in the works of Floquet and Bloch[Blo29].

Let (M, μ) be a σ -finite³ measure space and \mathcal{H} a separable Hilbert space, then we define a *constant fiber direct integral* $L^2(M, d\mu; \mathcal{H})$ to be the set of measurable vector-valued functions $f: X \rightarrow \mathcal{H}$, i.e. $\langle y, f(\cdot) \rangle_{\mathcal{H}}$ is a complex-valued measurable function for each $y \in \mathcal{H}$ (this is also called weak measurability), satisfying

$$\int_M \|f(m)\|_{\mathcal{H}}^2 d\mu(m) < \infty. \quad (81)$$

In the literature we can also find $L^2(M, d\mu; \mathcal{H})$ written as $\int_M^{\oplus} \mathcal{H} d\mu$ or $\int_M \oplus \mathcal{H} d\mu$, and we refer to the applications in the next section to get a better understanding of this choice. The integral can be defined since $\|f(\cdot)\|_{\mathcal{H}}^2$ is also measurable. We prove this in the following lemma. We denote by $\mathcal{B}(\mathbb{R})$ the Borel σ -algebra.

Lemma 4. *Let f and g be vector-valued measurable function from a measure space (M, μ) to a separable Hilbert space \mathcal{H} , then $\|f(\cdot)\|_{\mathcal{H}}^2$ and $\langle f(\cdot), g(\cdot) \rangle_{\mathcal{H}}$ are measurable.*

Proof. We will show that for any $b \geq 0$, $I := \{m \in M: \|f(m)\|_{\mathcal{H}}^2 \leq b\} \in \mathcal{B}(\mathbb{R})$. Since \mathcal{H} is separable, we can find a countable orthonormal basis $\{u_n\}_{n \in \mathbb{N}}$ of \mathcal{H} [RS80, Theorem II.7]. Let

$$A_n := \left\{ m \in M: \sum_{j=1}^n |\langle u_n, f(m) \rangle_{\mathcal{H}}|^2 \leq b \right\},$$

we know that $A_n \in \mathcal{B}(\mathbb{R})$ since $|\cdot|^2$ is a measurable function and finite sums of measurable functions are measurable. For any $m \in M$, we have by [RS80, Theorem II.6]

$$\|f(m)\|_{\mathcal{H}}^2 = \sum_{n=1}^{\infty} |\langle u_n, f(m) \rangle_{\mathcal{H}}|^2,$$

which implies that $I = \bigcap_{n \in \mathbb{N}} A_n$ which is measurable. Hence, $\|f(\cdot)\|_{\mathcal{H}}^2$ is measurable.

To prove that $\langle f(\cdot), g(\cdot) \rangle_{\mathcal{H}}$ is measurable we rely on the fact that

$$\operatorname{Re} \langle f, g \rangle_{\mathcal{H}} = \frac{1}{2} \|f\|_{\mathcal{H}}^2 + \frac{1}{2} \|g\|_{\mathcal{H}}^2 - \frac{1}{2} \|f + g\|_{\mathcal{H}}^2,$$

$$\operatorname{Im} \langle f, g \rangle_{\mathcal{H}} = \frac{1}{2} \|f\|_{\mathcal{H}}^2 + \frac{1}{2} \|g\|_{\mathcal{H}}^2 - \frac{1}{2} \|if + g\|_{\mathcal{H}}^2,$$

and the properties of Borel measurable functions. □

³ X can be written as a countable union of sets with finite measure.

Due to the previous lemma we can define the inner product

$$\langle f, g \rangle_{L^2(M, d\mu; \mathcal{H})} = \int_M \langle f(m), g(m) \rangle_{\mathcal{H}} d\mu(m). \quad (82)$$

Then $(L^2(M, d\mu; \mathcal{H}), \langle \cdot, \cdot \rangle_{L^2(M, d\mu; \mathcal{H})})$ is a Hilbert space (see [Dix81, Proposition 5(i)], alternatively, one can adapt the proof for $L^2(\mathbb{R})$ found in [Bre11, Theorem 4.8]).

Lemma 5. *Let $f, g \in L^2(M, d\mu; \mathcal{H})$ and $\{u_n\}_{n \in \mathbb{N}}$ be a orthonormal basis for \mathcal{H} . Then*

$$\sum_{n=1}^N \langle u_n, g(\cdot) \rangle_{\mathcal{H}} u_n \rightarrow g \quad (83)$$

as $N \rightarrow \infty$ in $L^2(M, d\mu; \mathcal{H})$ and

$$\langle f, g \rangle_{L^2(M, d\mu; \mathcal{H})} = \sum_{n=1}^{\infty} \int_M \langle f(m), u_n \rangle_{\mathcal{H}} \langle u_n, g(m) \rangle_{\mathcal{H}} d\mu(m) \quad (84)$$

Proof. For any $m \in M$, let $g_N(m) := \sum_{n=1}^N \langle u_n, g(m) \rangle_{\mathcal{H}} u_n$. By [RS80, Theorem II.6] we know that $\lim_{N \rightarrow \infty} g_N(m) = g(m)$ and $\lim_{N \rightarrow \infty} \|g_N(m)\|_{\mathcal{H}}^2 = \|g(m)\|_{\mathcal{H}}^2$. Since $\|g_N(m)\|_{\mathcal{H}}^2 = \sum_{n=1}^N |\langle u_n, g(m) \rangle_{\mathcal{H}}|^2 \leq \|g(m)\|_{\mathcal{H}}^2$ then the convergence in (83) follows from dominated convergence theorem [RS80, Theorem I.16].

From [Wei80, Theorem 3.7] we know that for any $m \in M$

$$\langle f(m), g(m) \rangle_{\mathcal{H}} = \sum_{n=1}^{\infty} \langle f(m), u_n \rangle_{\mathcal{H}} \langle u_n, g(m) \rangle_{\mathcal{H}}.$$

Integrating both sides we obtain

$$\int_M \langle f(m), g(m) \rangle_{\mathcal{H}} d\mu(m) = \int_M \sum_{n=1}^{\infty} \langle f(m), u_n \rangle_{\mathcal{H}} \langle u_n, g(m) \rangle_{\mathcal{H}} d\mu(m).$$

Hence, (84) follows from Fubini's theorem since

$$\begin{aligned} \left| \int_M \langle f(m), g(m) \rangle_{\mathcal{H}} d\mu(m) \right| &\leq \int_M |\langle f(m), g(m) \rangle_{\mathcal{H}}| d\mu(m) \\ &\leq \int_M \|f(m)\|_{\mathcal{H}} \|g(m)\|_{\mathcal{H}} d\mu(m) \\ &\leq \|f\|_{L^2(M, d\mu; \mathcal{H})} \|g\|_{L^2(M, d\mu; \mathcal{H})} < \infty, \end{aligned}$$

where we have used Cauchy–Schwarz and Hölder's inequalities. \square

A function $G(\cdot): M \rightarrow \mathcal{L}(\mathcal{H})$ is called *measurable* if for each $u, v \in \mathcal{H}$, the function $\langle u, G(\cdot)v \rangle_{\mathcal{H}}$ is measurable. Let $G(\cdot)$ be measurable and define

$$\|G(\cdot)\|_{\infty} := \text{ess sup} \|G(m)\|_{\mathcal{L}(\mathcal{H})}, \quad (85)$$

and $L^{\infty}(M, d\mu; \mathcal{H})$ as the space (equivalence class of almost everywhere equal) of measurable functions from M to $\mathcal{L}(\mathcal{H})$ with $\|G\|_{\infty} < \infty$.

A bounded operator G on $L^2(M, d\mu; \mathcal{H})$ is *decomposed* by the direct integral decomposition if there is a function $G(\cdot)$ in $L^{\infty}(M, d\mu; \mathcal{H})$ such that for all $f \in L^2(M, d\mu; \mathcal{H})$ we have that

$$(Gf)(m) = G(m)f(m), \quad (86)$$

where this equality should be considered almost everywhere. If that is the case, we call G *decomposable* and $G(m)$ the *fibers* of G . Following the notation usually found in the literature, we also write

$$G = \int_M^{\oplus} G(m) d\mu(m) \quad (87)$$

for a decomposable operator G with fibers $G(m)$. The next result is [RS78, Theorem XIII.83] which tells us that each $G(\cdot) \in L^{\infty}(M, d\mu; \mathcal{H})$ is associated with a unique decomposable operator.

Theorem 8. *If $G(\cdot) \in L^{\infty}(M, d\mu; \mathcal{H})$, then there is a unique decomposable operator $G \in \mathcal{L}(L^2(M, d\mu; \mathcal{H}))$ with $(Gu)(m) = G(m)u(m)$. Moreover, $\|G\|_{\mathcal{L}(L^2(M, d\mu; \mathcal{H}))} = \|G(\cdot)\|_{\infty}$.*

Proof. The uniqueness of G follows from (86). Let $f \in L^2(M, d\mu; \mathcal{H})$, and define a linear operator G by (86), so $(Gf)(m) = G(m)f(m)$.

- $G: L^2(M, d\mu; \mathcal{H}) \rightarrow L^2(M, d\mu; \mathcal{H})$.

Let $f \in L^2(M, d\mu; \mathcal{H})$ and $\{u_n\}_{n \in \mathbb{N}}$ be an ONB of \mathcal{H} . By adapting [RS80, Theorem II.6] we have $G(m)f(m) = \sum_{n=1}^{\infty} \langle u_n, f(m) \rangle_{\mathcal{H}} G(m)u_n$ almost everywhere. By assumption $G(\cdot)$ is measurable which means that $w_N(\cdot) := \sum_{n=1}^N \langle u_n, f(\cdot) \rangle_{\mathcal{H}} G(\cdot)u_n$ is weakly measurable since for any $y \in \mathcal{H}$

$$\langle y, w_N \rangle_{\mathcal{H}} = \sum_{n=1}^N \langle u_n, f(m) \rangle_{\mathcal{H}} \langle y, G(m)u_n \rangle_{\mathcal{H}}$$

is measurable because it is a finite sum of products of measurable functions. By

Lemma 4, we have that $\|w_n(\cdot)\|_{\mathcal{H}}^2$ is measurable. Moreover, for any $N \in \mathbb{N}$

$$\begin{aligned}
\int_{\mathcal{M}} \|w_N(\cdot)\|_{\mathcal{H}}^2 d\mu &= \int_{\mathcal{M}} \|G(m) \sum_{n=1}^N \langle u_n, f(m) \rangle_{\mathcal{H}} u_n\|_{\mathcal{H}}^2 d\mu(m) \\
&\leq \|G(\cdot)\|_{\infty} \int_{\mathcal{M}} \left\| \sum_{n=1}^N \langle u_n, f(m) \rangle_{\mathcal{H}} u_n \right\|_{\mathcal{H}}^2 d\mu(m) \\
&= \|G(\cdot)\|_{\infty} \int_{\mathcal{M}} \sum_{n=1}^N |\langle u_n, f(m) \rangle_{\mathcal{H}}|^2 d\mu(m) \\
&\leq \|G(\cdot)\|_{\infty} \|f\|_{L^2(\mathcal{M}, d\mu; \mathcal{H})}^2. \tag{88}
\end{aligned}$$

Similarly, for $N > K$ we have that

$$\begin{aligned}
\int_{\mathcal{M}} \|w_N(\cdot) - w_K(\cdot)\|_{\mathcal{H}}^2 d\mu &= \int_{\mathcal{M}} \|G(m) \sum_{n=K+1}^N \langle u_n, f(m) \rangle_{\mathcal{H}} u_n\|_{\mathcal{H}}^2 d\mu(m) \\
&\leq \|G(\cdot)\|_{\infty} \int_{\mathcal{M}} \left\| \sum_{n=K+1}^N \langle u_n, f(m) \rangle_{\mathcal{H}} u_n \right\|_{\mathcal{H}}^2 d\mu(m) \\
&= \|G(\cdot)\|_{\infty} \sum_{n=K+1}^N \int_{\mathcal{M}} |\langle u_n, f(m) \rangle_{\mathcal{H}}|^2 d\mu(m). \tag{89}
\end{aligned}$$

Combining (88) and (89) we have that $\{w_n(\cdot)\}_{n \in \mathbb{N}}$ is a Cauchy sequence in $L^2(\mathcal{M}, d\mu; \mathcal{H})$ since by (84) we know that

$$\sum_{n=1}^{\infty} \int_{\mathcal{M}} |\langle u_n, f(m) \rangle_{\mathcal{H}}|^2 d\mu(m) = \|f\|_{L^2(\mathcal{M}, d\mu; \mathcal{H})}^2 < \infty.$$

Hence, there is a $w \in L^2(\mathcal{M}, d\mu; \mathcal{H})$ such that $w_n(\cdot) \rightarrow w(\cdot)$ in $L^2(\mathcal{M}, d\mu; \mathcal{H})$. Note that $w_n(m)$ converges almost everywhere to $G(m)f(m)$ in \mathcal{H} . Thus, $G(\cdot)f(\cdot) \in L^2(\mathcal{M}, d\mu; \mathcal{H})$.

- $\|G\|_{\mathcal{L}(L^2(\mathcal{M}, d\mu; \mathcal{H}))} = \|G(\cdot)\|_{\infty}$.

