Non-binary and linear precoded faster-than-Nyquist signaling

Rusek, Fredrik; Anderson, John B

Published in:
IEEE Transactions on Communications

DOI:
10.1109/TCOMM.2008.060075

Published: 2008-01-01

Citation for published version (APA):

General rights
Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

• Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
• You may not further distribute the material or use it for any profit-making activity or commercial gain
• You may freely distribute the URL identifying the publication in the public portal

Take down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.
Non Binary and Precoded Faster Than Nyquist Signaling

Fredrik Rusek and John B. Anderson

Abstract—Faster than Nyquist (FTN) signaling is an important method of narrowband coding. The concept is extended here to non binary signal constellations; these are much more bandwidth efficient than binary ones. A powerful method of finding the minimum distance for binary and non binary FTN is presented. Precoding FTN transmissions with short linear filters proves to be an effective way to gain distance. A Shannon limit to bit error rate is derived that applies for FTN. Tests of an M-algorithm receiver are performed and compared to this limit.

Index Terms—Coded modulation, Mazo limit, faster than Nyquist, bandwidth efficient coding.

I. INTRODUCTION

The concept of Faster Than Nyquist (FTN) signaling is well established. If a pulse amplitude modulation (PAM) signal \( \sum a[n]v(t-nT) \) is based on an orthogonal pulse \( v(t) \), the pulses can be packed closer than the Nyquist rate \( 1/T \) without suffering any distance loss. In a bandpass system two quadrature signals can be used.

The result is a much more bandwidth-efficient coding system. Mazo showed [1] that for binary sinc pulses the symbol time can be reduced to \( 0.802T \) without suffering any loss in minimum Euclidean distance. We refer to this value as the Mazo limit. An introduction to the philosophy of Mazo signaling has been given in [2]. At first, FTN signaling seems to contradict the Nyquist limit and so it is useful to review how it works. Nyquist pulse signal design is based on orthogonality. There exist about \( 2WT \) orthogonal signals in \( W \) positive Hertz and \( T \) seconds. By means of filters matched to each one, data values that modulate each can be maximum likelihood detected independently, and therefore about \( 2WT \) symbols can be transmitted. If \( v = \sqrt{T/W}\text{sinc}(t/T) \) and there are \( N \) pulses each in the I and Q baseband channels, the product \( 2WT \) tends in ratio to \( 2(1/T)(NT) = 2N \). The sinc pulses thus carry as many symbols as any orthogonal pulse train can carry.

If the aim is to achieve asymptotically the same error rate, without necessarily using orthogonal pulses, then the sinc pulses can arrive faster than \( 1/T \). A more complex maximum likelihood sequence estimation (MLSE) receiver is required because of intersymbol interference. A similar phenomenon occurs for other \( T \)-orthogonal pulses, and the limits for root raised cosine (RC) pulses with excess bandwidth were derived in [3]. Efficient receivers for FTN signaling were presented there for the first time. Methods of computing the minimum distance of FTN signaling can be found in [4] and [5]. Mazo-type limits can be derived for pulse shapes that are not orthogonal for any \( T \) [6]. Mazo limit phenomena turn up in other places as well, for example, in constant-envelope coded modulation; see [7] and references therein. Precoding strategies for FTN were studied in [8] and [9].

The non binary case has not been studied as much, and its minimum distances are still an open problem. In this paper we develop an efficient method of finding minimum distances for non binary FTN. Distances for short (4–8 tap) optimal precoding filters with quaternary as well as binary FTN are also studied.

The paper is organized as follows. In section II we give the system model and in section III we derive the algorithm used to search for the minimum Euclidean distance. In section IV a method to find optimal precoding filters is presented. Numerical results and capacity calculations are given in section V and VI. Decoding is discussed in section VII.

II. SYSTEM MODEL

Consider a baseband PAM system based on a \( T \)-orthogonal pulse \( \psi(t) \). We are mostly interested in \( \psi(t) \) being a root RC pulse with excess bandwidth \( \alpha \). When \( \alpha = 0 \), \( \psi(t) \) is a sinc pulse. The one sided bandwidth of \( \psi(t) \) is \( W = (1+\alpha)/(2T) \). The transmitted signal is

\[
s_n(t) = \sum_{n=-\infty}^{\infty} a[n]\psi(t-n\tau T), \quad \tau \leq 1
\]

where \( a[n] \) are independent identically distributed data symbols and \( 1/\tau T \) is the signaling rate. We assume \( \psi(t) \) to be unit energy, i.e. \( \int_{-\infty}^{\infty} |\psi(t)|^2 dt = 1 \). For \( T \)-orthogonal pulses the system will not suffer from intersymbol interference (ISI) when \( \tau = 1 \). For \( \tau < 1 \) we say that we have FTN signaling, and ISI is unavoidable. The normalized bandwidth consumption is

\[
nbw = \tau T \frac{1+\alpha}{2T} \frac{1}{\log_2 M_a} \text{ Hz}/\text{bit}/s,
\]

where \( M_a \) is the data alphabet size.

The optimum receiver should filter the received signal \( s_n(t) + n(t) \), where \( n(t) \) is additive white Gaussian noise (AWGN) with spectral density \( N_0/2 \), with a filter matched to \( \psi(t) \) [3]. This should be followed by sampling every \( \tau T \)

Paper approved by H. Leib, the Editor for Communication and Information Theory of the IEEE Communications Society. Manuscript received February 8, 2006; revised November 23, 2006 and April 4, 2007. This work was supported in part by the Swedish Research Council (VFR), grant number 621-2003-3210.

The authors are with the Department of Information Technology, Lund University, Box 118 SE-221 00 Lund, Sweden (e-mail: {anderson, fredrik}@it.lth.se).