From (88) we get that $\|G(\cdot)f(\cdot)\|_{\mathcal{L}(L^2(\mathcal{M}, d\mu; \mathcal{H}))} \leq \|G(\cdot)\|_{\infty} \|f(\cdot)\|_{L^2(\mathcal{M}, d\mu; \mathcal{H})}$ for all $f \in L^2(\mathcal{M}, d\mu; \mathcal{H})$. Hence, $\|G\|_{\mathcal{L}(L^2(\mathcal{M}, d\mu; \mathcal{H}))} \leq \|G(\cdot)\|_{\infty}$. For the converse inequality, we can take a dense subset $\{v_n\}_{n \in \mathbb{N}}$ of the unit ball in \mathcal{H} since \mathcal{H} is separable. Let $f \in L^1(\mathcal{M}, d\mu)$ then we can find $g, h \in L^2(\mathcal{M}, d\mu)$ such that $f = gh$ and $\|g\|_{L^2(\mathcal{M}, d\mu)}^2 = \|h\|_{L^2(\mathcal{M}, d\mu)}^2 = \|f\|_{L^1(\mathcal{M}, d\mu)}$, for example one can take $g(m) = |f(m)|^{\frac{1}{2}} e^{i \text{Arg}(f(m))}$ and $h(m) = |f(m)|^{\frac{1}{2}}$. Fix $i, j \in \mathbb{N}$ and consider $u = \bar{g}v_i$ and $w = hv_j$. Then, by

Cauchy–Schwarz inequality,

$$\begin{aligned}
\left| \int_{\mathcal{M}} f(m) \langle v_i, G(m)v_j \rangle_{\mathcal{H}} d\mu \right| &= \left| \int_{\mathcal{M}} \overline{\langle g(m)v_i, G(m)v_j h(m) \rangle_{\mathcal{H}}} d\mu \right| \\
&= |\langle u, Gv \rangle_{L^2(\mathcal{M}, d\mu; \mathcal{H})}| \\
&\leq \|G\|_{\mathcal{L}(L^2(\mathcal{M}, d\mu; \mathcal{H}))} \|u\|_{L^2(\mathcal{M}, d\mu; \mathcal{H})} \|w\|_{L^2(\mathcal{M}, d\mu; \mathcal{H})} \\
&\leq \|G\|_{\mathcal{L}(L^2(\mathcal{M}, d\mu; \mathcal{H}))} \int_{\mathcal{M}} |f(m)| d\mu(m),
\end{aligned}$$

since $\|v_n\|_{\mathcal{H}} \leq 1$ for all $n \in \mathbb{N}$. The function $f \in L^1(\mathcal{M}, d\mu)$ was arbitrary, so using the fact that the dual of $L^1(\mathcal{M}, d\mu)$ is $L^\infty(\mathcal{M}, d\mu)$ (see [Bre11, Theorem 4.14]) we have that $|\langle v_i, G(m)v_j \rangle_{\mathcal{H}}| \leq \|G\|_{\mathcal{L}(L^2(\mathcal{M}, d\mu; \mathcal{H}))}$ almost everywhere. Then $\|G(\cdot)\|_\infty \leq \|G\|_{\mathcal{L}(L^2(\mathcal{M}, d\mu; \mathcal{H}))}$ follows by density. \square

Similarly, as for the bounded case, we say that a function $G(\cdot)$ from \mathcal{M} to the self-adjoint operators on a Hilbert space \mathcal{H} is *measurable* if and only if $(i + G(\cdot))^{-1} : \mathcal{M} \rightarrow \mathcal{L}(\mathcal{H})$ is measurable. Given such function $G(\cdot)$, we define the operator G acting on $L^2(\mathcal{M}, d\mu; \mathcal{H}) = \int_{\mathcal{M}}^{\oplus} \mathcal{H} d\mu$ as

$$(Gu)(m) = G(m)u(m)$$

with domain

$$\begin{aligned}
\mathcal{D}(G) &= \left\{ u \in L^2(\mathcal{M}, d\mu; \mathcal{H}) : u(m) \in \mathcal{D}(G(m)) \text{ a.e.} \right. \\
&\quad \left. \text{and } \int_{\mathcal{M}} \|G(m)u(m)\|_{\mathcal{H}}^2 d\mu(m) < \infty \right\}. \quad (90)
\end{aligned}$$

Following the previous notation we write $G = \int_{\mathcal{M}}^{\oplus} G(m) d\mu(m)$. Next, we formulate the main properties of G given in [RS78, Theorem XIII.85] which are going to be useful for us.

Theorem 9. *Let $G = \int_{\mathcal{M}}^{\oplus} G(m) d\mu(m)$ where $G(\cdot)$ are measurable and $G(m)$ is self-adjoint for each m . Then*

(a) G is self-adjoint.

(b) For any (possibly unbounded) Borel measurable function $F : \mathbb{R} \rightarrow \mathbb{R}$

$$F(G) = \int_{\mathcal{M}}^{\oplus} F(G(m)) d\mu(m). \quad (91)$$

(c) $\lambda \in \sigma(G)$ if, and only if, for all $\varepsilon > 0$

$$\mu(\{m : \sigma(G(m)) \cap (\lambda - \varepsilon, \lambda + \varepsilon) \neq \emptyset\}) > 0.$$

Proof. (a) We first prove that G is densely defined. Let $v \in \mathcal{D}(G)^\perp$ then

$$\int_M \langle v(m), u(m) \rangle_{\mathcal{H}} d\mu(m) = 0,$$

for all $u \in \mathcal{D}(G)$. Let δ be any measurable set of M and define $\tilde{u}(m) = \chi_\delta(m)u(m)$ for all $m \in M$ where $\chi_\delta(m) = 1$ if $m \in \delta$ and zero otherwise. Note that $\tilde{u} \in \mathcal{D}(G)$, so we have that

$$\int_\delta \langle v(m), u(m) \rangle_{\mathcal{H}} d\mu(m) = 0,$$

for any measurable set δ of M . By Radon–Nikodym theorem (see [RS80, Theorem I.19]) we have that for any $m \in M$ $\langle v(m), u(m) \rangle_{\mathcal{H}} = 0$ almost everywhere. By assumption, $u(m) \in \mathcal{D}(G(m))$ almost everywhere and $\mathcal{D}(G(m))^\perp = \{0\}$. Hence, $v = 0$ almost everywhere.

For $u, v \in \mathcal{D}(G)$, using that $G(m)$ is self-adjoint for each m , we have

$$\begin{aligned} \langle v, Gu \rangle_{L^2(M, d\mu; \mathcal{H})} &= \int_M \langle v(m), G(m)u(m) \rangle_{\mathcal{H}} d\mu(m) \\ &= \int_M \langle G(m)v(m), u(m) \rangle_{\mathcal{H}} d\mu(m) \\ &= \langle Gv, u \rangle_{L^2(M, d\mu; \mathcal{H})}, \end{aligned}$$

implying that G is a symmetric operator. Using [CR21, Proposition 2.64] it is enough to prove $\text{Ran}(G \pm i) = \mathcal{H}$ in order to prove self-adjointness. Consider $C(m) := (G(m) + i)^{-1}$. By assumption, we know that $C(m)$ is measurable and $\|C(m)\| \leq \frac{1}{|\text{Im}(i)|} = 1$ by adapting [Dav95, Theorem I.2.10] since $G(m)$ is self-adjoint. Then by Theorem 8, we know that there is a unique decomposable operator $C \in \mathcal{L}(L^2(M, d\mu; \mathcal{H}))$ with

$$(Cf)(m) = C(m)f(m),$$

for any $f \in L^2(M, d\mu; \mathcal{H})$. Let $g = Cf$ for $f \in L^2(M, d\mu; \mathcal{H})$, then almost everywhere we have that $g(m) \in \text{Ran } C(m) = \mathcal{D}(G(m))$ and

$$\begin{aligned} \|G(m)g(m)\|_{\mathcal{H}} &= \|G(m)C(m)f(m)\|_{\mathcal{H}} \\ &\leq \|f(m)\|_{\mathcal{H}} + \|-iC(m)f(m)\|_{\mathcal{H}} \\ &\leq 2\|f(m)\|_{\mathcal{H}} \end{aligned}$$

since $\|C(m)\| \leq 1$. This means that $g \in \mathcal{D}(G)$. In addition, $(G + i)g = f$, where $f \in L^2(M, d\mu; \mathcal{H})$ was arbitrarily chosen. Hence, $\text{Ran } (G + i) = L^2(M, d\mu; \mathcal{H})$. To show that $\text{Ran } (G - i) = L^2(M, d\mu; \mathcal{H})$ note that for any $u, v \in \mathcal{H}$ and $m \in M$

$$\begin{aligned} \langle u, (i + G(m))^{-1}v \rangle_{\mathcal{H}} &= \langle (-i + G(m))(-i + G(m))^{-1}u, (i + G(m))^{-1}v \rangle_{\mathcal{H}} \\ &= \langle (-i + G(m))^{-1}u, v \rangle_{\mathcal{H}}, \end{aligned}$$

giving that $(-i + G(m))^{-1} = C(m)^*$ which is weakly measurable since for any $u, v \in \mathcal{H}$

$$\langle u, (i + G(\cdot))^{-1} v \rangle_{\mathcal{H}} = \overline{\langle v, (-i + G(\cdot))^{-1} u \rangle_{\mathcal{H}}}$$

which is measurable because the complex conjugate of a measurable function is a measurable function. Showing $\text{Ran}(G - i) = L^2(M, d\mu; \mathcal{H})$ is done analogously as for $(G + i)$.

(b) See [DNSZ16, Proposition 4.2].

(c) This follows from (b) taking $F(x) = \chi_{(\lambda - \varepsilon, \lambda + \varepsilon)}(x)$ where $\varepsilon > 0$ is arbitrary, then by the properties of the functional calculus we have

$$E_G(\lambda - \varepsilon, \lambda + \varepsilon) = \int_M^{\oplus} E_G(m)(\lambda - \varepsilon, \lambda + \varepsilon) d\mu(m).$$

By [Sch12, Proposition 5.10] we have that $\lambda \in \sigma(G)$ if and only if $E_G(\lambda - \varepsilon, \lambda + \varepsilon) \neq 0$, so the result follows from the fact that

$$\int_M^{\oplus} E_G(m)(\lambda - \varepsilon, \lambda + \varepsilon) d\mu(m) = 0,$$

if and only if $E_G(m)(\lambda - \varepsilon, \lambda + \varepsilon) = 0$ almost everywhere. \square

3.2 Examples of constant fiber direct integrals

We adapt the results in [RS78, Section XIII.16] and [GS20, Section 6.6] to give two different examples of constant direct integral decompositions involving the magnetic Laplacian.

3.2.1 Neumann Magnetic Laplacian on the unit disc

For this first example we follow [KL22, Section 3] and use the same notation as in Paper III. Let $\Omega := D(0, 1) = \{x \in \mathbb{R}^2 : \|x\| < 1\}$, $A(x_1, x_2) = \frac{1}{2}(-x_2, x_1)$ be the vector magnetic potential generating the unit magnetic field $\text{curl} A = 1$ and $b > 0$ be the strength of the magnetic field. We consider the closed quadratic form

$$q_b(u) := \int_{\Omega} |(-i\nabla - bA)u|^2 dx, \quad (92)$$

with domain $\mathcal{D}(q_b) = H^1(\Omega)$. The Neumann realization of the magnetic Laplacian in Ω is the self-adjoint operator, which we denote by H_b , associated with $q_{m,b}$ using Theorem 2. Recall that H_b can be characterized by (26) as

$$\left. \begin{aligned} \mathcal{D}(H_b) &= \{u \in H^1(\Omega) : \exists w \in L^2(\Omega) \text{ such that } q_b(v, u) = \langle v, w \rangle_{L^2(\Omega)} \forall v \in \mathcal{D}(q_b)\}, \\ H_b u &= w. \end{aligned} \right\} \quad (93)$$

Since the domain is rotationally symmetric, we can work in polar coordinates and use Fourier series in the angular variable.

Consider the operator $U : L^2(\Omega) \rightarrow \int_{\mathbb{Z}}^{\oplus} L^2((0, 1), r \, dr) \, d\mu$, where μ is a sum of point measures at \mathbb{Z} , given by

$$U(u)(m) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(r, \theta) e^{-im\theta} \, d\theta, \quad (94)$$

where we define this operator acting on $C_c^\infty(\Omega)$. In Section 2.4.2, we introduced for $m \in \mathbb{Z}$ the closed quadratic form

$$q_{m,1}^b = \int_0^1 \left(|u'(r)|^2 + \left(\frac{m}{r} - \frac{br}{2} \right)^2 |u(r)|^2 \right) r \, dr$$

with domain $\mathcal{D}(q_{m,1}^b) = \{u \in L^2((0, 1), r \, dr) : u', mu/r \in L^2((0, 1), r \, dr)\}$. We denote by $H_{m,1}^b$ the self-adjoint operator associated with $q_{m,1}^b$ via Theorem 2.

Proposition 5. *Let U be defined as in (94), then U is well-defined on $C_c^\infty(\Omega)$ and uniquely extendable to a unitary operator acting on $L^2(\Omega)$. Moreover,*

$$H_b = U^{-1} \left(\int_{\mathbb{Z}}^{\oplus} H_{m,1}^b \, d\mu(m) \right) U. \quad (95)$$

Proof. Let $\mathcal{H} := L^2((0, 1), r \, dr)$ and $u \in C_c^\infty(\Omega)$ then using Fubini and Parseval theorems we get

$$\begin{aligned} \|U(u)\|_{L^2(\mathbb{Z}, d\mu; \mathcal{H})}^2 &= \sum_{m \in \mathbb{Z}} \|U(u)(m)\|_{\mathcal{H}}^2 \\ &= \int_0^1 \sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \left| \int_0^{2\pi} u(r, \theta) e^{-im\theta} \, d\theta \right|^2 r \, dr \\ &= \int_0^1 \int_0^{2\pi} |u(r, \theta)|^2 r \, d\theta \, dr \\ &= \|u\|_{L^2(\Omega)}^2. \end{aligned}$$

Then we have that U is well-defined and, using density arguments, it has a unique extension to an isometry between $L^2(\Omega)$ and $\int_{\mathbb{Z}}^{\oplus} L^2((0, 1), r \, dr) \, d\mu$. To prove that U is unitary we are left to show that $\text{Ran}(U) = \int_{\mathbb{Z}}^{\oplus} L^2((0, 1), r \, dr) \, d\mu$. We will do this by computing the adjoint U^* . For $v \in \int_{\mathbb{Z}}^{\oplus} L^2((0, 1), r \, dr) \, d\mu$ consider

$$U^*(v) := \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} v(m) e^{im\theta}. \quad (96)$$

Let $u \in L^2(\Omega)$ and $v \in \int_{\mathbb{Z}}^{\oplus} L^2((0, 1), r dr) d\mu$ then by Fubini theorem we have

$$\begin{aligned}
\langle v, U(u) \rangle_{L^2(\mathbb{Z}, d\mu; \mathcal{H})} &= \sum_{m \in \mathbb{Z}} \int_0^1 \overline{v(m)} U(u)(m) r dr \\
&= \sum_{m \in \mathbb{Z}} \int_0^1 \overline{v(m)} \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(r, \theta) e^{-im\theta} d\theta r dr \\
&= \int_0^1 \left(\int_0^{2\pi} \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} \overline{v(m)} e^{-im\theta} u(r, \theta) d\theta \right) r dr \\
&= \langle U^* v, u \rangle_{L^2(\Omega)}.
\end{aligned}$$

which means that U^* is the adjoint operator of U . Let $u \in L^2(\Omega)$ then

$$\begin{aligned}
U^*(U(u)) &= U^* \left(\frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(r, \theta) e^{-i(\cdot)\theta} d\theta \right) \\
&= \sum_{m \in \mathbb{Z}} \frac{1}{2\pi} \left(\int_0^{2\pi} u(r, \theta) e^{-im\theta} d\theta \right) e^{im\theta} \\
&= u.
\end{aligned}$$

On the other hand, if $v \in \int_{\mathbb{Z}}^{\oplus} L^2((0, 1), r dr) d\mu$ then

$$\begin{aligned}
U(U^*(v))(m) &= U \left(\frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} v(k) e^{ik\theta} \right) (m) \\
&= \frac{1}{2\pi} \int_0^{2\pi} v(m) d\theta \\
&= v(m),
\end{aligned}$$

where we have used $\int_0^{2\pi} e^{i(m-k)\theta} d\theta = 2\pi \delta_{mk}$. Hence, U is a unitary operator (see [Wei80, Theorem 4.34]).

In order to lighten the notation we define $H := \int_{\mathbb{Z}}^{\oplus} H_{m,1}^b d\mu(m)$. To prove (95), we divide the proof in two parts:

I. $U^{-1}(\mathcal{D}(H)) \subseteq \mathcal{D}(H_b)$. From (90) we have that

$$\mathcal{D}(H) = \left\{ u \in \int_{\mathbb{Z}}^{\oplus} L^2((0, 1), r dr) d\mu : u(m) \in \mathcal{D}(H_{m,1}^b) \text{ a.e.} \right.$$

$$\left. \text{and } \int_{\mathbb{Z}} \|H_{m,1}^b u(m)\|_{L^2((0,1), r dr)}^2 d\mu(m) < \infty \right\}. \quad (97)$$

Let $u \in \mathcal{D}(H)$, using polar coordinates, Fubini and Parseval theorem⁴

$$\begin{aligned} \|U^{-1}u\|_{H^1(\Omega)}^2 &= \int_{\Omega} (|\nabla(U^{-1}u)| + |U^{-1}u|) dx \\ &= \sum_{m \in \mathbb{Z}} \int_0^1 (|\partial_r u(m)|^2 + \left(\frac{m^2}{r^2} + 1\right) |u(m)|^2) r dr < \infty, \end{aligned}$$

since $u \in \mathcal{D}(H)$, $q_{m,1}^b(u(m)) = \langle u(m), H_{m,1}^b u(m) \rangle$ for all $m \in \mathbb{Z}$ and

$$\frac{m^2}{r^2} \leq 2 \left(\frac{m}{r} - \frac{br}{2} \right)^2 + \frac{br^2}{2}.$$

It is only left to show that there exists a $w \in L^2(\Omega)$ such that $q_b(v, U^{-1}u) = \langle v, w \rangle_{L^2(\Omega)}$ for all $v \in H^1(\Omega)$. The proof is similar to the one before, and it is based in the fact that for $v \in H^1(\Omega)$ we can consider its Fourier series with respect to the angular component

$$v(r, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} v_m(r) e^{im\theta},$$

where $v_m(r)$ are the Fourier coefficients for $m \in \mathbb{Z}$. It can be shown that $v_m \in \mathcal{D}(q_{m,1}^b)$ for all $m \in \mathbb{Z}$ and, in polar coordinates, $w(r, \theta) = \frac{1}{\sqrt{2\pi}} \sum_{m \in \mathbb{Z}} H_{m,1}^b u(m) e^{im\theta}$.

2. $H_b U^{-1} = U^{-1}H$. Since $A(x_1, x_2) = \frac{1}{2}(-x_2, x_1) = \frac{r}{2} \hat{e}_\theta$

$$\begin{aligned} (-i\nabla - A)^2 &= -\Delta + 2i \underbrace{\nabla \cdot A}_{=0} + iA \cdot \nabla + |A|^2 \\ &= -\frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \left(-i \frac{\partial}{\partial \theta} - \frac{br}{2} \right)^2. \end{aligned}$$

Thus, $H_b U^{-1} = U^{-1}H$ follows from a straightforward computation finishing the proof. \square

From (95) and Theorem 9 we have

$$\sigma(H_b) = \sigma\left(U^{-1} \int_{\mathbb{Z}}^{\oplus} H_{m,1}^b d\mu(m) U\right) = \sigma\left(\int_{\mathbb{Z}}^{\oplus} H_{m,1}^b d\mu(m)\right) = \bigcup_{m \in \mathbb{Z}} \sigma(H_{m,1}^b).$$

Remark. As pointed out in [RS78] and [KL22], we can identify

$$\int_{\mathbb{Z}}^{\oplus} L^2((0, 1), r dr) d\mu \simeq \bigoplus_{m \in \mathbb{Z}} L^2((0, 1), r dr).$$

⁴See also [Wei80, Chapter 3] using that $\{\frac{e^{im\theta}}{\sqrt{2\pi}} : m \in \mathbb{Z}\}$ is an ONB of $L^2(0, 2\pi)$.

3.2.2 Magnetic Laplacian with step magnetic field on the plane

For this second example, we follow the notation of Paper II and consider the magnetic Laplacian in \mathbb{R}^2

$$\mathcal{L}_{b,0} = (-i\nabla + \mathcal{A}_{b,0})^2, \quad (98)$$

with domain

$$\mathcal{D}(\mathcal{L}_{b,0}) = \{u \in L^2(\mathbb{R}^2) : (-i\nabla + \mathcal{A}_{b,0})u \in L^2(\mathbb{R}^2, \mathbb{R}^2) \text{ and } (-i\nabla + \mathcal{A}_{b,0})^2 u \in L^2(\mathbb{R}^2)\},$$

and $\mathcal{A}_{b,0} := (A_{b,0}, 0)$ with

$$A_{b,0}(x_1, x_2) = \begin{cases} -x_2 & \text{if } x_2 \geq 0 \\ -bx_2 & \text{if } x_2 < 0 \end{cases}, \quad (99)$$

and $b \in [-1, 0)$. We recall the family of operators introduced in Section 2.4.1

$$h_b^\xi = -\frac{d^2}{dx_2^2} + V_b(\xi, x_2),$$

with domain $\mathcal{D}(h_b^\xi) = \{u \in B^1(\mathbb{R}) : h_b^\xi u \in L^2(\mathbb{R})\}$ and

$$V_b(\xi, x_2) = \begin{cases} (bx_2 + \xi)^2, & x_2 < 0 \\ (x_2 + \xi)^2, & x_2 > 0 \end{cases},$$

for $b \in [-1, 0)$. The operator $\mathcal{L}_{b,0}$ is invariant under translations in the x_1 -direction, and this allows to consider the partial Fourier transform $U: L^2(\mathbb{R}^2) \rightarrow \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) d\xi$, given by

$$\mathcal{F}(u)(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x_1, x_2) e^{-i\xi x_1} dx_1, \quad (100)$$

where we define this operator acting on $C_c^\infty(\mathbb{R}^2)$ (we can also define the operator acting on $\mathcal{S}(\mathbb{R}^2)$).

Proposition 6. *Let \mathcal{F} be defined as in (100), then \mathcal{F} is well-defined on $C_c^\infty(\Omega)$ and uniquely extendable to a unitary operator acting on $L^2(\mathbb{R}^2)$. Moreover,*

$$\mathcal{F}\mathcal{L}_{b,0}\mathcal{F}^{-1} = \int_{\mathbb{R}}^{\oplus} h_b^\xi d\xi. \quad (101)$$

Proof. This proof can be seen as a continuous analogue of the proof of Proposition 5 where we use Plancherel theorem instead of Parseval theorem. We omit the proof of \mathcal{F} being well-defined and uniquely extendable to a unitary operator acting on $L^2(\mathbb{R})$ since it can be

proven in a similar way as done in Proposition 5 for U . To prove that \mathcal{F} is unitary, we need to show that $\text{Ran}(\mathcal{F}) = \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) \, d\xi$. As in Proposition 5, we will compute the adjoint \mathcal{F}^* . For $v \in \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) \, d\xi$ consider

$$\mathcal{F}^*(v) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} v(\xi) e^{i\xi x_1} \, d\xi, \quad (102)$$

which is the partial inverse Fourier transform. Using Fubini theorem we have for $v \in \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) \, d\xi$

$$\begin{aligned} \langle v, \mathcal{F}(u) \rangle_{L^2(\mathbb{Z}, d\xi; \mathcal{H})} &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \overline{v(\xi)} \left(\int_{\mathbb{R}} u(x_1, x_2) e^{-i\xi x_1} \, dx_1 \right) dx_2 \, d\xi \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} \left(\int_{\mathbb{R}} \overline{v(\xi) e^{i\xi x_1}} \, d\xi \right) u(x_1, x_2) \, dx_1 \, dx_2 \\ &= \langle \mathcal{F}^* v, u \rangle_{L^2(\mathbb{R})}. \end{aligned}$$

The equalities $\mathcal{F}\mathcal{F}^* = I_{\int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) \, d\xi}$ and $\mathcal{F}^*\mathcal{F} = I_{L^2(\mathbb{R}^2)}$ are a consequence of Fourier inversion theorem. Hence, by [Wei80, Theorem 4.34], we have that \mathcal{F} is unitary.

The proof of (101) can be divided in two parts. As in Proposition 5 we set $H := \int_{\mathbb{R}}^{\oplus} \mathfrak{h}_b^\xi \, d\xi$ to lighten the notation.

- I. $\mathcal{F}(\mathcal{D}(\mathcal{L}_{b,0})) \subseteq \mathcal{D}(H)$. Since $\mathcal{A}_{b,0} \in H_{\text{loc}}^1(\mathbb{R}^2, \mathbb{R}^2)$, we know by [LS81] that $\mathcal{L}_{b,0}$ is essentially self-adjoint on $C_c^\infty(\mathbb{R}^2)$ meaning that it suffices to show that $\mathcal{F}(u) \in \mathcal{D}(H)$ for $u \in C_c^\infty(\mathbb{R}^2)$. From (90) we have that

$$\mathcal{D}(H) = \left\{ u \in \int_{\mathbb{R}}^{\oplus} L^2(\mathbb{R}) \, d\xi : u(\xi) \in \mathcal{D}(\mathfrak{h}_b^\xi) \text{ a.e.} \right. \\ \left. \text{and } \int_{\mathbb{R}} \|\mathfrak{h}_b^\xi u(\xi)\|_{L^2(\mathbb{R})}^2 \, d\xi < \infty \right\}. \quad (103)$$

We show that $\int_{\mathbb{R}} \|\mathfrak{h}_b^\xi u(\xi)\|_{L^2(\mathbb{R})}^2 \, d\xi < \infty$. Observe that $u(\xi) \in \mathcal{D}(\mathfrak{h}_b^\xi)$ a.e. can be proven similarly. Using Fubini and Plancherel theorems we have

$$\begin{aligned} \int_{\mathbb{R}} \|\mathfrak{h}_b^\xi \mathcal{F}(u)\|_{L^2(\mathbb{R})}^2 \, d\xi &= \int_{\mathbb{R}} \int_{\mathbb{R}} \left| \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi}} \mathfrak{h}_b^\xi u(x_1, x_2) e^{-i\xi x_1} \, dx_1 \right|^2 \, d\xi \, dx_2 \\ &= \int_{\mathbb{R}} \int_{\mathbb{R}} |(-i\nabla + \mathcal{A}_b)^2 u(x_1, x_2)|^2 \, dx_1 \, dx_2 \\ &= \|(-i\nabla + \mathcal{A}_b)^2 u\|_{L^2(\mathbb{R}^2)}^2 < \infty, \end{aligned}$$

where we have used in the second equality that $\mathcal{F}(\partial_{x_1} u)(\xi) = i\xi \mathcal{F}(u)(\xi)$.

2. $H\mathcal{F} = \mathcal{F}\mathcal{L}_{b,0}$. Let $u \in C_c^\infty(\mathbb{R}^2)$ then, using again $\mathcal{F}(\partial_{x_1} u)(\xi) = i\xi\mathcal{F}(u)(\xi)$

$$\begin{aligned}\mathcal{F}(\mathcal{L}_{b,0}u)(\xi) &= \mathcal{F}\left(-\frac{\partial^2}{\partial x_2^2}u + \left(-i\frac{\partial}{\partial x_1} + A_{b,0}\right)^2u\right)(\xi) \\ &= -\frac{\partial^2}{\partial x_2^2}\mathcal{F}(u)(\xi) + (\xi + A_{b,0})^2\mathcal{F}(u)(\xi) \\ &= \mathfrak{h}_b^\xi\mathcal{F}(u)(\xi),\end{aligned}$$

which finishes the proof. \square

From (101)

$$\sigma(\mathcal{L}_{b,0}) = \sigma(\mathcal{F}\mathcal{L}_{b,0}\mathcal{F}^{-1}) = \sigma\left(\int_{\mathbb{R}}^{\oplus} \mathfrak{h}_b^\xi d\xi\right) = \overline{\bigcup_{\xi \in \mathbb{R}} \sigma(\mathfrak{h}_b^\xi)},$$

since by Theorem 9 we have the resolvent relation $\mathcal{F}R(\lambda, \mathcal{L}_{b,0})\mathcal{F}^{-1} = \int_{\mathbb{R}}^{\oplus} R(\lambda, \mathfrak{h}_b^\xi) d\xi$.

4 The Laplacian and the de Rham complex

In Paper IV, we introduce a method using differential forms and the de Rham complex as a tool to deal with inequalities between the Dirichlet and the Neumann Laplacian eigenvalues on bounded domains $\Omega \subset \mathbb{R}^n$. In this section, we present the most important concepts in order to define the de Rham complex. In Paper IV, we take one more step following the work of Brüning and Lesch [BL92] and compare eigenvalues related to the de Rham complex with boundary conditions. For clarity, we restrict ourselves to smooth manifolds and real vector spaces, although a lot of results carry over domains with less regularity and complexified vector spaces (see Section 3 of Paper IV). We follow [Lee13, GH19, War83] throughout this section.

4.1 Smooth manifolds and tangent space

We say that M is an *n -dimensional topological manifold* if it is a second countable, Hausdorff and locally Euclidean topological space of dimension n (each point has a neighbourhood homeomorphic to an open set of \mathbb{R}^n). The pair (U, φ) is called a *coordinate system* on M if φ is a homeomorphism of a connected open set $U \subset M$ onto an open subset of \mathbb{R}^n . Sometimes we write (U, x_1, \dots, x_n) where $x_i = r_i \circ \varphi$ and $r_i(y) = y_i$ is the i :th coordinate function where $y = (y_1, \dots, y_n) \in \mathbb{R}^n$. M is an *n -dimensional topological manifold with boundary* if it is a second countable and Hausdorff space where each point $p \in M$ has a

neighbourhood homeomorphic either to an open set of \mathbb{R}^n or to a (relatively) open set of the n -dimensional half-space $\mathbb{H}^n := \{(y_1, \dots, y_n) \in \mathbb{R}^d : y_n \geq 0\}$.

An n -dimensional smooth manifold is a pair (M, \mathcal{F}) consisting of an n -dimensional topological manifold M together with a collection of coordinate systems $\mathcal{F} = \{(U_i, \varphi_i) : i \in I\}$ where

- (a) $\bigcup_{i \in I} U_i = M$.
- (b) $\varphi_i \circ \varphi_j^{-1}$ is C^∞ for all $i, j \in I$.
- (c) The collection \mathcal{F} is maximal with respect to (b), in other words if (U, φ) is a coordinate system such that $\varphi \circ \varphi_i^{-1}$ and $\varphi_i \circ \varphi^{-1}$ are C^∞ for all $i \in I$, then $(U, \varphi) \in \mathcal{F}$.

A \mathcal{F} fulfilling the three conditions above is called *smooth structure* on M . We will typically denote a smooth manifold (M, \mathcal{F}) simply by M and from now on, M will always be a smooth manifold. In the case of a manifold with boundary, we can define similarly an n -dimensional smooth manifold with boundary (see [Lee13, Chapter 1] for more details).

Let $U \subseteq M$ be an open set, then $f : U \rightarrow \mathbb{R}^n$ is a C^∞ function on U (we denote it by $f \in C^\infty(U)$) if $f \circ \varphi^{-1}$ is C^∞ for each $(U, \varphi) \in \mathcal{F}$. Let (M, \mathcal{F}_M) and (N, \mathcal{F}_N) be smooth manifolds, a continuous map $\psi : M \rightarrow N$ is *differentiable of class C^∞* , we denote it by $\psi \in C^\infty(M, N)$, if $\varphi \circ \psi \circ \tau^{-1}$ is C^∞ for each coordinate system $(U, \tau) \in \mathcal{F}_M$ and $(V, \varphi) \in \mathcal{F}_N$.

A linear map $v : C^\infty(M) \rightarrow \mathbb{R}$ is called a *derivation at p* if it satisfies

$$v(fg) = f(p)v(g) + g(p)v(f) \quad \text{for all } f, g \in C^\infty(M). \quad (104)$$

The set of all derivations of $C^\infty(M)$ at p , which we denote by $T_p M$, is a vector space called the *tangent space to M at p* . An element of $T_p M$ is called a *tangent vector at p* . For alternative definitions of the tangent space we refer to [Lee13, Chapter 3].