Digital Object Identifier 10.1109/TCOMM.2008.060075.
second and a decoding algorithm to mitigate the effects of the ISI. The system model is illustrated in figure 1. For MLSE reception, it can be shown that there exist constants $K_1$ and $K_2$ such that the probability of a symbol error $P_s$ can be bounded by [16]

$$K_1 Q\left(\sqrt{\frac{d_{\text{min}}^2 E_b}{N_0}}\right) \leq P_s \leq K_2 Q\left(\sqrt{\frac{d_{\text{min}}^2 E_b}{N_0}}\right).$$  (3)

These inequalities are tight for large $E_b/N_0$, and $d_{\text{min}}^2$ thus drives the asymptotic error probability and is a measure of a systems noise immunity. The square Euclidean distance, henceforth simply called “distance”, between the (real) data sequences $a$ and $a'$ is

$$d^2(a, a') = \frac{1}{2E_b} \int_{-\infty}^{\infty} |s_a(t) - s_{a'}(t)|^2 dt$$

$$= \frac{1}{2E_b} \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} (a[n] - a'[n])\psi(t - n\tau T)^2 dt$$

$$= \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \rho_{\psi}[n] e[n] = d^2(e),$$  (4)

where

$$\rho_{\psi}[n] = \int_{-\infty}^{\infty} \psi(t)\psi(t + n\tau T) dt$$  (5)

is the autocorrelation of the continuous pulse $\psi(t)$ at samples spaced $\tau T$ seconds, and $e[n] = (a[n] - a'[n])/\sqrt{2E_b}$. An important fact is that (4) takes the linear form

$$d^2(e) = \sum_{n=-\infty}^{\infty} r_e[n]^2$$  (6)

where

$$r_e[n] = \sum_{k=-\infty}^{\infty} e[n]e[n + k].$$  (7)

By concatenating an outer code to the FTN signals the $d_{\text{min}}^2$ here can be significantly increased, which will reduce BER. There may also be a bandwidth expansion, so it will be important to compare systems with similar bandwidth in what follows.

Mazo’s claim that it is possible to transmit at $0.802/\tau T$ for $\alpha = 0$ without distance loss was proven rigorously in [4]. The results for $\alpha > 0$, given in [3], were obtained by an exhaustive search out to 14 error symbols. For the nonbinary case little is known. Finding minimum distances by searching is very hard for large alphabet sizes since there are $|E|^L$ error events of length $L$ for the error symbol alphabet $E$. For 8 PAM, $|E| = 15$, and searching out to length 14 as in [3] gives $3 \times 10^{16}$ error sequences which is beyond our computation capability. Therefore symmetry properties of the error measures are important. In [8] the following hypothesis was stated: If $|\rho_{\psi}[1]| \gg |\rho_{\psi}[n]|, n > 1$ then the error event causing the minimum distance should be one where the symbols alternate in sign. Another hypothesis is that for low enough bandwidth the worst error event is a zero sum event, i.e. $\sum_{n} e[n] = 0$. Low enough bandwidth means typically $\text{nbw} < 0.2 \text{ Hz/bit/s}$; see [7] and references therein. Both these hypotheses heavily reduce the computation. However, we can never be sure that they are valid and in our numerical results we give an example where a search based on the first hypothesis does not produce the minimum distance of the system. We will show that the second assumption gives too small reduction to be really useful.

We therefore take a new look at the problem of efficiently finding the minimum distance for non binary alphabets. The notation used is as follows:

- $e$ discrete vector of symbols, with $n$th element $e[n]$
- $\rho_{\psi}[n]$ $\tau T$-sampled autocorrelation of a continuous pulse $\psi(t)$
- $g_b[n]$ autocorrelation of a discrete sequence $b[n]$
- $T_{\text{Nor}}$ truncation to the first $N + 1$ symbols of $e$
- $d_{\text{min}}^2(e)$ distance of $e$ calculated by (4) using $\rho_{\psi}[n]$
- $u \ast v$ convolution of $u$ and $v$
- $u^*[n]$ complex conjugate of $u[n]$
- $\text{supp}(e)$ support of $e$

### III. Finding the Minimum Distance

We start by describing a different but closely related problem; we will then transform our original ISI problem into the new one. Consider the finite causal ISI tap sequence $b[n]$. The transmitted signal for data symbols $a$ and generator sequence $b$ is

$$x_{a:b}[k] = \sum_{n=-\infty}^{\infty} a[n]b[k - n].$$  (8)

The distance between two data signals is

$$d^2(a, a') = \frac{1}{2E_b} \sum_{n=-\infty}^{\infty} |x_{a:b}[n] - x_{a':b}[n]|^2$$

$$= \sum_{m,n=-\infty}^{\infty} e[m]g_b[n - m]e[n] = d^2(e),$$  (9)

where

$$g_b[k] = \sum_{n=-\infty}^{\infty} b[n]b[n + k].$$  (10)

It can be shown [10] that the $Z$ transform of $g_b[k]$ can be written as

$$G_b(z) = c e^{\sum_{i=1}^{N_z} (1 - \zeta_i z^{-1})(1 - \zeta_i^* z)},$$  (11)

where $\zeta_i$ and $\zeta_i^*$ are the zeros of $G_b(z)$ and $c$ is a normalization constant. From (11) we see that it is always possible to construct

$$H(z) = c e^{\sum_{i=1}^{N_z} (1 - \zeta_i z^{-1})},$$  (12)
such that
\[ H^*(1/z^*) = c^* \prod_{i=1}^{N_z} (1 - \zeta_i^* z), \tag{13} \]
and
\[ G_b(z) = H(z)H^*(1/z^*). \tag{14} \]

Let \( H[n] \) be a sequence obtained by the inverse \( z \)-transform of \( H(z) \), i.e., \( H[n] = Z^{-1}\{H(z)\} \); note that since there is a great degree of freedom when constructing \( H \) we have in general \( b[n] \neq h[n] \). We say that \( h[n] \) and \( H(z) \) are obtained from the spectral factorization of \( G_b(z) \) \cite{10}. Since \( H^*[−n] = Z^{-1}\{H^*(1/z^*)\} \) it follows that
\[ g_b[k] = \sum_{n=-\infty}^{\infty} h[n]h[n + k]. \tag{15} \]