Let $\psi \in C^\infty(M, N)$, the *differential of ψ at $p \in M$* is the linear map

$$d\psi_p : T_p M \rightarrow T_{\psi(p)} N \quad (105)$$

defined by $d\psi_p(v)(g) = v(g \circ \psi)$ for any $g \in C^\infty(N)$ and $v \in T_p M$. Let (U, φ) be a coordinate system, we have that $d\varphi : T_p M \rightarrow T_{\varphi(p)} \mathbb{R}^n$ is an isomorphism. Moreover, $\left\{ \frac{\partial}{\partial x_i} \Big|_{\varphi(p)}, \dots, \frac{\partial}{\partial x_n} \Big|_{\varphi(p)} \right\}$ is a basis of $T_{\varphi(p)}(\mathbb{R}^n)$ where $\frac{\partial}{\partial x_i}$ is the partial derivative in the i :th coordinate of \mathbb{R}^n . Hence, using the differential of $d\varphi$ we can define for $i \in \{1, \dots, n\}$

$$\frac{\partial}{\partial x_i} \Big|_p = (d\varphi_p)^{-1} \left(\frac{\partial}{\partial x_i} \Big|_{\varphi(p)} \right) = d(\varphi^{-1})_{\varphi(p)} \left(\frac{\partial}{\partial x_i} \Big|_{\varphi(p)} \right), \quad (106)$$

which acts on functions f which are C^∞ on a neighbourhood of p as

$$\frac{\partial}{\partial x_i} \Big|_p (f) = \frac{\partial}{\partial r_i} \Big|_{\varphi(p)} (f \circ \varphi^{-1}).$$

Thus, $\left\{ \frac{\partial}{\partial x_1} \Big|_p, \dots, \frac{\partial}{\partial x_n} \Big|_p \right\}$ is a basis of T_p^*M . For a function C^∞ function $f : M \rightarrow \mathbb{R}$, we write $df|_p$ to refer to the element of $(T_p^*M)^*$ defined by

$$df(v) = v(f),$$

for any $v \in T_pM$. We write df instead of $df|_p$ unless it is not clear from context. Taking this into account we find that $\{dx_1, \dots, dx_n\}$ is the basis of $(T_pM)^*$ with

$$dx_j \left(\frac{\partial}{\partial x_i} \Big|_p \right) = \frac{\partial}{\partial x_i} \Big|_p (x_j) = \frac{\partial}{\partial r_i} \Big|_{\varphi(p)} \underbrace{(x_j \circ \varphi^{-1})}_{=r_j} = \delta_{ij}.$$

4.2 Multilinear algebra

To define the concept of differential form, we need to establish some facts about multilinear algebra. Let V be an n -dimensional vector space and let V^k be the set of all k -tuples (v_1, \dots, v_k) where $v_1, \dots, v_k \in V$. A map $T : V^k \rightarrow \mathbb{R}$ is a k -tensor if T is linear in each of its variables. We denote by $\mathcal{T}^k(V)$ the set of all k -tensors in V , we set $\mathcal{T}^0(V) = \mathbb{R}$. If $T_1 \in \mathcal{T}^k(V)$ and $T_2 \in \mathcal{T}^l(V)$ we define the *tensor product of T_1 and T_2* , we denote it by $T_1 \otimes T_2$, as the $k + l$ -tensor given by

$$(T_1 \otimes T_2)(v_1, \dots, v_{k+l}) = T_1(v_1, \dots, v_k) T_2(v_{k+1}, \dots, v_{k+l}).$$

A tensor $T \in \mathcal{T}^k(V)$ is called *decomposable* if we can find $l_i \in V^*$ for $i \in \{1, \dots, k\}$ such that

$$T = l_1 \otimes \dots \otimes l_k.$$

Let $\{e_1, \dots, e_n\}$ be a basis of V and $\{e_1^*, \dots, e_n^*\}$ be the dual basis of V^* , then the k -tensors

$$e_i^* := e_{i_1}^* \otimes \dots \otimes e_{i_k}^*,$$

where we consider every multi-index $I = \{i_1, \dots, i_k\}$, form a basis of $\mathcal{T}^k(V)$ (see [GH19, Theorem 1.3.13]). Let S_k be the permutation group (or symmetric group on k letters), define $T^\tau \in \mathcal{T}^k(V)$ as

$$T^\tau(v_1, \dots, v_k) = T(v_{\tau^{-1}(1)}, \dots, v_{\tau^{-1}(k)}).$$

A k -tensor $T \in \mathcal{T}^k(V)$ is *alternating* if $T^\tau = \text{sgn}(\tau)T$ and we denote by $A^k(V)$ the set of alternating k -tensors (see [GH19, Equation 1.4.8] for the definition of the sign of a permutation). If $\{e_1, \dots, e_n\}$ is a basis of V , then the k -tensors

$$\text{Alt}(e_I) := \sum_{\tau \in S_k} \text{sgn}(\tau) (e_I^*)^\tau,$$

where $e_I^* = e_{i_1}^* \otimes \dots \otimes e_{i_k}^*$ with $I = (i_1, \dots, i_k)$ and $i_1 < \dots < i_k$, form a basis for $\mathcal{A}^k(V)$ (see [GH19, Proposition 1.4.26]). We define

$$\mathcal{F}^k(V) := \text{Span}\{v \in \mathcal{F}^k(V) : v = l_1 \otimes \dots \otimes l_k \text{ with } l_1, \dots, l_k \in V^* \text{ and } l_i = l_{i+1} \text{ for some } i\}.$$

We set $\mathcal{F}^1(V) := 0$. $\mathcal{F}^k(V)$ is a linear subspace of $\mathcal{F}^k(V)$ and every $T \in \mathcal{F}^k(V)$ can be written uniquely as $T = T_1 + T_2$ with $T_1 \in \mathcal{A}^k(V)$ and $T_2 \in \mathcal{F}^k(V)$. We define the quotient space

$$\Lambda^k(V^*) := \mathcal{F}^k(V) / \mathcal{F}^k(V). \quad (107)$$

The linear map $\pi : \mathcal{F}^k(V) \rightarrow \Lambda^k(V^*)$, $T \mapsto T + \mathcal{F}^k(V)$ has kernel $\mathcal{F}^k(V)$ and maps $\mathcal{A}^k(V)$ bijectively onto $\Lambda^k(V^*)$. Let $\omega_i \in \Lambda^{k_i}(V^*)$ for $i \in \{1, 2\}$, then we can find $T_1, T_2 \in \mathcal{F}^k(V)$ with $\omega_i = \pi(T_i)$ for $i \in \{1, 2\}$. The *wedge product* $\omega_1 \wedge \omega_2$ is an element of $\Lambda^{k_1+k_2}(V^*)$ defined by

$$\omega_1 \wedge \omega_2 := \pi(T_1 \otimes T_2). \quad (108)$$

Using π and the wedge product we see that the elements $e_{i_1}^* \wedge \dots \wedge e_{i_k}^* = \pi(e_I^*)$ with $I = (i_1, \dots, i_k)$ and $i_1 < \dots < i_k$ form a basis of $\Lambda^k(V^*)$ (see [GH19, Theorem 1.6.13]). Define $\Lambda(V^*) := \bigoplus_{k=0}^n \Lambda^k(V^*)$.

Let L be a one-dimensional vector space. An *orientation* is a choice of one of the connected components of $L \setminus \{0\}$. Let V be an n -dimensional vector space. An *orientation* of V is an orientation of $\Lambda^n(V^*)$. Assume that V^* is a real inner product space and denote by $\langle \cdot, \cdot \rangle_{V^*} : V^* \times V^* \rightarrow \mathbb{R}$ the inner product. We can extend the inner product to $\Lambda(V^*)$ by setting the inner product to zero if the elements are of different degree and

$$\langle v_1 \wedge \dots \wedge v_l, u_1 \wedge \dots \wedge u_l \rangle = \det \langle v_i, u_j \rangle_{V^*},$$

and extending by bilinearity to $\Lambda^k(V^*)$. With respect to this inner product the basis $e_{i_1}^* \wedge \dots \wedge e_{i_k}^* = \pi(e_I^*)$ with $I = (i_1, \dots, i_k)$ and $i_1 < \dots < i_k$ is an orthonormal basis of $\Lambda^k(V^*)$. Having this in mind, there is a well-defined linear transformation called the *Hodge star* or *Hodge star operator*

$$\star : \Lambda(V^*) \rightarrow \Lambda(V^*), \quad (109)$$

defined acting in any orthonormal basis $\{e_1^*, \dots, e_n^*\}$ of V^* as

$$\star(1) = \pm e_1^* \wedge \dots \wedge e_n^*, \quad \star(e_1^* \wedge \dots \wedge e_n^*) = \pm 1, \quad \star(e_1^* \wedge \dots \wedge e_l^*) = \pm e_{l+1}^* \wedge \dots \wedge e_n^*, \quad (110)$$

where one takes the $+$ sign if $e_1^* \wedge \dots \wedge e_n^*$ lies in the component of $\Lambda^n(V^*)$ chosen by the orientation, and $-$ otherwise. Note that $\star : \Lambda^k(V^*) \rightarrow \Lambda^{n-k}(V^*)$ and $\star\star = (-1)^{k(n-k)}$ when acting on $\Lambda^k(V^*)$.

4.3 Differential forms

Let M be a smooth manifold, we define

1. $TM := \bigcup_{p \in M} T_p M$ the tangent bundle over M .
2. $T^*M := \bigcup_{p \in M} (T_p M)^*$ the cotangent bundle over M .
3. $\Lambda^k T^*M := \bigcup_{p \in M} \Lambda^k((T_p M)^*)$ exterior k bundle over M .
4. $\Lambda^* T^*M := \bigoplus_{k=0}^n \Lambda^k T^*M$ exterior algebra bundle over M .

All the spaces defined above have natural manifold structures and the bases of $T_p M$ and $(T_p M)^*$ yield bases of these bundles (see [War83]). Using the results from the previous section, a basis of $\Lambda^k((T_p M)^*)$ is given by $\{dx_{i_1} \wedge \dots \wedge dx_{i_k} : i_1 < \dots < i_k\}$.

Remark. There are natural projections from the spaces defined above onto M , for example for TM we have

$$\pi : TM \rightarrow M, \quad \pi(v) = p \quad \text{if } v \in T_p M.$$

In the literature, when talking about more general vector bundles, one identifies the bundle with this surjective map. In our case, we identify TM with $TM \rightarrow M$.

A (smooth) vector field is a C^∞ mapping of M into TM and a k -form is a C^∞ mapping from M to $\Lambda^k T^*M$. Next, we define an important operator acting on differential forms: the exterior differential d .

Theorem 10. *There exists a unique linear mapping $d : \Lambda^* T^*M \rightarrow \Lambda^* T^*M$ such that*

1. $d(\omega_1 \wedge \omega_2) = d\omega_1 \wedge \omega_2 + (-1)^k \omega_1 \wedge d\omega_2$ for $\omega_1 \in \Lambda^k T^*M$ and $\omega_2 \in \Lambda^* T^*M$, we say that d is an anti-derivation.
2. $d^2 = 0$.
3. For $f \in C^\infty(M) = C^\infty(M, \Lambda^0 T^*M)$, df is the differential of f .

For a proof we refer to [War83, Theorem 2.20]. An important remark coming from the proof is that if we take a coordinate system (U, x_1, \dots, x_n) where U is an open neighbourhood of $p \in M$, we can express a k -form as $\omega|_{\mathcal{D}(\omega) \cap U} = \sum a_\Phi dx_\Phi$ where $a_\Phi \in C^\infty(U \cap \mathcal{D}(\omega))$, Φ runs over all subsets of $\{1, \dots, n\}$ of length k and dx_Φ is given by $dx_{i_1} \wedge \dots \wedge dx_{i_k}$ when $\Phi = \{i_1 < \dots < i_k\}$ (or constant function 1 when $\Phi = \emptyset$) then $d\omega$ is given locally at p as

$$d\omega_p = \sum da_\Phi|_p \wedge dx_\Phi|_p.$$

In particular, if the exterior differential is acting on k -forms we write $d_k : C^\infty(M, \Lambda^k T^*M) \rightarrow C^\infty(M, \Lambda^{k+1} T^*M)$.

4.4 The de Rham complex

A *Riemannian structure* on a smooth manifold M is a smooth choice of a positive definite inner product $\langle \cdot, \cdot \rangle_p$ on each tangent space $T_p M$. By smooth we mean that for any vector fields $X, Y \in C^\infty(M, TM)$ then $\langle X, Y \rangle \in C^\infty(M)$. A *Riemannian manifold* is a smooth manifold together with a Riemannian structure. Let $p \in M$, then we can find a neighbourhood U and a collection $\{e_1, \dots, e_n\}$ of C^∞ vector field on U forming an orthonormal basis of the tangent space at each $p \in U$. This collection $\{e_1, \dots, e_n\}$ is called a *local orthonormal frame field*. Since there is a natural isomorphism $\varphi: T_p M \rightarrow (T_p M)^*$ given by $v \mapsto \varphi(v)$ where $\varphi(v)(u) = \langle v, u \rangle_p$ we have that $(T_p M)^*$ inherits an inner product (see [War83, Section 4.10] for more details). Let $\{e_1, \dots, e_n\}$ be a local orthonormal frame field on U , and consider $\omega_1, \dots, \omega_n$ the 1-forms such that $\omega_i(e_j) = \delta_{ij}$ on U . Then $\{\omega_1, \dots, \omega_n\}$ form a *local orthonormal coframe field*.

We say that a smooth manifold M is *orientable* if $\Lambda^n T^* M \setminus M$ has two connected components. Being orientable means to be able to choose an orientation on each $(T_p M)^*$ consistently. An *orientation on M* is a choice of one of the two components of $\Lambda^n T^* M \setminus M$. In this section, M is a compact Riemannian orientable manifold. To shorten notation we define $\Omega^k(M) := C^\infty(M, \Lambda^k T^* M)$ the vector space of k -forms in M . A local orthonormal coframe field $\{\omega_1, \dots, \omega_n\}$ on a neighbourhood U of M is *oriented* if $\omega_1 \wedge \dots \wedge \omega_n$ belongs to the orientation on each point of U . If we choose a local oriented orthonormal coframe field at each point $p \in M$, then the n -forms $\omega_1 \wedge \dots \wedge \omega_n$ agree on the overlaps of the neighbourhoods were they are defined, so they determine a globally defined n -form on M which does not vanish anywhere. We call this form the *volume form* and denote it by dV .

Consider the operator $\delta_k: \Omega^{k+1}(M) \rightarrow \Omega^k(M)$ given by

$$\delta_k := (-1)^{n_{k+1}} * d_{n-k-1} * . \quad (\text{III})$$

Using δ_k we can define the *Hodge Laplacian operator* $\Delta_k: \Omega^k(M) \rightarrow \Omega^k(M)$ as the linear operator

$$\Delta_k = \delta_k d_k + d_{k-1} \delta_{k-1} \quad (\text{II2})$$

for $k \in \{0, 1, \dots, n\}$. Note that on $\Omega^0(\mathbb{R}^n)$ we have that $\Delta_0 = -\sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator acting on $C^\infty(\mathbb{R}^n)$. We define an inner product on $\Omega^k(M)$ given by

$$\langle \omega_1, \omega_2 \rangle := \int_M \omega_1 \wedge * \omega_2 = \int_M \langle \omega_1, \omega_2 \rangle_{\Lambda^k T^* M} dV,$$

for $\omega_1, \omega_2 \in \Omega^k(M)$ where dV is the volume form. Let $\delta: \oplus_{k=0}^n \Omega^k(M) \rightarrow \oplus_{k=0}^n \Omega^k(M)$ be the operator acting as δ_k on $\Omega^k(M)$, then

$$\langle du, v \rangle = \langle u, \delta v \rangle$$

meaning that δ is the *adjoint* of d on $\bigoplus_{k=0}^n \Omega^k(M)$ (see [War83, Proposition 6.2] for a proof). As a consequence, Δ_k is formally self-adjoint

$$\langle \Delta_k u, v \rangle = \langle u, \Delta_k v \rangle,$$

for $u, v \in \Omega^k(M)$ and $k \in \{1, \dots, n\}$, and bounded from below since

$$\langle \Delta_k u, u \rangle = \langle d_k u, d_k u \rangle + \langle \delta_{k-1} u, \delta_{k-1} u \rangle,$$

for $u \in \Omega^k(M)$. A k -form $\omega \in \Omega^k(M)$ is called *closed* if $d\omega = 0$ and it is called *exact* if there is a $(k-1)$ -form v such that $\omega = dv$. Since $d^2 = 0$, every exact form is closed. The k -th *de Rham cohomology group* is the quotient space

$$H_{\text{dR}}^k(M) := \{\text{closed } k\text{-forms}\} / \{\text{exact } k\text{-form}\} = \ker d_k / \text{Ran } d_{k-1}. \quad (\text{II3})$$

The elements of $H_{\text{dR}}^k(M)$ are called *de Rham cohomology classes*. The *de Rham complex* of a smooth manifold M is the sequence

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \dots \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M) \xrightarrow{d} \dots \xrightarrow{d} \Omega^n(M) \xrightarrow{d} 0, \quad (\text{II4})$$

where $d \circ d: \Omega^i(M) \rightarrow \Omega^{i+2}(M)$ is the zero map for each $i \in \{0, \dots, n-1\}$. By the Hodge Decomposition Theorem [War83, Theorem 6.8] we know that

$$\Omega^k(M) = d(\Omega^{k-1}(M)) \oplus \delta(\Omega^{k+1}(M)) \oplus H^k, \quad (\text{II5})$$

where $H^k := \{\omega \in \Omega^k(M) : \Delta_k \omega = 0\}$ are the so-called *harmonic k -forms*. From this decomposition it is clear that $\text{Ran } d_{k-1} \subset \ker d_k$.

Theorem II (Hodge theorem). *Each de Rham cohomology class on a compact oriented Riemannian manifold M contains a unique harmonic form.*

See [War83, Theorem 6.11] for a proof. The Hodge theorem implies that H^k is isomorphic to $H_{\text{dR}}^k(M)$, which gives that all the de Rham cohomology groups for compact, orientable and smooth manifolds are finite dimensional.

5 Summary of research papers

In this section we introduce the research papers, which constitute the main scientific contribution of this thesis.

Paper I - Non-monotonicity of the first eigenvalue for the 3D magnetic Robin Laplacian

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain and consider the magnetic Laplacian

$$P_{\gamma,B} = (-i\nabla + BA)^2, \quad (\text{II6})$$

with domain

$$\mathcal{D}(P_{\gamma,B}) = \{u \in H^2(\Omega) : in \cdot (-i\nabla + BA) + \gamma u = 0 \text{ on } \partial\Omega\},$$

where $\gamma \in \mathbb{R}$ is the Robin parameter, n the unit outward pointing normal vector of $\partial\Omega$, A is the magnetic potential and B is a constant playing the role of the strength of the magnetic field. Unless it is specified, we assume that Ω is simply connected. Let $\lambda(\gamma, B)$ be the lowest eigenvalue of $P_{\gamma,B}$. We are interested in the following question:

Is $B \mapsto \lambda(\gamma, B)$ a monotone increasing function for all $B > 0$?

Weak magnetic fields. For small $B > 0$ the answer is positive because introducing a magnetic field raises the ground state energy due to the diamagnetic inequality

Theorem 12 (Diamagnetic inequality). *Let $A \in L^2_{\text{loc}}(\mathbb{R}^n)$ and $f \in L^2_{\text{loc}}(\mathbb{R}^n)$ such that $(-i\nabla + A)f \in L^2_{\text{loc}}(\mathbb{R}^n)$ then*

$$|\nabla|f|| \leq |(-i\nabla + BA)f| \quad (\text{II7})$$

almost everywhere.

This inequality first appeared in [Kat72] and we refer to [FH10, Theorem 2.1.1] for a proof.

Strong magnetic fields. For large B , we are interested in finding some $B_0 > 0$ such that $B \mapsto \lambda(\gamma, B)$ is a monotone increasing function for $B \in [B_0, \infty)$, this also known in the literature as *strong diamagnetism*. This regime has been intensively studied because it is intimately connected with the computation of the third critical field $H_{C_3}(\kappa)$ defined in (6). As initially discussed in Section 1.1, if we have strong diamagnetism for $\lambda_1(0, \kappa HA)$ for large enough Ginzburg–Landau parameter κ , the equation

$$\lambda_1(\gamma, \kappa HA) = \kappa^2,$$

has a unique solution which allows us to compute $H_3(\kappa)$. We list below the main results regarding strong diamagnetism as well as counterexamples for strong magnetic fields.

- **Strong diamagnetism in constant magnetic fields.** Assume that $\text{curl } A = 1$ and $\gamma = 0$. If $\Omega \subset \mathbb{R}^2$ we have strong diamagnetism for regular domains [HMO1, FH07] and domains with corners [BNF07]. If $\Omega \subset \mathbb{R}^3$, then we have strong diamagnetism for generic domains [FH09] and for the unit ball [FP11]. Except for domains with corners, we have that $H_3(\kappa) \sim \Theta_0^{-1}\kappa$ where Θ_0 is the de Gennes constant introduced in Section 2.4.1. Lower order terms in κ can be obtained and they depend on the curvature of the domain. If we consider a fixed $\gamma \neq 0$, then we have analogous results in two-dimensional domains for $H_3(\kappa)$ since the Robin parameter γ does not appear in the leading term [Pano3, Kaco6]. In domains presenting corners, we have $H_3(\kappa) \sim \Theta / \min_{s \in \partial\Omega} \mu_1(\alpha_s)$ where $\mu_1(\alpha_s)$ is the lowest eigenvalue of a modal operator of the sector corresponding to the corner of smaller angle in Ω .
- **Strong diamagnetism in variable magnetic fields.** Let $\Omega \subset \mathbb{R}^2$ be a bounded and smooth domain, assume that the magnetic field is strictly positive and has a nondegenerate minimum B' achieved on the boundary. If $0 < \Theta_0 B' < \min_{x \in \Omega} B(x)$, then $\lambda(0, b)$ is monotone for large B and $H_3(\kappa) \sim \frac{\kappa}{\Theta_0 B'}$ [Ray09]. Recently, the analogue result in three dimensions was proven in [AAHR25]. For results considering discontinuous magnetic fields see [Ass21].
- **Counterexamples.** There are several ways to obtain counterexamples to strong diamagnetism:
 - (i) **Specific topological properties.** The main counterexample of strong diamagnetism was observed experimentally by Little and Parks in 1962 [LP62] when considering a thin superconducting film deposited on a cylindrical insulating support. When the applied magnetic field was increased, they observed oscillations in the electrical resistance when the superconductor was at a fixed temperature close to T_C . Since Ω is not simply connected, it is possible to prove that superconductivity is not lost at a unique value of the applied magnetic field [Erd97, FPS15]. A generalization of a thin annulus are the so-called thin domains (see [HK21]) where similar results can be obtained.
 - (ii) **Varying magnetic field.** The cases treated before with varying magnetic fields required a minimum of the magnetic field to be attained a unique point. If one consider $\text{curl } A(x) = \delta + (1 - |x|^2)\delta$ for $\delta > 0$ small enough, then $B \mapsto \lambda(0, B)$ is not monotone for large B [FPS15].
 - (iii) **Robin boundary condition with large γ .** Another way to obtain that $\lambda(\gamma, B)$ is non-monotone with respect to B is considering $\gamma \rightarrow +\infty$. This allows us to use γ as the asymptotic parameter instead of considering weak or strong magnetic fields. This approach was used⁵ in [KPS20] to prove that for $\Omega \subset \mathbb{R}^2$ being the

⁵In [KPS20] the opposite sign convention for γ is used, so the reader can find $\gamma \rightarrow -\infty$ there. However, we follow the sign convention proposed in [HKR22] and take $\gamma \rightarrow +\infty$.

unit disc

$$\lambda(\gamma, B) = -\gamma^2 + \gamma + \inf_{m \in \mathbb{Z}} \left(m - \frac{B}{2} \right)^2 - \frac{1}{2} + o(1) \quad \text{as } \gamma \rightarrow +\infty.$$

In Paper I, we use this last approach to provide one of the first counterexamples to monotonicity of $B \mapsto \lambda(\gamma, B)$ in three dimensions. Let $P_{\gamma, B}$ be the Robin magnetic Laplacian introduced in (116) and $\Omega = \{x \in \mathbb{R}^3 : \|x\| < 1\}$ be the unit ball, it was shown in [HKR22] that, as $\gamma \rightarrow +\infty$

$$\lambda(\gamma, B) = -\gamma^2 + 2\gamma + \mathfrak{e}(B) + o(1), \quad (118)$$

where $\mathfrak{e}(B) = \inf_{m \in \mathbb{Z}} \lambda_m(B)$ and

$$\lambda_{m, B} = \inf_{f \in \mathcal{D}(q_{m, B})} \frac{q_{m, B}(f)}{\|f\|_{\mathcal{H}}^2},$$

with $\mathcal{H} = L^2((0, \pi), \sin \theta \, d\theta)$ and

$$q_{m, B}(f) = \int_0^\pi (|f'(\theta)|^2 + \left(\frac{m}{\sin \theta} - \frac{B}{2} \right)^2 |f(\theta)|^2) \sin \theta \, d\theta$$

with domain

$$\mathcal{D}(q_{m, B}) = \begin{cases} \{f \in \mathcal{H} : \frac{1}{\sin \theta} f, f' \in \mathcal{H}\} & \text{if } m \neq 0 \\ \{f \in \mathcal{H} : f' \in \mathcal{H}\} & \text{if } m = 0. \end{cases}$$

We establish that the function $B \rightarrow \mathfrak{e}(B)$ is non-monotone which together with (118) gives the non-monotonicity of $B \mapsto \lambda(\gamma, B)$.

Paper II - Discrete spectrum of the magnetic Laplacian on almost flat magnetic barriers

In Section 1.1, we motivated why the study of the spectral properties of the magnetic Laplacian is relevant to describe superconductors. In paper II, we focus on a modal operator for domains $\Omega \subset \mathbb{R}^2$ which turns out to be useful to describe the asymptotics of the lowest eigenvalue of the magnetic Laplacian under piecewise constant strong magnetic fields with Dirichlet boundary condition. Recall that the Robin magnetic Laplacian $P_{\gamma, B} = (-i\nabla + BA)^2$ was introduced in (116). We have replaced κH by B in (7) to lighten the notation.

The case $\gamma = 0$ of the Neumann magnetic Laplacian $P_{0, B}$ corresponds to a superconductor surrounded by an insulator or vacuum [DG18, Kaco6]. If the superconductor is surrounded by a material in normal state or another superconductor then this leads to consider $\gamma \neq 0$ [Kaco6]. In the extreme case $\gamma = +\infty$ we have a Dirichlet boundary condition where the order parameter is zero on the boundary.

In order to prove the asymptotics of the lowest eigenvalue of the magnetic Laplacian for strong magnetic fields we use effective, or modal, operators. These operators are simpler than the original operator, but still allow us to capture the leading terms of the asymptotics of the original operator. In most cases, we use the constant fiber direct integral decomposition explained in section 3 to end up with a simpler operator. The eigenvalue asymptotics are obtained using a matching upper and lower bound. The upper bound is based on the construction of a trial state using the information coming from the effective operators and the min-max principle (see Theorem 5), combined with Agmon estimates [Agm82, Hel88]. For the lower bound, we use a partition of unity $(\chi_j)_{j \in J} \subset C^\infty(\mathbb{R}^n)$ which allows us to rewrite the quadratic form $q_{\gamma,B}$ associated to $P_{\gamma,B}$ as

$$q_{\gamma,B}(u) = \sum_{j \in J} q_{\gamma,B}(\chi_j u) - \sum_{j \in J} \|\nabla \chi_j u\|_{L^2(\Omega)}^2, \quad (119)$$

where we choose $\text{supp } \chi_j \subset D(z_j, B^{-\rho})$ with $z_j \in \overline{\Omega}$ and $\sum_{j \in J} |\nabla \chi_j| \leq CB^{2\rho}$ for a suitable $0 < \rho < 1$. For an overview of this method we refer to [FH10, Ray17]. In particular, if $D(z_j, B^{-\rho}) \cap \partial\Omega \neq \emptyset$, then one uses boundary coordinates [FH10, Appendix F] to “stretch” the boundary into a flat one.

Asymptotics of the magnetic Laplacian for a strong constant applied magnetic field.

For smooth domains, the asymptotics of $\lambda(\gamma, B)$ for Dirichlet boundary condition appear (in a more general situation) in works of Melin [Mel71], Hörmander [Hör07, Chapter 22.3] and Helffer–Morame [HM96] where

$$\lambda(+\infty, B) = B + o(B) \quad \text{as } B \rightarrow +\infty. \quad (120)$$

For the Neumann case, Lu and Pan [LP99a] found

$$\lambda(0, B) = \Theta_0 B + o(B) \quad \text{as } B \rightarrow +\infty, \quad (121)$$

where Θ_0 is the de Gennes constant introduced in Section 2.4.1.

In the Neumann case, there are two effective operators which need to be studied.

1. The magnetic Laplacian with constant magnetic field equal to 1 on \mathbb{R}^2 , also known as the Landau Hamiltonian, we denote it by $P_{0,1}(\mathbb{R}^2)$.
2. The Neumann magnetic Laplacian with constant magnetic field equal to 1 on the half-plane $\mathbb{R} \times \mathbb{R}_+$, we denote it by $P_{0,1}(\mathbb{R} \times \mathbb{R}_+)$. This operator is related, using partial Fourier transform and constant direct integral decomposition, with the operators \mathfrak{H}^ξ , introduced in Section 2.4.1. This is the effective operator where the leading term $\Theta_0 B$ in (121) comes from.

If the boundary of Ω is not smooth and presents corners, then a new model operator needs to be studied (see Figure 7). This corresponds to the magnetic Laplacian with constant magnetic field equal to 1 in the infinite sector of angle α , which we denote by $P_{0,1}(\Omega_\alpha)$. In this case we have by the works of Bonnaillie-Noël [Bono3, Bono5].

$$\lambda(0, B) = B \min_{i \in I} \mu(\alpha_i) + o(B) \quad \text{as } B \rightarrow +\infty, \quad (122)$$

where $\mu(\alpha_i)$ corresponds to the lowest eigenvalue of the magnetic Laplacian with constant magnetic field in Ω_{α_i} where $\alpha_i < \pi$. From numerical computations[Bono3], we know that $\mu(\alpha)$ is monotone in α and $\mu(\alpha) < \Theta_0$ for all $\alpha < \pi$. However, this last inequality has only been proven for $\alpha \leq 0.595\pi$ [Bono5, ELPO18]. Recently, Bonnaillie-Noël, Fournais, Kachmar and Raymond[BNFKR22] were able to prove $\mu(\pi - \delta) < \Theta_0$ for δ small enough.