If \( H(z) \) is obtained from the factorization with \( |\zeta_i| \leq 1, \forall i \), the sequence obtained by the inverse \( z \)-transform is minimum phase and is denoted \( h_{\text{mp}}[n] \). The minimum distance of all \( h[n] \) including \( b[n] \) and \( h_{\text{mp}}[n] \) are equal since they have the same autocorrelation sequence. But more effective bounds will stem from \( h_{\text{mp}}[n] \) since it is minimum phase. Henceforth we construct all tap sequences such that they are minimum phase, and \( b[n] \) will be taken as the factorization \( h_{\text{mp}}[n] \).

An efficient branch and bound algorithm to find the minimum distance of an ISI sequence is given in \cite{14}. The algorithm works as follows. For a given error event \( e \), a lower bound on distance for all error events starting with the same symbols as \( e \) is found. This lower bound is then compared to an upper bound for \( d_{\text{min}}^2 \); when the lower bound is larger than the upper, the whole tree emanating from \( e \) is removed. The lower bound is lemma 1 below. This bound can be further sharpened but this is omitted here since we will eventually replace the lemma with another.

**Lemma 1:** Given a generator sequence \( h[n] \) and a particular error sequence \( e_b[n] \), if \( A_N(e_b) \) is the set of error sequences
\[ A_N(e_b) = \{ e : T_Ne = T_Ne_b \}, \tag{16} \]
then a lower bound for these is
\[ l_N^2(e_b) \defeq \sum_{n=0}^{N} |x_{e_b}h[n]|^2 \leq \min_e \{ d^2(e) \}. \tag{17} \]

The lemma implies that if \( l_N^2(e_b) \) is larger than some known upper bound \( d_{\text{ub}}^2 \) to \( d_{\text{min}}^2 \), then all events in the set \( A_N(e_b) \) can be eliminated from the search for \( d_{\text{min}}^2 \) as previously mentioned. Note that the distance of any error event gives an upper bound to \( d_{\text{min}}^2 \). Based on the sequence \( h[n] \) and lemma 1 it is straightforward to set up a branch and bound algorithm to solve for \( d_{\text{min}}^2 \). Any \( h[n] \) giving the same autocorrelation \( g \) may be used, but the minimum phase one will be most effective in curtailing the search.

We now return to our original problem: given an arbitrary time continuous pulse \( \psi(t) \), find the minimum distance \( d_{\text{min}}^2 \). From (6) we see that if the error event support is limited to \( L+1 \) error symbols then the distance of an event only depends on \( \{\rho_{\psi}[-L], \rho_{\psi}[-L+1], \ldots, \rho_{\psi}[L]\} \). If we only consider events of length \( L+1 \) we actually only find upper bounds to \( d_{\text{min}}^2 \). But if \( L \) is large, say 20 or so, we are confident that the result is valid. This is motivated by the fact that the \( d_{\text{min}}^2 \) achieving error events turned out to be much shorter than the search length \( (L+1) \) used in forthcoming sections. In the sequel we write \( d_{\text{ub}}^2 \) when we mean upper bound to \( d_{\text{min}}^2 \).

Since \( d_{\text{min}}^2 \) for \( \psi(t) \) only depends on a finite sequence of autocorrelation values there is in principle nothing that differs this problem from finding \( d_{\text{min}}^2 \) for a discrete tap sequence. We could try to find a sequence \( b[n] \) having an autocorrelation sequence equal to
\[ g_b[n] = \begin{cases} \rho_{\psi}[n], & n \leq L \\ 0, & \text{otherwise} \end{cases} \tag{18} \]

But this truncated \( g_b[n] \) is in general not a valid autocorrelation sequence and consequently no sequence \( b[n] \) need exist such that \( g_b[n] = b[n]\ast b[−n] \). Thus lemma 1 cannot be used.

Note that if we truncate the pulse \( \psi(t) \) to a certain length, the (finite) autocorrelation stemming from the truncation is valid; then the method to come is unnecessary. However, the obtained result is then only a (good) approximation. If we want to avoid truncations and seek distances for infinite pulse shapes we need the method below. Furthermore, using our approach the problem of escaped distance (see \cite{7}, chapter 6) is completely avoided.

The following lemma gives a sufficient and necessary condition for a sequence to be a valid autocorrelation sequence.

**Lemma 2:** A sequence \( g[n] \) with Hermitian symmetry is a valid autocorrelation sequence if and only if
\[ G(e^{i\omega}) = \sum_{k=-\infty}^{\infty} g[k]e^{-i\omega k} \geq 0, \text{ for all } \omega \in (−\pi, \pi). \tag{19} \]

Now let \( g_b \) be as in (18) and take
\[ \theta = \min_{\omega \in (−\pi, \pi)} G_b(e^{i\omega}). \tag{20} \]
The case \( \theta \geq 0 \) implies that a sequence \( b[n] \) can be found from \( g_b \) and consequently the algorithm in \cite{14} can be applied. Take \( \theta < 0 \) and define a new autocorrelation sequence \( g_{b'} \) such that
\[ g_{b'}[n] = \begin{cases} g_b[n] - \theta, & n = 0 \\ g_b[n], & n \neq 0. \end{cases} \tag{21} \]
From \( g_{b'}[n] \) it is now possible to obtain a sequence \( b'[n] \) through spectral factorization since \( G_{b'}(\omega) \geq 0, \omega \in (−\pi, \pi) \). However, the distance of an error event \( e[n] \) calculated using \( g_{b'} \) is not equal to the distance calculated using \( g_b \) (or \( \rho_{\psi} \)). In fact, from (6),
\[ d_{\text{ub}}^2(e) = d_{\text{ub}}^2(e) + \theta e[0]. \tag{22} \]