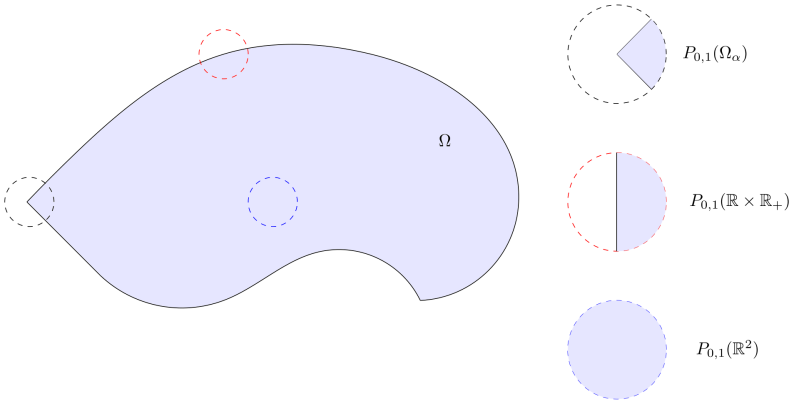


Figure 7: Visual representation on how the partition of unity is chosen for a general domain with corners. We identify in different colours the three case we can encounter.

Asymptotics of the magnetic Laplacian for a strong piecewise constant applied magnetic field.

Let $\Omega \subset \mathbb{R}^2$ be a smooth and bounded domain such that we can write $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ where Γ is a smooth curve which separates Ω_1 and Ω_2 (see Figure 8 for a visual representation of this case). Consider a piecewise constant magnetic field given by $\text{curl } \mathcal{A} = \mathbf{1}_{\Omega_1} + b\mathbf{1}_{\Omega_2}$ where $b \in (-1, 0)$. In this situation, we need six effective operators (see Figure 8) which need to be considered for the Neumann realization and three for the Dirichlet one. We denote by $\lambda^D(b, B)$ and $\lambda^N(b, B)$ the lowest eigenvalue of the Dirichlet and Neumann realization.

- i. The magnetic Laplacian on \mathbb{R}^2 with constant magnetic field equal to 1 and b , which we denote by $P_{0,1}(\mathbb{R}^2)$ and $P_{0,b}(\mathbb{R}^2)$ respectively.

2. The Neumann magnetic Laplacian on the half-plane $\mathbb{R} \times \mathbb{R}_+$ with constant magnetic field equal to 1 and to b , which we denote it by $P_{0,1}(\mathbb{R} \times \mathbb{R}_+)$ and $P_{0,b}(\mathbb{R} \times \mathbb{R}_+)$ respectively.
3. The magnetic Laplacian on \mathbb{R}^2 with piecewise constant magnetic field $\text{curl } A = \mathbb{1}_{\{x_1 < 0\}} + b\mathbb{1}_{\{x_1 > 0\}}$, which we denote as P_{mb} . This operator is related with the family of operators \mathfrak{h}_b^{ξ} (see Section 3.2).
4. The Neumann magnetic on the half-plane $\mathbb{R} \times \mathbb{R}_+$ with a step magnetic field $\text{curl } A = \mathbb{1}_{\{0 < \theta < \alpha\}} + b\mathbb{1}_{\{\alpha < \theta < \pi\}}$ where (r, θ) are the polar coordinates. This effective operator was introduced in [Ass21] and we denote it by $P_{0,b}(\alpha)$.

From the works of Assaad [Ass19, Ass21] we have

$$\lambda^D(b, B) = |b|B + o(B) \quad \text{as } B \rightarrow +\infty, \quad (123)$$

and for α close to $\frac{\pi}{2}$ and b close to -1 (see [Ass21] for the precise admissible region)

$$\lambda^N(b, B) = \min \mu(\alpha_j, b)B + o(B) \quad \text{as } B \rightarrow +\infty, \quad (124)$$

where $\mu(\alpha_j, B)$ is the lowest eigenvalue of the effective operator $P_{0,b}(\alpha_j)$ for admissible α_j .

In paper II, we study the situation where Γ , also called the magnetic barrier, is not smooth. In particular, we assume that Γ presents a corner of angle $\pi + \delta$ seen from Ω_1 (see Figure 9). We make this definition more precise. Consider the function g_δ defined as

$$g_\delta(x_1) = \begin{cases} x_1 \tan \delta & \text{if } x_1 \leq 0 \\ 0 & \text{if } x_1 > 0. \end{cases} \quad (125)$$

Let

$$\Omega_{1,\delta} := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 < g_\delta(x_1)\}, \quad (126)$$

and

$$\Omega_{2,\delta} := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > g_\delta(x_1)\}. \quad (127)$$

We consider the magnetic Laplacian in \mathbb{R}^2

$$\mathcal{L}_{b,\delta} = (-i\nabla + \mathcal{A}_{b,\delta})^2, \quad (128)$$

with domain

$$\mathcal{D}(\mathcal{L}_{b,\delta}) = \{u \in L^2(\mathbb{R}^2) : (-i\nabla + \mathcal{A}_{b,\delta})u \in L^2(\mathbb{R}^2, \mathbb{R}^2) \text{ and } (-i\nabla + \mathcal{A}_{b,\delta})^2 u \in L^2(\mathbb{R}^2)\},$$

where $\mathcal{A}_{b,\delta} \in H_{\text{loc}}^1(\mathbb{R}^2, \mathbb{R}^2)$ satisfies

$$\text{curl } \mathcal{A}_{b,\delta} = \mathbb{1}_{\Omega_{1,\delta}} + b\mathbb{1}_{\Omega_{2,\delta}},$$

with $b \in (-1, 0)$. Let $\lambda_b(\delta) := \inf \sigma(\mathcal{L}_{b,\delta})$, then adapting the construction in [BNFKR22], which is based in the ideas introduced in [CG21], we are able to prove the following results.

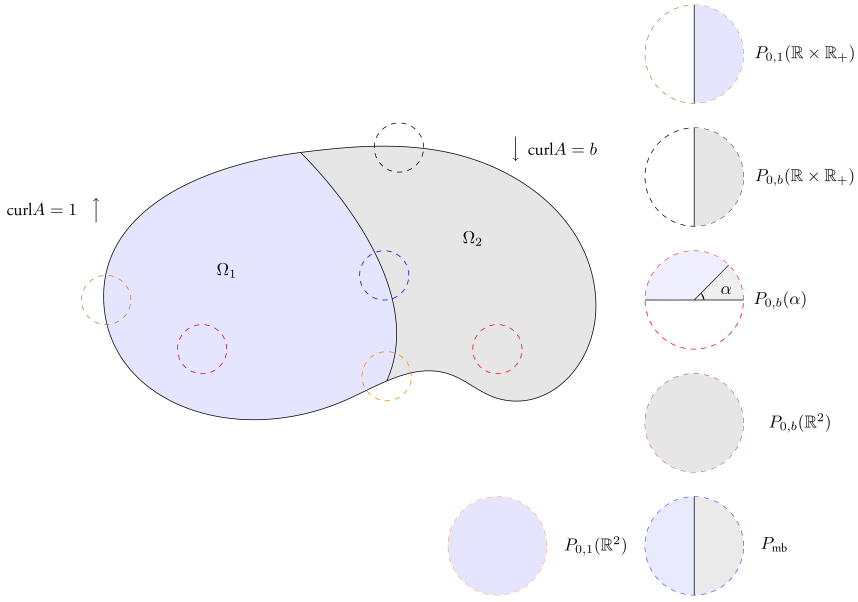


Figure 8: Visual representation on how the partition of unity is chosen for a smooth domain with a step magnetic field. We identify in different dashed circles the six possible modal operators we can encounter.

Theorem 13. *There exists a $\delta_0 \in (0, \frac{\pi}{2})$ such that, for all $\delta \in (0, \delta_0)$,*

$$\lambda_b(\delta) \leq \beta_b - C^2 \delta^2 + o(\delta^2) < \beta_b, \quad (129)$$

where C is a constant related to a spectral parameter of the operator \mathfrak{h}_b^ξ introduced in Section 2.4.I.

An important remark is that the construction fails if we swap the region with magnetic field b with the region being 1 because the constant C changes sign.

Moreover, this theorem implies that

$$\lambda^D(b, B) = B \lambda_b(\delta) + o(B) \quad \text{as } B \rightarrow +\infty, \quad (130)$$

for smooth domains with a magnetic barrier Γ which has an almost flat corner.

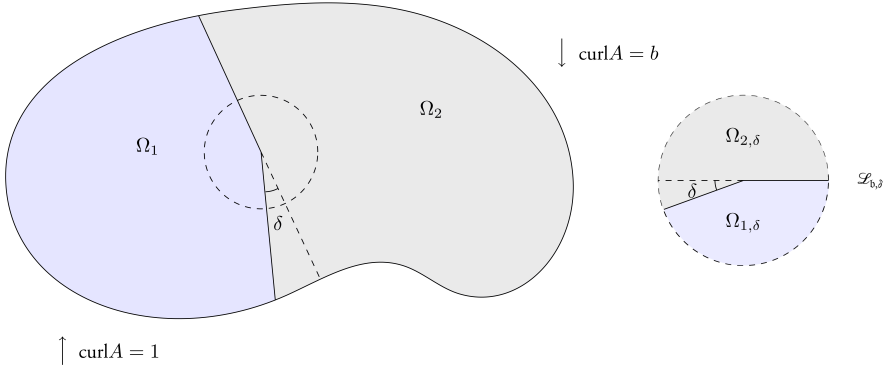


Figure 9: Figure of the type of the domains considered in Paper II where $\mathcal{L}_{b,\delta}$ is used as a modal operator.

Paper III - The magnetic Laplacian on the disc for strong magnetic fields

We consider the magnetic Laplacian⁶

$$H_B(R) = (-i\nabla - BA)^2 \quad (131)$$

on the disc $\Omega_R = \{x \in \mathbb{R}^2 : |x| < R\}$ of radius R where the magnetic potential $A(x_1, x_2) = \frac{1}{2}(-x_2, x_1)$ generates a constant magnetic field $\text{curl } A = 1$. As explained in Section 1.1, the spectral properties of this operator are connected with superconductivity, for example in determine the third critical field for infinite cylindrical samples under a constant magnetic field parallel to the axis.

We consider $H_B(R)$ with Dirichlet and Neumann boundary conditions written in polar coordinates (r, θ) . We denote $H_B^D(R)$ the Dirichlet realization and $H_B^N(R)$ Neumann respectively. Using the L^2 -unitary transform

$$U_R u(r, \theta) := u(r/R, \theta)$$

we have that $H_B(R)$ is unitarily equivalent to $R^{-2}H_{BR^2}(1)$ which reduces the study to the unit disc. We denote $H_B(1)$ by H_B to save some notation and Ω_1 by Ω . Using the constant direct fiber decomposition introduced in Section 3.2, we can decompose $u \in L^2(\Omega)$ as

$$u(r, \theta) = \sum_{m \in \mathbb{Z}} u_m(r) \frac{e^{-im\theta}}{\sqrt{2\pi}},$$

⁶Here we used a different sign convention for the magnetic Laplacian but changing the sign of the magnetic field does not change the physics of the problem.

where

$$u_m(r) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} u(r, \theta) e^{im\theta} d\theta,$$

and $\sum_{m \in \mathbb{Z}} \|u_m\|^2 < +\infty$. This decomposition allows us to decompose our original operators as

$$H_B^D = \bigoplus_{m \in \mathbb{Z}} H_{m,B}^D \text{ and } H_B^N = \bigoplus_{m \in \mathbb{Z}} H_{m,B}^N.$$

Moreover, as discussed in Section 3, $H_{m,B}^D$ and $H_{m,B}^N$ are the Dirichlet, respectively Neumann, realization acting on $L^2((0, 1), r dr)$ of the differential operator

$$H_{m,B} = -\frac{d^2}{dr^2} - \frac{1}{r} \frac{d}{dr} + \left(\frac{m}{r} - \frac{Br}{2} \right)^2. \quad (132)$$

More precisely, for $m \neq 0$, the domains of $H_{m,B}^D$ and $H_{m,B}^N$ are

$$\begin{aligned} \mathcal{D}(H_{m,B}^D) &= \{u: u, u/r, u', H_{m,B}u \in L^2((0, 1), r dr), u(1) = 0\}, \\ \mathcal{D}(H_{m,B}^N) &= \{u: u, u/r, u', H_{m,B}u \in L^2((0, 1), r dr), u'(1) = 0\}, \end{aligned}$$

whereas

$$\begin{aligned} \mathcal{D}(H_{0,B}^D) &= \{u: u, H_{0,B}u \in L^2((0, 1), r dr), u(1) = 0 \text{ and } \lim_{r \rightarrow 0^+} \frac{u(r)}{\ln r} = 0\}, \\ \mathcal{D}(H_{0,B}^N) &= \{u: u, H_{0,B}u \in L^2((0, 1), r dr), u'(1) = 0 \text{ and } \lim_{r \rightarrow 0^+} \frac{u(r)}{\ln r} = 0\}. \end{aligned}$$

Let $\lambda_n(H_{m,B}^D)$ and $\lambda_n(H_{m,B}^N)$ be the n -th eigenvalues of $H_{m,B}^D$ and $H_{m,B}^N$ respectively. The study of the eigenvalues of $H_{m,B}^N$ traces back to the work of Saint-James [SJ65]. As it is stated in [SJ65] the computation of the eigenvalues of $H_{m,B}^N$ involves special functions which are hard to compute and numerical computations were provided instead.

We are interested in the asymptotics of $\lambda_n(H_{m,B}^D)$ and $\lambda_n(H_{m,B}^N)$ for $B \rightarrow +\infty$. Results concerning the Dirichlet case $\lambda_n(H_{m,B}^D)$ were obtained by Erdős, Helffer and Persson-Sundqvist for $n = 1$ and $m \geq 0$. Recently, Baur and Weidl [BW25] obtained a two term eigenvalue asymptotics for all $n \in \mathbb{N}$ and $m \in \mathbb{Z}$

$$\begin{aligned} \lambda_n(H_{m,B}^D) &= (2n - 1 + |m| - m)B \\ &\quad + e^{-\frac{B}{2}} \frac{B^{2n+m}}{(n-1)!(m+n-1)!2^{2(n-1)+m}} \left(1 + \mathcal{O}(B^{-1}) \right). \quad (133) \end{aligned}$$

The idea of the proof was the use of special functions to reduce the problem to find the zeroes of a particular Kummer's confluent hypergeometric function. This approach is more

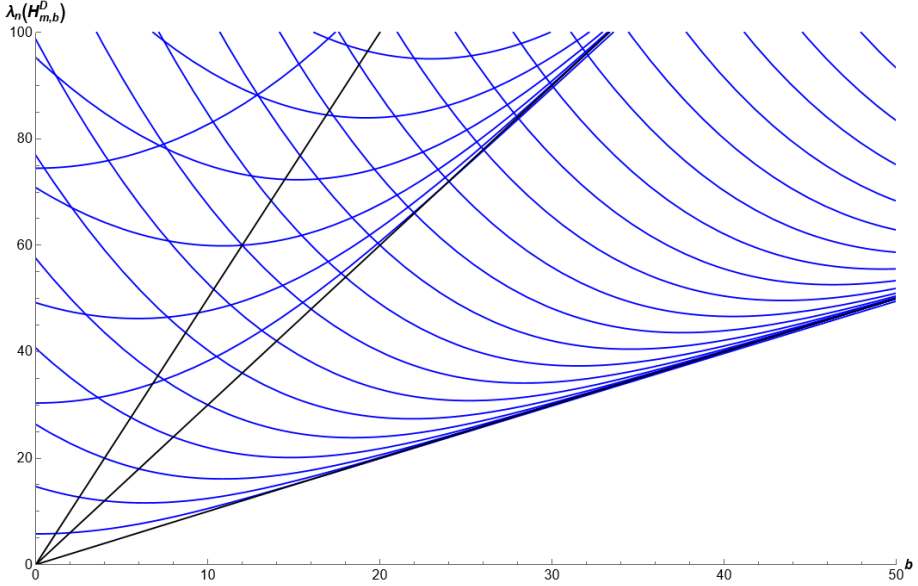


Figure 10: Numerical computations of $\lambda_n(H_{m,b}^D)$ (blue) for $0 \leq m \leq 25$ and $1 \leq n \leq 3$ together with the curves $b, 3b$ and $5b$ in black.

complicated in the Neumann case since one has to study a combination of a confluent hypergeometric function and its derivative. In Paper III, we extend the result from Baur and Weidl to give a two terms asymptotics of $\lambda_n(H_{m,B}^N)$.

Theorem 14. *Given $n \in \mathbb{N}$ and $m \in \mathbb{Z}$, if $B \rightarrow +\infty$, it holds that*

$$\lambda_n(H_{m,B}^N) = (2n - 1 + |m| - m)B - e^{-\frac{B}{2}} \frac{B^{2n+m}}{(n-1)!(m+n-1)!2^{2(n-1)+m}} \left(1 + \mathcal{O}(B^{-1})\right). \quad (134)$$

Remark. We obtain an analogue result for Robin boundary conditions $u'(1) = \gamma u(1)$ for fixed $\gamma \in \mathbb{R}$ and we can also recover the Dirichlet case (133) with the same approach.

The proof of Theorem 14 is divided in an upper and a lower bound.

- Upper bound. We use the min-max principle (see Theorem 5) with the trial state

$$u_{m,n}(r) := r^m L_{n-1}^m \left(\frac{Br^2}{2} \right) \left(C_1 e^{\frac{B}{4}(1-r^2)} + C_2 e^{-\frac{B}{4}(1-r^2)} \right),$$

where C_1, C_2 are constant to determine so that $u_{m,n} \in \mathcal{D}(H_{m,B}^N)$ and L_{n-1}^m are the associated Laguerre polynomials.

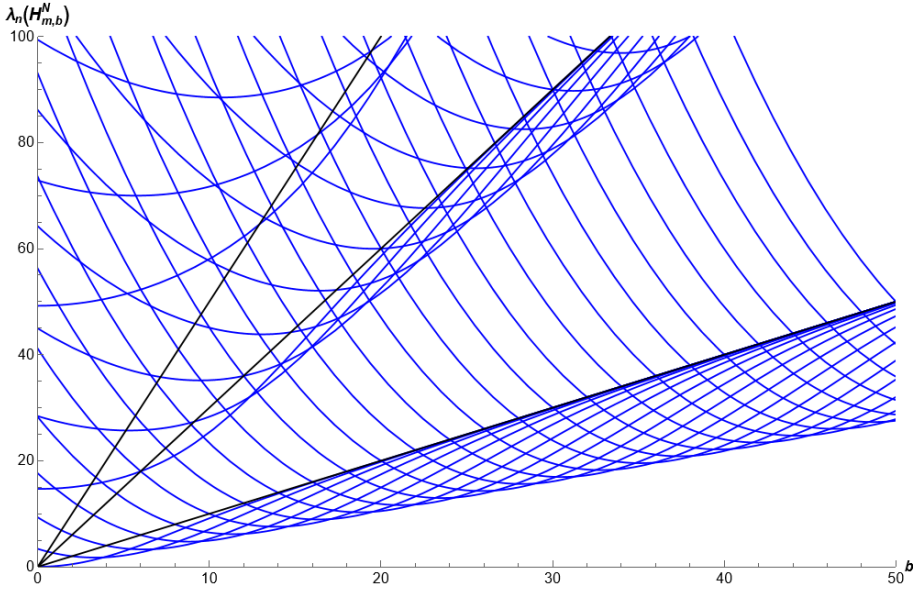


Figure 11: Numerical computations of $\lambda_n(H_{m,b}^N)$ (blue) for $0 \leq m \leq 25$ and $1 \leq n \leq 3$ together with the curves b , $3b$ and $5b$ in black.

- Lower bound. We use the Temple inequality (see Theorem 6) and the lower bound given in the following proposition which is proven using a suitable partition of unity to relate $H_{m,B}^N$ with two self-adjoint operators with Dirichlet boundary conditions at $r = \frac{3}{4}$ and $r = \frac{1}{2}$.

Proposition 7. *For any integer $m \geq 0$, there exist $C, B_0 > 0$ such that, for $n = 1, 2, 3, \dots$ and $B \geq B_0$, we have*

$$\lambda_n(H_{m,B}^N) \geq (2n - 1)B - C.$$

Paper IV - Reproving Friedlander's inequality with the de Rham complex

Let $\Omega \subset \mathbb{R}^n$ be an open connected, bounded domain with Lipschitz boundary with $n \geq 2$. The Dirichlet and the Neumann Laplacian in Ω , which we denote by Δ_D and Δ_N respectively, have discrete spectrum and we denote by $\lambda_j(\Delta_D)$ and $\lambda_j(\Delta_N)$ the j :th eigenvalue of Δ_D and Δ_N . Inequalities between $\lambda_j(\Delta_D)$ and $\lambda_j(\Delta_N)$ trace back to the work of Pólya for $n = 2$ [Pól52] where the inequality $\lambda_2(\Delta_N) < \lambda_1(\Delta_D)$ was proven. The case $n = 1$ is completely understood due to the interlacing property of the zeroes of successive eigenfunctions for conventional Sturm–Liouville operators [Inc44, Chapter 10]. *Friedlander's inequality*

$$\lambda_{j+1}(\Delta_N) \leq \lambda_j(\Delta_D) \tag{I35}$$

for $j = 1, 2, \dots$, was proven by Friedlander in 1991 [Fri91] for general C^1 domains. The smoothness assumption was removed and the inequality was proven to be strict by Filonov in [Filo4]. Recently, Rohleder [Roh25] proved that for any simply connected, bounded, Lipschitz domain in \mathbb{R}^2

$$\lambda_{j+2}(\Delta_N) \leq \lambda_j(\Delta_D), \quad (136)$$

for $j = 1, 2, \dots$. The main contribution of this paper is the method we introduce. We combine the ideas of [Roh25] with the de Rham complex [BL92] into a common framework that gives a new proof of Friedlander's inequality (135) and a generalization of Rohleder's results on the curl-curl operator [Roh24]. Moreover, we provide a short proof of (136) in dimension 2. In Section 4, we introduced the de Rham complex, but $\Omega \subset \mathbb{R}^n$ has boundary, so we require an appropriate boundary condition. We consider the de Rham complex

$$0 \rightarrow \mathcal{H}_0 \xrightarrow{d_0} \mathcal{H}_1 \xrightarrow{d_1} \dots \mathcal{H}_{n-1} \xrightarrow{d_{n-1}} \mathcal{H}_d \rightarrow 0,$$

where $\mathcal{H}_k := L^2(M; \wedge^k T^*M)$ for $k \in \{0, 1, \dots, n\}$ and $d_{k,\max}$ is the maximal realization of the exterior differential introduced in Theorem 10. We shall write $d_{k,\max}$ for the maximal realization of d_k , i.e. $u \in \mathcal{D}(d_{k,\max})$ if and only if $u \in L^2(M; \wedge^k T^*M)$ satisfies that $du \in L^2(M; \wedge^{k+1} T^*M)$ where the exterior differential is applied in a distributional sense. Similarly, as in (112), but now with a boundary condition, we consider the Laplacian

$$\Delta_{k,a} := d_{k,\max}^* d_{k,\max} + d_{k-1,\max} d_{k-1,\max}^*. \quad (137)$$

Note that $d_{k-1,\max}^*$ is the minimal realization of δ_{k-1} , where δ_{k-1} was defined in (111). It is important to observe that

$$\Delta_{0,a} = d_{0,\max}^* d_{0,\max} = \Delta_N$$

is defined from the quadratic form with domain $H^1(M)$, so it is the Neumann realization of the Hodge Laplacian on 0-forms. Moreover,

$$\Delta_{n,a} = d_{n,\max} d_{n,\max}^* = \Delta_D,$$

is defined from the quadratic form with domain $H_0^1(M, \wedge^n T^*M)$, so it is the Dirichlet realization of the Hodge Laplacian on n -forms. Indeed, the Hodge star operator $L^2(M) \rightarrow L^2(M; \wedge^n T^*M)$, $f \mapsto f \star (1) = f dV$ implements a canonical identification of $\Delta_{n,a}$ with the Dirichlet realization of the Laplacian on 0-forms. For a k -form ω we can near the boundary write

$$\omega = \omega_1 + dx_\nu \wedge \omega_2, \quad (138)$$

where x_ν is the inwards pointing normal coordinate near the boundary and ω_1 and ω_2 take values in $\wedge^k T^* \partial M$ and $\wedge^{k-1} T^* \partial M$ respectively. Following [Gil95, Section 2.7.1] we can define the absolute boundary condition B_a by

$$B_a \omega := \omega_2|_{\partial M} = 0.$$

The quadratic form associated with $\Delta_{k,a}$ is given by

$$q_{k,a}(\omega) := \int_M (|d_k \omega|^2 + |\delta_{k-1} \omega|^2) dx,$$

with domain

$$\mathcal{D}(q_{k,a}) := \mathcal{D}(d_{k,\max}) \cap \mathcal{D}(d_{k-1,\max}^*) \equiv \{u \in H^1(M; \wedge^k T^* M) : B_a u = 0\}.$$

An important tool to prove the main results of this paper is Glazman's lemma, which can be seen as a reformulation of the min-max principle introduced in Theorem 5. Let q_T be the quadratic form associated with a positive self-adjoint operator T , we define the *Rayleigh quotient* $R[u] := q_T(u) / \langle u, u \rangle$ where $u \in \mathcal{D}(q_T) \setminus \{0\}$.

Lemma 6 (Glazman's lemma). *Let T be a positive self-adjoint operator with discrete spectrum acting in a Hilbert space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ and q_T the quadratic form associated with T . Then*

$$N(T, \lambda) = \max_{\substack{V \subseteq \mathcal{D}(q_T) \\ R[u] \leq \lambda \forall u \in V \setminus \{0\}}} \dim V,$$

where $N(T, \lambda) := \#\{j : \lambda_j(T) \leq \lambda\}$ is the counting function of T .

Combining the ideas from [Roh25] applied to the de Rham complex we can show the following two lemmas.

Lemma 7. *Let $\Omega \subset \mathbb{R}^n$ be a connected and bounded domain with Lipschitz boundary with $n \geq 2$ and $(\Delta_{k,a})_{k=0,\dots,n}$ be the corresponding Hodge Laplacians on k -forms with absolute boundary conditions. For $\lambda \geq 0$ and $0 \leq k \leq n$ we have that*

$$\binom{n}{k} N(\Delta_{n,a}, \lambda) \leq N(\Delta_{k,a}, \lambda) \leq \binom{n}{k} N(\Delta_{0,a}, \lambda).$$

Lemma 8. *Let $\Omega \subset \mathbb{R}^n$ be a connected and bounded domain with Lipschitz boundary, with $n \geq 2$, and let $(\Delta_{k,a})_{k=0,\dots,n}$ be the corresponding Hodge Laplacians on k -forms with absolute boundary conditions. For $\lambda \geq 0$,*

$$n N(\Delta_{n,a}, \lambda) + \dim \ker(\Delta_{n,a} - \lambda) \leq N(\Delta_{n-1,a}, \lambda).$$

The combination of these two lemmas gives

$$nN(\Delta_D, \lambda) + \dim \ker(\Delta_D - \lambda) \leq nN(\Delta_N, \lambda),$$

which by using Glazman's lemma proves Friedlander's inequality. Equation (136) and the results concerning the curl-curl operator can be shown using Lemma 8 for $n = 2$ and $n = 3$ identifying $\Delta_{n-1,a}$ and its spectrum with an appropriate operator.

6 Open questions

6.1 Counting eigenvalues below higher Landau levels

Let $\Omega \subset \mathbb{R}^2$ be a bounded, connected and smooth domain with constant magnetic field $B > 0$. Consider the Neumann realization of the magnetic Laplacian in Ω

$$H_B^\Omega = (-i\nabla - BA)^2 \quad (139)$$

where the magnetic potential A solves $\text{curl } A = 1$. Since Ω is bounded, H_B^Ω has discrete spectrum and we denote by $\lambda_j(H_B^\Omega)$ the j :th eigenvalue. We define the magnetic flux on Ω by

$$\Phi := \frac{1}{2\pi} \int_{\Omega} B \text{curl } A(x) \, dx,$$

and the counting function $N(H_B^\Omega, \lambda) = \#\{j : \lambda_j(H_B^\Omega) \leq \lambda\}$. In [FFG⁺25], it was proven that for Ω simply connected we have the isochoric inequality

$$N(H_B^\Omega, B) \geq N(H_B^{D(0,1)}, B) = [\Phi] \quad (140)$$

As already pointed out in [FFG⁺25, Remark 2.3] it would be interesting to count the number of eigenvalues below higher Landau levels $(2n - 1)B$ for $n = 1, 2, 3, \dots$

Due to the previous inequality and as already discussed in [FFG⁺25], getting a bound for $N(H_B^{D(0,1)}, (2n - 1)B)$ would be a good starting point. Let $H_{m,B}^N$ be the fiber operators given in (132) with Neumann boundary condition, then using Theorem 1 of Paper III we can show the following proposition for the j :th eigenvalue $\lambda_j(H_{m,B}^N)$ of $H_{m,B}^N$.