Due to (22) lemma 1 does not hold and consequently the algorithm to find \( d_{\text{min}}^2 \) must be modified. First let \( \epsilon \) denote the largest energy among the error symbols in \( E \), i.e.
\[ \epsilon = \max |e[n]|^2, \text{ } e[n] \in E. \tag{23} \]
We now initialize every error event with the “distance” \( \epsilon(L+1)\theta \). This corresponds to the worst case of \( \theta e[0] \) in (22) for a given search length \( L \). We can modify lemma 1 into

**Lemma 3:** Let the set \( A_N(e_b) = \{ e : T_Ne = T_Ne_b, \text{supp}(e) \leq L+1 \}, \) with \( N \leq L \). For \( g_b \) as in (18)
and a particular error sequence \( e_s[n] \), a lower bound for set \( \mathcal{A}_N(e_s) \) is

\[
\lambda^2_N(e_s) = \sum_{n=0}^{N} |x_{e_s}[b'][n]|^2 + \epsilon(L + 1)\theta + \min_{\epsilon \in \mathcal{A}_N(e_s)} \{d^2_{gb}(e)\}.
\]

Proof: According to (22) we can write \( d^2_{gb}(e) \) as

\[
d^2_{gb}(e) = \sum_{n=0}^{N} |x_{e}[b'][n]|^2 + \sum_{n=N+1}^{\infty} |x_{e}[b'][n]|^2 + \theta \sum_{n=0}^{L} e^2[n] + \sum_{n=0}^{\infty} \epsilon^2[n] - \epsilon \theta \leq \lambda^2_N(e_s) + \sum_{n=N+1}^{\infty} |x_{e}[b'][n]|^2 + \sum_{n=N+1}^{\infty} \epsilon^2[n] - \epsilon \theta.
\]

Since \( \epsilon \geq e^2[n], \theta \leq 0 \) implies that the last term of (25) is always nonnegative, \( |x_{e}[b'][n]|^2 \) being nonnegative, we have \( \lambda^2_N(e_s) \leq d^2_{gb}(e) \). All sequences in \( \mathcal{A}_N(e_s) \) have the same first \( N + 1 \) components as \( e_s \) and we have for all \( e \in \mathcal{A}_N(e_s) \)

\[
\lambda^2_N(e_s) = \lambda^2_N(e_s) \leq d^2_{gb}(e),
\]

which especially implies that \( \lambda^2_N(e_s) \leq \min_{e \in \mathcal{A}_N(e_s)} \{d^2_{gb}(e)\} \) and the proof is complete.

Now we further sharpen lemma 3. Let

\[
d^2_{\text{min},gb} = \min_{e : \supp(e) \leq n} \{d^2_{gb}(e)\}, \quad n \text{ integer.}
\]

We can then prove the following lemma:

**Lemma 4**: Define the set \( \mathcal{B}_N(e_s) = \{e : T_N(e) = T_N e_s, T_N e \neq e, \supp(e) \leq L + 1\} \), with \( N \leq L \).

Let \( \Delta_N(e) = d^2_{gb}(T_N e) - \lambda^2_N(e) - (L - N)\theta \). Then for \( e \in \mathcal{B}_N(e_s) \) and \( \Delta_N(e) < d_{\text{min},gb}[L + 1 - N] \) we have

\[
d^2_{gb}(e) \geq d^2_{\text{min},gb}[L + 1 - N] + d_{\text{min},gb}[L + 1 - N] \Delta_N(e) + d^2_{gb}(T_N e) - (L - N)\theta \epsilon.
\]

And for the case \( \Delta_N(e) \geq d_{\text{min},gb}[L + 1 - N] \) we have

\[
d^2_{gb}(e) \geq \lambda^2_N(e_s).
\]

This lemma is proved in appendix B. The lemma is a modification of a lemma in [15].

It is now straightforward to set up a branch and bound algorithm to find the minimum distance of the system. For lemma 4 to be useful, it should be easy to find \( d^2_{\text{min},gb}[n] \) compared to \( d^2_{\text{min},gb}[n] \). From our experience this is always the case; since \( g_b \) is a valid autocorrelation sequence the search method in [14] can be applied. The root node (depth 0), should be initialized by \( \epsilon(L + 1)\theta \) and the branch metric at depth \( k \) is

\[
|x_{e}[b'][k]|^2 + (\epsilon^2[k] - \epsilon)\theta.
\]

If the expressions for \( d^2_{\text{min},gb}(e_s) \) in lemma 4 are larger at some node than an upper bound to \( d^2_{\text{min}} \), the entire tree beyond that node can be removed from the search.

**IV. PRECODED FASTER THAN NYQUIST SIGNALING**

In this section we improve \( d^2_{\text{min}} \) by precoding the input data sequence. In the literature there is a rich variety of coding/precoding methods for ISI and partial response signaling (PRS) channels; see [11]–[13] and references therein. Most of these methods work by applying some sort of rate decreasing outer code and possibly a precoder; sometimes the precoder is a linear filter. In [3] constraint coding was used to increase the distance, but the systems there could never achieve the full antipodal distance of an uncoded system.

The task of the precoder is usually to ease the decoding burden at the expense of the bit error rate (BER); examples are the famous Tomlinson-Harashima precoder and the Laroia-Trettter-Farvardin precoder from [12]. We will in a sense do the opposite, use a rate 1 PRS precoder in order to improve the BER, at the cost of decoding complexity. This strategy was used in [8], with an argument tracing back to Forney [16], but the results are apparently incorrect. A method of designing optimal linear filters with respect to \( d^2_{\text{min}} \) is given in [14]. The scope of that paper was to design optimal PRS codes based on orthogonal interpolation pulses \( \psi(t) \). A technique called partial spectrum mapping was used in [15] for designing bandlimited filters; this technique is essentially what we make use of next. But since the algorithm in section 3 was unknown it was not possible to find \( d^2_{\text{min}} \) and it could only be estimated by means of a heavy search. Furthermore, how to implement the pulses obtained via partial spectral mapping was not described; there was no concept of an underlying pulse form.

Assume that the input data sequence is convolved with a finite tap sequence \( b[n] \). Alternatively this can be seen as a linear modulation of the data sequence with a different pulse \( \phi(t) \); this is illustrated in figure 2. The transmitted signal is

\[
d[n] = \sum_{\phi[n]} b[n] \phi(t) n(t) \psi(t) = r(t) \psi(t) = t = n \tau T
\]

![Fig. 2. System model for precoded FTN signaling. The input data symbols are spaced \( \tau T \) seconds apart.](image)

1Our outcomes differ from [8] in a number of ways in the sequel, most often because the error events explored in [8] were too short. Some brief examples are as follows. (i) The distance of 0.90235 marked by an asterisk in table III stems from the error sequence 2,−2,0, repeated 15 times, an event of length 44; [8] suggests 0.9778, which stems from a shorter event. (ii) Combining the dibinary pulse (37) having \( \rho = 0.65 \) with the 2-tap prefilter 1.4302, −1.4302 leads not to signals with \( \rho^2 = 1 \), but to \( \rho^2 = 0.3665 \); it is achieved by the error event 2,−2,0,0,0,2,−2,0,0,0,0,−2,2. Furthermore, for optimum 2-tap prefilters, \( d^2_{\text{min}} \) cannot equal 2 for any \( \rho \leq 0.867 \). (iii) The paper proposes another system in which the transfer function is \( H(f) = \sqrt{\pi T/2f} \cos(\pi T f/\rho) \); \( |f| < \rho/(2T) \), \( \rho \leq 1 \). It is claimed that \( \rho = 0.60 \) and the 3-tap prefilter 1.3727, −1.3727, 1.3727 lead to \( d^2_{\text{min}} = 2 \), but we find that \( d^2_{\text{min}} = 0.7027 \), achieved by the event 2,−2,2,−2,0,0,2,−2,2,0,0,−2,2. Furthermore, for the best 3-tap prefilters, \( d^2_{\text{min}} \) cannot equal 2 for \( \rho \leq 0.775 \).
TABLE I
THE OBTAINED MINIMUM DISTANCES FOR 4 AND 8 PAM. FOR 4 PAM THE
SEARCH LENGTH L WAS 25 FOR ALL α. FOR 8 PAM L WAS 20 FOR ALL α.

<table>
<thead>
<tr>
<th>α \ M_a</th>
<th>10%</th>
<th>20%</th>
<th>30%</th>
</tr>
</thead>
<tbody>
<tr>
<td>τ \ M_a</td>
<td>4</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>.8</td>
<td>.45</td>
<td>.27</td>
<td>.45</td>
</tr>
<tr>
<td>.75</td>
<td>.708</td>
<td>.204</td>
<td>.45</td>
</tr>
<tr>
<td>.70</td>
<td>.593</td>
<td>.131</td>
<td>.677</td>
</tr>
<tr>
<td>.65</td>
<td>.558</td>
<td>.0885</td>
<td>.547</td>
</tr>
<tr>
<td>60</td>
<td>.151</td>
<td>.0479</td>
<td>.254</td>
</tr>
<tr>
<td>.55</td>
<td>.129</td>
<td>.151</td>
<td>.0393</td>
</tr>
<tr>
<td>.50</td>
<td>.131</td>
<td>.0212</td>
<td>.147</td>
</tr>
<tr>
<td>.45</td>
<td>.102</td>
<td>.128</td>
<td>.0160</td>
</tr>
</tbody>
</table>

given by

\[ s_a(t) = \sum_{n=-\infty}^{\infty} a[n] \phi(t - n\tau T), \quad (31) \]

where \( \phi(t) \) is

\[ \phi(t) = \sum_{n=0}^{L_b-1} b[n] \psi(t - n\tau T), \quad (32) \]

and \( L_b \) is the support of \( b[n] \). It can be shown that the
autocorrelation \( \rho_\phi[n] \) is

\[ \rho_\phi[n] = g_b[n] * \rho_\psi[n]. \quad (33) \]

It can be shown that the distance of an error event is

\[ d^2(e) = \sum_{n=1-L_b}^{L_b-1} g_b[n] \mu_e[n], \quad (34) \]

where

\[ \mu_e[n] = \sum_{k=1-L_b}^{\infty} r_e[k] \rho_\psi[k - n], \quad (35) \]

with \( r_e[k] \) defined in (7). The energy of \( \phi(t) \) should equal 1; this is equivalent to \( \rho_\phi[0] = 1 \), or from (33)

\[ \sum_{k=1-L_b}^{L_b-1} g_b[k] \rho_\psi[k] = 1. \quad (36) \]

We now have linear equations both for distance (34) and energy normalization (36). Together with the linear condition given in lemma 2 we can solve for the optimal sequence \( b[n] \) for a given pulse \( \psi(t) \). For the procedure used we refer to [14]. To find \( d^2_{\min} \) we use the branch and bound algorithm in section 3.

V. NUMERICAL RESULTS

We start with \( d^2_{\min} \) for the uncoded root RC FTN case in
TABLE I. We present results for 4 and 8 PAM and \( \alpha = 10, 20 \)
and 30%. Recall that the matched filter bounds are .8 and 2/7
for 4 and 8 PAM. The Mazo limits, i.e. the \( \tau \) where \( d^2_{\min} \) falls
below the matched filter bound for the first time, are the same
for 2,4 and 8 PAM and are \( \tau = .7032 \) for \( \alpha = 30\% \).