Proposition 8. *Let $n \in \mathbb{N}$, then for $m \geq 0$, $\lambda_{n+1}(H_{m,B}^N) > (2n - 1)B$ and, for $m < 0$, $\lambda_l(H_{m,B}^N) > (2n - 1)B$ if $l + |m| > n$.*

Let

$$J_{B,n} = \{(j, m) : \lambda_j(H_{m,B}^N(R)) < (2n - 1)B\}.$$

Due to the previous proposition we can expand $J_{B,n}$ as follows

$$J_{B,n} = \bigcup_{j=1}^n J_{B,n,+}^{(j)} \cup \bigcup_{m=1}^{n-1} \{l : \lambda_l(H_{-m,B}^N) < 2(n - 1)B \text{ and } 1 \leq l \leq n - m\}$$

where $J_{B,n,+}^{(j)} = \{m : m \geq 0 \text{ and } \lambda_j(H_{m,B}^N) < (2n - 1)B\}$. The second component is finite by Proposition 8, so for large B it would not contribute to the leading term. We are counting

when an eigenvalue crosses the Landau level (see Figure 11). The general solution of the differential equation $H_{m,B} = (2n - 1)B$, which is not singular at the origin, is given by

$$u(r) = Cr^m L_{n-1}^m \left(\frac{Br^2}{2} \right) e^{-\frac{Br^2}{4}}, \quad (141)$$

for $m \geq 0$ where L_{n-1}^m is the associated Laguerre polynomial and C is a constant. Imposing the Neumann boundary condition $u'(1) = 0$ we obtain

$$\left(m - \frac{B}{2} \right) L_{n-1}^m \left(\frac{B}{2} \right) + B (L_{n-1}^m)' \left(\frac{B}{2} \right) = 0. \quad (142)$$

This means that to compute the counting function $N(H_A^{D(0,1)}, (2n - 1)B)$ it is enough to count the roots of (142) with respect to B since as we have seen in Section 3.2

$$\sigma(H_B^{D(0,1)}) = \bigcup_{m \in \mathbb{Z}} \sigma(H_{m,B}^N).$$

Let r_1, \dots, r_n be the roots of the polynomial $a_n m^n + \dots + a_1 m + a_0$, then by Vieta's formulas we know that

$$\sum_{i=1}^n r_i = -\frac{a_{n-1}}{a_n} = \Phi - \frac{n(n-1)}{2} + (n-1)\Phi = n\Phi - \frac{n(n-1)}{2}. \quad (143)$$

Applying this to the polynomial in (142) we get that the sum of the roots is given by $n\Phi - n(n-1)/2$ where the flux $\Phi = \frac{B}{2}$ in the case of the unit disc. If we ensure that all the roots r_i are positive for Φ large enough, then

$$\sum_{j=1}^n \#J_{b,n,+}^{(j)} = n[\Phi] - \frac{n(n-1)}{2}, \quad (144)$$

where the ceiling function $[\cdot]$ is taking to add the eigenvalues coming from $m = 0$. Moreover, $\frac{n(n-1)}{2}$ takes into account the finite number of roots coming from $\bigcup_{m=1}^{n-1} \{l : \lambda_l(H_{-m,B}) < 2(n-1)B \text{ and } 1 \leq l \leq n-m\}$, so if B is large enough to make all the eigenvalues coming from negative m to have crossed the Landau level $(2n-1)B$ then we would have

$$N(H_B^{D(0,1)}, (2n-1)B) = (2n-1)[\Phi], \quad (145)$$

Proving that for each $n \in \mathbb{N}$ there is exists a $B > 0$ big enough such that all roots in (142) are positive can be done for a fixed n using Sturm chains⁷ but so far no method has proven to be effective to tackle this problem for all $n \in \mathbb{N}$. However, this partial result suggests the following conjecture.

⁷See [Per51] for an introduction on Sturm chains and Sturm's theorem.

Conjecture 1. *Let $n \in \mathbb{N}$, then for $\Omega \subset \mathbb{R}^2$ simply connected there exists a $B(n) > 0$ such that for all $B > B(n)$*

$$N(H_B^\Omega, (2n-1)B) \geq N(H_b^{D(0,1)}, (2n-1)B) = n[\Phi] \quad (146)$$

Another conjecture related with the eigenvalues of the Neumann magnetic Laplacian on the disc is the following.

Conjecture 2. *For all $B > 0$*

$$\lambda_1(H_B^{D(0,1)}) < \Theta_0 B. \quad (147)$$

This conjecture is observed already in the work of Saint-James [SJ65] where numerical results indicate that the conjecture holds. Moreover, the conjecture is known to be true for weak and strong magnetic fields (see [FH10, HL25] and reference therein). However, no result for all $B > 0$ is known. This result together with the reverse Faber–Krahn inequality conjectured by Fournais and Helffer [FH19] would give

$$\lambda_1(H_B^\Omega) \leq \lambda(H_B^{\Omega^*}), \quad (148)$$

where $\Omega \subset \mathbb{R}^2$ is an open, bounded, simply connected and smooth set and Ω^* is the disc such that $|\Omega^*| = |\Omega|$. This reverse Faber–Krahn inequality has been proven for weak magnetic fields in [CLPS24], which implies that $\lambda_1(H_B^\Omega) < \Theta_0 B$ for small $B > 0$.

6.2 Existence of discrete spectrum for modal operators in domains with corners

As already discussed in the summary of Paper II, if $\Omega \subset \mathbb{R}^2$ is a domain with corners the asymptotics of the lowest eigenvalue $\lambda(0, B)$ of the Neumann magnetic Laplacian for strong magnetic fields is given by

$$\lambda(0, B) = B \min_{i \in I} \mu(\alpha_i) + o(B) \quad \text{as } B \rightarrow +\infty,$$

where $\mu(\alpha_i)$ corresponds to the lowest eigenvalue of the magnetic Laplacian with constant magnetic field in the infinite sector of angle α_i denoted by $P_{0,1}(\Omega_{\alpha_i})$. By Persson’s lemma (see Theorem 7 and [Bono3, Section 3.2]) we have that

$$\inf \sigma_{\text{ess}}(P_{0,1}(\Omega_\alpha)) = \Theta_0,$$

for any $\alpha \in (0, 2\pi)$ where $\Theta_0 \approx 0.59$ is the de Gennes constant discussed in Section 2.4.1. We mentioned before that numerical results [BNFo7, Figure 1] suggest that $\mu(\alpha) < \Theta_0$ and $\alpha \mapsto \mu(\alpha)$ is an increasing function for $\alpha \in (0, \pi)$ with $\mu(\pi) = \Theta_0$. Moreover, for angles $\alpha > \pi$, numerical results indicate that there are no eigenvalues below the infimum of

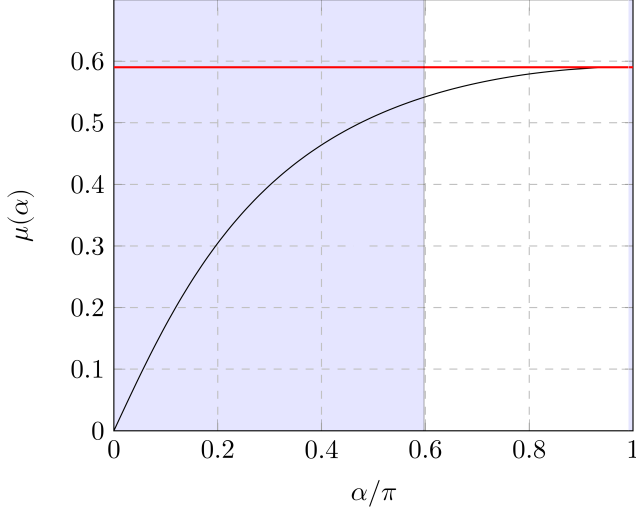


Figure 12: Plot of $\mu(\alpha)$ with respect to $\frac{\alpha}{\pi}$ for $\alpha \in [0, \pi]$. We used the numerical computations in [BNDMV07, Table 1]. The red line represents the bottom of the essential spectrum given by Θ_0 and the blue area denotes the α for which the existence of discrete spectrum has been proven.

the essential spectrum. However, this last inequality has only been proven for $\alpha \leq 0.595\pi$ [Bono05, ELPO18] and for $\alpha = \pi - \delta$ for $\delta > 0$ small enough [BNFKR22]. Proving the existence of discrete spectrum for angles $\alpha \in (0.595\pi, \pi - \delta)$ remains open as well as the monotonicity of $\alpha \mapsto \mu(\alpha)$ which would imply the existence of discrete spectrum in the missing interval too.

There have been several trial states constructed to show that $\mu(\alpha) < \frac{\pi}{2}$. A new possible approach uses the change of variables

$$s = x_1 + \cot \frac{\alpha}{2} x_2 \text{ and } t = x_1 - \cot \frac{\alpha}{2} x_2,$$

to send the sector Ω_α to $\mathbb{R}_+ \times \mathbb{R}_+$ and be able to work with a domain which does not depend on α . A trial state of the form

$$u_M(s, t) := \phi(ks)\phi(kt)e^{i(a(s-t)+b(s^2-t^2))} \quad (149)$$

where k, a, b are constants with $k > 0$ and ϕ is the eigenfunction of the lowest eigenvalue of $\mathfrak{h}_0^{\tilde{\zeta}}$ introduced in Section 2.4.1. This trial state gives $\mu(\alpha) < \Theta_0$ for $\alpha < \pi/2$ choosing the appropriate constants and works better than the trial state of [ELPO18] for small α . However, it does not improve the results in [ELPO18].

Another approach is using the idea behind the trial state chosen by Jadallah in [Jado1] where it was proven that $\mu(\frac{\pi}{2}) < \Theta_0$. The trial state is given in polar coordinates (r, θ) by

$$u(r, \theta) := e^{-\frac{r^2}{2}s(\theta)} e^{i(rp(\theta) + \frac{1}{2}r^2g(\theta) + \frac{1}{3}r^3q(\theta))}, \quad (150)$$

where $s(\theta) = a_0(5 + \sin^2(2\theta))$, $g(\theta) = \frac{1}{4}\sin(4\theta)$, $p(\theta) = a_1\cos(2\theta)$ and $q(\theta) = a_2\cos(6\theta)$, where a_0, a_1 and a_2 are suitable constants. Combining the trial state of Jadallah with the

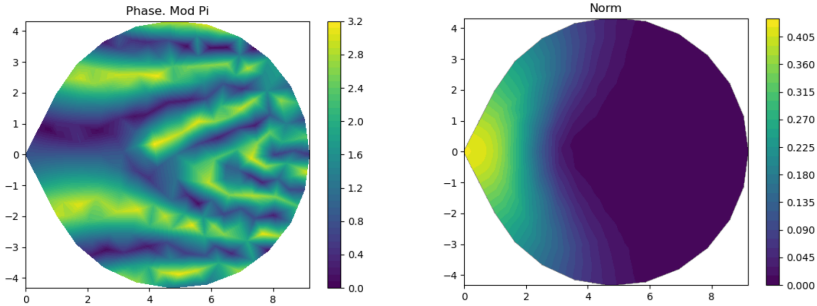


Figure 13: Phase and norm of the numerical computations for $\alpha = 0.7\pi$. Note that the phase looks almost linear. The right-hand side of the plots start to be more unreliable since the norm of the function is very small far from the origin.

intuition we get from numerical computations (see Figure 13) we can consider

$$u_\alpha(r, \theta) := e^{-r^2v(r,\theta)} e^{i\psi_n(r,\theta)}, \quad (151)$$

where $v(r, \theta) = a_0 + a_1\cos(\frac{\pi\theta}{\alpha})^2$ and $\psi_n = \sum_{i=1}^n b_i r^i \sin(\frac{i\pi\theta}{\alpha})$ where a_i and b_i are suitable constants and $n \in \mathbb{N}$. Computations using Mathematica have shown that we could use this trial state to get below Θ_0 for angles $\alpha \leq 0.57\pi$, which still does not improve the results in [ELPO18].

6.3 Clustering of eigenvalues for non-smooth magnetic perturbations of the Landau Hamiltonian

In 2008 Rozenblum and Tashchian [RT08] showed that the Landau levels split when perturbing the Landau Hamiltonian $P_{0,1}(\mathbb{R}^2)$ with a smooth compactly supported perturbation of the magnetic field. In other words, they showed that an infinite number of eigenvalues split away from the Landau levels and they converge superexponentially to the Landau levels. The approach they used was based on the study of approximate creation and annihilation operators. The upper estimate required only C_c^2 perturbations but the lower estimates require more regularity if we consider higher Landau Levels. The case of a moderately decaying magnetic field has also been studied in [RT09] by the same authors.

We would like to consider piecewise smooth compactly supported perturbations. This case is interesting in the context of surface superconductivity when considering piecewise constant applied magnetic fields. The results on step magnetic fields assume that the magnetic barrier divides the domain in simply connected pieces (see Figure 8 for an example). However, it would be interesting to consider another case when the magnetic barrier is a closed loop (see Figure 14). Studying the eigenvalue clustering around the lowest Landau level for less regular compactly supported perturbations would be a good starting point to get a better understanding of this case.

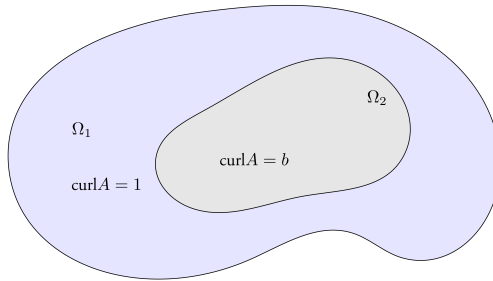


Figure 14: Example of the type of domain we are interested in studying where the region with constant magnetic field b is surrounded by the region with constant magnetic field 1.

6.4 More general inequalities between Dirichlet and Neumann eigenvalues of the Laplacian

In Paper IV we gave a proof of Friedlander's inequality (135)

$$\lambda_{j+1}(\Delta_N) \leq \lambda_j(\Delta_D)$$

for $j = 1, 2, \dots$. In a beautiful argument, Filonov[Filo4] was able to prove for $j = 1, 2, \dots$

$$\lambda_{j+1}(\Delta_N) < \lambda_j(\Delta_D), \tag{152}$$

with minimal regularity assumption on the domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$. Levine and Weinberger [LW86] proved

$$\lambda_{j+n}(\Delta) \leq \lambda_j(\Delta_D) \tag{153}$$

for $j = 1, 2, \dots$ and Ω being a convex bounded domain. In [Roh25], Rohleder proved (153) for $n = 2$ for simply connected domain with no convexity assumption. It was conjectured in [BLP09] that

$$\lambda_{j+n}(\Delta_N) < \lambda_j(\Delta_D) \tag{154}$$

holds for all domains $\Omega \subset \mathbb{R}^n$ with no convexity assumption. This conjecture was established using the unit ball as the edge case for this inequality. Further inspection of zeroes of the Bessel function indicates that the optimal shift could be a binomial or sum of binomial coefficients [CMS19]. The de Rham complex introduces large binomial coefficients, so the method introduced in Paper IV could be useful to get closer to the optimal shift.

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Scientific publications

Author contributions

Co-authors and supervisors are abbreviated as follows:

Magnus Fries (MF), Magnus Goffeng (MG), Ayman Kachmar (AK) and Mikael Persson Sundqvist (MPS).

Paper I: Non-monotonicity of the first eigenvalue for the 3D magnetic Robin Laplacian

I was the sole author of this paper. The problem was discussed with AK and I carried out the formulation and solution of the paper under the supervision of MPS.

Paper II: Discrete spectrum of the magnetic Laplacian on almost flat magnetic barriers

I was the sole author of this paper. The problem was discussed with AK and I carried out the formulation and solution of the paper under the supervision of MPS.

Paper III: The magnetic Laplacian on the disc for strong magnetic fields

AK and MPS suggested the problem. MPS offered guidance about the trial state to be used. I carried out the formulation and solution of the paper under the supervision of MPS. I wrote the first draft of the paper which was later modified by AK adding the appendix, which I also contributed to in later revisions.

Paper iv: Reproving Friedlander's inequality with the de Rham complex

I suggested the problem and provided the corresponding bibliography. In collaboration with MF and MG, I contributed to the formulation and solution of the problem as well as the writing. I contributed to the writing of all sections except section 3.



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