A comparison between 2,4 and 8 PAM is shown in figure 3
for \( \alpha = 30\% \). It is seen that there is an optimal alphabet size
for each bandwidth. For example, at nbw .015 Hz/bit/s there
is roughly 3 dB gain by using 8 PAM instead of 4 PAM. A
similar result for Butterworth pulses is reported in [6].

We give some results for the precoded FTN signaling next.
We only consider only \( \alpha = 30\% \). Results for 2 PAM with
\( L_b = 4, 6, 8 \) and 4 PAM with \( L_b = 4, 6 \) are given in table II.
Especially note the 4.8 dB gain by using a precoding filter of support 6 for 4 PAM and \( \tau = .55 \).

We also found \( d^2_{\min} \) for frequency scaled versions of the
dubinary pulse, as proposed for binary transmission in [8].
The transfer function of the dubinary pulse is

\[ |\Psi(f)| = \sqrt{2T/\rho \cos(\pi Tf/\rho)}, \quad |f| < (\rho/2T), \quad \rho \leq 1. \quad (37) \]

The normalized bandwidth is nbw = \( \rho/2 \log_2 M_a \) Hz/bit/s.
Results are given in table III. Some of these differ from [8].

To see the strength of the \( d^2_{\min} \) algorithm we compare the
effort of finding \( d^2_{\min} \) for 8 PAM, \( \alpha = 10\% \), \( \tau = .60 \) with an
exhaustive search. We searched over all events with support
\( \leq 20 \). For an exhaustive search this implies testing \( 15^{20} \approx
3.33 \times 10^{23} \) events but our algorithm only considered \( \approx 6 \times
10^7 \). Using generating functions one can show that the number
events fulfilling the zero sum assumption for \( M_a \)
in which \( P \) is now \( \int |H(f)|^2 \, df \). Some calculations with (38) show that the stopband of \( H(f) \) has a major effect on the capacity of narrowband signaling, even though its power is small.

A Shannon limit to BER gives the lowest BER of coding schemes with a given PSD, as a function of \( E_b/N_0 \). It is derived as follows. Consider a coded modulation with PSD \( |H(f)|^2 \) that carries \( R_{\text{ber}} \) binary data bits/s. If \( R_{\text{ber}} \geq C_H \), standard rate-distortion theory tells us that \( R_{\text{ber}} \) can be compressed to \( C_H \) in the ratio

\[
\frac{R_{\text{ber}}}{C_H} = 1 - h_B(\beta),
\]

where \( \beta \) is the resulting error rate of the compression and \( h_B(\beta) \) is the binary entropy function. The channel carries the compressed data nearly perfectly at rate \( C_H \). We now fix the system rate \( R_{\text{ber}} \) and scale \( |H(f)|^2 \) by a parameter \( \gamma > 0 \). This scales \( E_b/N_0 = P/N_0 R_{\text{ber}} \) to \( \gamma E_b/N_0 \), and eqs. (38)–(39) yield a BER \( \beta \) for the new \( \gamma E_b/N_0 \). We thus obtain a relationship between \( \beta \) and \( E_b/N_0 \) that is parameterized in \( \gamma \). The highest allowed \( E_b/N_0 \) is the one for which \( C_H \) is the given \( R_{\text{ber}} \).

The Shannon BER limit is the ultimate limit for any coding scheme having PSD \( |H(f)|^2 \), but some coded modulations may be bounded away from this limit. Consider TCM, multilevel coding and many other types of coded modulation, in which the signals have the form \( s(t) = \sum a_n v(t - nT) \) with \( T \)-orthogonal \( v(t) \). The \( \{a_n\} \) can be thought of as real valued code letters. If the \( \{a_n\} \) are uncorrelated and zero mean, the PSD of \( s(t) \) has the same shape as \( |V(f)|^2 \). As shown by Nyquist, useful orthogonal pulse PSDs obey a symmetry condition about the frequency \( W = 1/2T \). The most common pulse is the root RC, with nominal passband \([0, W] \) Hz and stopband \([W, (1 + \alpha)/2T] \). The most narrowband orthogonal pulse is \( \text{sinc}(t/T) \), with flat PSD and \( \alpha = 0 \).

It can be shown that \( C_V \) in (38) always increases compared to the brickwall \( C_W \) when a pulse with Nyquist’s symmetry and \( \alpha > 0 \) is substituted for \( \text{sinc}(t/T) \). This is the fundamental reason why orthogonal-pulse schemes can be bounded away from their limit: The BER of these does not depend on \( \alpha \) but only on the fact that \( v(t) \) is orthogonal; in particular, \( \alpha \) can be zero, so these schemes must achieve a Shannon BER limit derived from the brickwall capacity \( C_V \). The general BER limit derived from a non-sinc \( |V(f)|^2 \) via eqs. (38)–(39) must lie strictly to the left in a plot of BER vs. \( E_b/N_0 \) like those in the next section. This opens an avenue for FTN schemes to perform better than orthogonal ones.

### VI. CAPACITY

In this section we derive the capacity of schemes like FTN and the Shannon bound to bit error rate for them. In section VII the BER bound for FTN will be compared to both actual decoding performance and to BER bounds for trellis-coded modulation (TCM) schemes based on RC pulses. The capacity calculation has appeared in the literature (see e.g. [21]) but the BER bound has apparently not. It is an important tool in the evaluation of FTN-like coding schemes, since it includes the effect of both energy and spectral density and it directly relates to an easily measured quantity.

According to classical Shannon theory, signals with \( W \) positive hertz, uniform power spectral density (PSD) and \( P \) Watts have capacity \( C_W = W \log_2[1 + P/N_0 W] \), where \( N_0/2 \) is the noise density. Elementary calculus extends this brickwall result to signals with an arbitrary PSD \( |H(f)|^2 \); the outcome is

\[
C_H = \int_0^\infty \log_2[1 + \frac{2 |H(f)|^2}{N_0}] \, df \quad \text{(bit/s)} \tag{38}
\]

2An example is uncoded binary transmission with \( \alpha = 30\% \), \( \tau = .45 \). Then \( \rho_0[1] = .6868 \) and \( \max_{n>1} |\rho_0[n]| = 1.796 \) so the conditions in the hypotheses are fulfilled, but the worst event is \( 2, -2, 0, 2, -2, 0, \ldots \).
equalization and nonlinear methods such as straightforward decision feedback are ordinarily ruled out.

Over the past 30 years much research has gone into finding good RSE strategies for the AWGN channel; see [17], [18] and references therein. For example a common strategy is reduced state sequence estimation (RSSE); this method works with a considerably smaller trellis than the original and obtains close to optimal performance as $E_b/N_0$ increases.

An efficient receiver structure well suited to binary FTN was recently proposed in [3]; the structure was based on the Ungerboeck observation model [19]. This structure could be generalized to $M_a$-ary signaling. But the first part of the receiver is a soft output truncated Viterbi algorithm [20], whose complexity grows as $M_a^{L_v}$ where $L_v$ denotes the truncation length of the ISI. Since we have precoded signals and significant FTN complexity, a large $L_v$ is probably needed in order to avoid too much residual ISI; therefore we believe that this receiver is in general too complex for quaternary and octal signaling with precoders as long as 4–6 taps.

In this paper a different strategy is tested, the simple $M$-algorithm. If the Ungerboeck model is used the $M$-algorithm is observed to work badly for large alphabets, such as 8 PAM. Therefore the whitened matched filter model (WMF) model [16] is assumed. Similar to [3] we found it hard to work with the WMF model when the impulse response of the root RC pulse is long, e.g. $80T$. Therefore we have done the following: a front end whitened matched receiver filter was determined for root RC pulses of length $20T$; all receiver tests are done with pulses of length $80T$ but the receiver filter is for the $20T$ pulse. This of course makes the decoder mismatched, but the noise variance emanating from the mismatch is small compared to the AWGN variance. We now have to decode backwards, which might be a drawback, since the decoding cannot start until the whole block has arrived. If the receiver filter is set up for forward decoding, it is no longer stable; this is due to the mismatching of the filters.

This receiver is simple and shows good performance. At each level in the trellis the $M$-algorithm keeps only the most promising paths. As usual the symbols are released with some delay (the decision depth); see [22] for a study of decision depths for ISI channels.

We have performed receiver tests for both 4- and 8PAM for uncoded, precoded and convolutionally encoded FTN systems; the tests are shown in figures 4–6. By convolutionally encoded FTN signaling we mean a scheme where the first part of the information sequence is the convolutionally encoded FTN, then the latter part is the uncoded, precoded or convolutionally encoded FTN signaling. The minimum distance of the convolutionally encoded scheme is 

$$d_{\min}^2 = 2\alpha K_1$$

where $a = 1 + 0.3/2, 55/3 = 0.1192$ Hz/bit/s. The $Q$-function estimates lie 1–2 dB to the left of the test results. This reference is based solely on $d_{\min}^2$ for 4PAM and only applies asymptotically. For 8PAM we included the multiplicity $K_1$, see eq. (3), of the $d_{\min}^2$. This reference becomes tight.

Figure 5 shows coding systems at rate 2 bits/channel use. An uncoded 4PAM system (decoder $M = 8$) is compared to a $(23,40)$ convolutionally encoded 8PAM system (decoder $M = 8$). The actual precoder used is

$$\{2.2418, -3.6121, 4.2673, -3.5962, 2.3436, -0.9159\}.$$
M = 80). Signal parameters are $\tau = 0.7$ and $\alpha = 0.3$. The convolutionally coded system is about 2 dB better at all $E_b/N_0$ tested, although it requires a decoder with 10 times the complexity. The figure shows the $d_{\min}$ reference for the uncoded 4PAM system, and it is a tight estimate at high $E_b/N_0$. The figure also shows Shannon BER limits both for the FTN pulse PSD and for 30% root RC-based TCM systems.

Figure 6 shows an uncoded rate 3 system with the same signal parameters as Figure 5. Once again, performance is compared to the $d_{\min}$ reference and the two Shannon BER limits. Agreement with the reference is again good. In both figures 5 and 6 we see that the uncoded FTN signaling lies roughly 6 dB from its Shannon limit at BER $10^{-5}$ but only 3–4 dB from the Shannon limit for competing methods such as TCM and multilevel coding. The convolutionally encoded scheme (Figure 5) actually gets within 1.5 dB of the competing method Shannon limit at BER $10^{-5}$.

VIII. CONCLUSIONS

We have proposed a powerful algorithm to search for the minimum distance of nonbinary FTN signaling. We are capable of searching out to a very large search length even for large alphabets. The algorithm is the extension to infinite time signals of an existing algorithm proposed for discrete signals in [15]. Furthermore, the method can be used for a general ISI signal, and no assumptions on the worst case error events are made, except for length. We also found the optimal precoding filter for binary and quaternary FTN transmissions; the distance algorithm was a crucial tool here. M-algorithm receiver tests give a reasonable verification of these distances. A major reason for the improved performance of FTN systems is their more favorable Shannon limit. In a future paper we will present a careful study of the FTN Shannon capacity.

APPENDIX: PROOF OF LEMMA 4

The proof of lemma 4 is a modification of the proof of lemma 7.4.2 in [15]. The modification is due to the extra term in (22). The proof requires lemma 5 below; except for notation, lemma 5 is identical to lemma 7.4.1 in [15] and therefore we give it without proof. Let $x_{e|b'}[n]$ be an error signal generated according to (8). Decompose $x_{e|b'}[n]$ into two parts

$$\bar{x}_{e|b'}[n, N] \triangleq \sum_{n=0}^{N} e[k]b'[n-k]$$

$$\hat{x}_{e|b'}[n, N] \triangleq \sum_{n=N+1}^{\infty} e[k]b'[n-k]$$

(40)

Then lemma 7.4.1 in [15] with our notation reads

**Lemma 5**: If $e_r$ is an error sequence such that $e_r \neq T_N e_r$ then

$$\sum_{N+1}^{\infty} |\hat{x}_{e|b'}[n, N]|^2 \geq d_{\min, g_{\theta'}}^2 [L+1-N]$$

(41)

We can now prove lemma 4.

**Proof of lemma 4.** According to (22) we can write $d_{gb}^2 (e_r)$ as shown in (42). By using $T_N e_r$ instead of $e_r$ in (42) we obtain

$$d_{gb}^2 (T_N e_r) = \lambda_N^2 (e_r) + \sum_{n=N+1}^{\infty} |\bar{x}_{e|b'}[n, N]|^2 + (L-N) \theta e$$

(43)

Following [15] we conclude as shown in (44). Instead of finding the minimum in (44) we lower bound it as shown in (45). The last inequality follows from the fact that there is more freedom to choose $z[n]$, than $\hat{x}_{e|b'}[n, N]$ in the minimization. That $||z|| \geq d_{\min, g_{\theta'}}[L+1-N]$ is clear from lemma 5. The minimization (45) is easy to solve; there are two types of solution depending whether the difference

$$\Delta_N^2 (e_r) \triangleq d_{gb}^2 (T_N e_r) - \lambda_N^2 (e_r) - (L-N) \theta e$$

$$= \sum_{n=N+1}^{\infty} |\bar{x}_{e|b'}[n, N]|^2$$

(46)

is larger or smaller than $d_{\min, g_{\theta'}}[L+1-N]$. The solution is

$$z[n] = \begin{cases} \max \{1, d_{\min, g_{\theta'}}[L+1-N]/\Delta_N (e_r)\} \hat{x}_{e|b'}[n, N], & n > N \end{cases}$$

(47)

The value of this solution is shown in (48). Inserting (48) into (44) gives (49), and the lemma is proved.

REFERENCES

\[ d_{gb}^2(e) = \sum_{n=0}^{N} |x_e[n]|^2 + \sum_{n=N+1}^{\infty} |x_e[n] + \hat{x}_{e|=}\hat{b}|^2 + \theta \sum_{n=0}^{L} e^2[n] \]

\[ = \lambda_N^2(e) + \sum_{n=N+1}^{\infty} |x_e[n] + \hat{x}_{e|=}\hat{b}|^2 + \sum_{n=N+1}^{L} (\hat{e}^2[n] - \epsilon \theta) \]  

(42)

\[ \min_{e \in B_N(e_r)} \{ d_{gb}^2(e) \} = \lambda_N^2(e) + \min_{e \in B_N(e_r)} \left\{ \sum_{n=N+1}^{\infty} |x_e[n] + \hat{x}_{e|=}\hat{b}|^2 + \sum_{n=N+1}^{L} (\hat{e}^2[n] - \epsilon \theta) \right\} \]

(43)

\[ \geq \min_{e \in B_N(e_r)} \left\{ \sum_{n=N+1}^{\infty} |x_e[n] + \hat{x}_{e|=}\hat{b}|^2 \right\} + \min_{e \in B_N(e_r)} \left\{ \sum_{n=N+1}^{L} (\hat{e}^2[n] - \epsilon \theta) \right\} \]

(45)

\[ \min_{\|x\| \geq d_{\min,gb}[L+1-N]} \left\{ \sum_{n=N+1}^{\infty} |x_e[n] + z[n]|^2 \right\} = (\max\{0, d_{\min,gb}[L+1-N] - \Delta_N(e_r)\})^2 \]

(48)

\[ d_{gb}^2(e) \geq \left\{ \begin{array}{ll}
\lambda_N^2(e), \\
\min_{\Delta_N(e) \geq d_{\min,gb}[L+1-N]} \left[ d_{gb}^2(T_N e) - (L-N) \right] - d_{gb}^2(e) + \Delta_N(e) \end{array} \right. \]

(49)


Fredrik Rusek was born in Lund, Sweden in 1978. He received the M.S. degree in electrical engineering in 2002 from Lund Institute of Technology in Sweden. From 2003 he is with the Department of Electrical and Information Technology at Lund Institute of Technology, where he is pursuing the Ph.D. degree. He will defend his Ph.D. thesis September 24, 2007. His research interests include modulation theory, coding theory, wireless communications and applied information theory.
John B. Anderson was born in New York State in 1945. He received the B.S., M.S. and Ph.D. degrees in electrical engineering from Cornell University in 1967, 1969 and 1972. During 1972-80 he was on the faculty of the Electrical and Computer Engineering Dept. at McMaster University in Canada, and during 1981-98 he was Professor in the Electrical, Computer and Systems Engineering Dept. at Rensselaer Polytechnic Institute. Since 1998 he has held the Ericsson Chair in Digital Communication at Lund Univ., Sweden. He has held visiting professorships at the Univ. of Calif., Berkeley (1978-79), Chalmers Univ., Sweden (1987), Queen’s Univ., Canada (1987), Deutsche Luft- und Raumfahrt, Germany (1991-92, 1995-96) and Tech. Univ. of Munich (1995-96). His research work is in coding and communication algorithms, bandwidth-efficient coding, and the application of these to data transmission and compression. He has served widely as a consultant in these fields. Presently, he is Director of the Swedish Strategic Research Foundation Center for High Speed Wireless Communication at Lund.

Dr. Anderson was a member of the IEEE Information Theory Society Board of Governors during 1980-87 and 2001-06, serving as the Society’s Vice-President (1983-84) and President (1985). In 1983 and 2006 he was Co-Chair of the IEEE International Symposium on Information Theory. He served during the 1990s as chair of Research Initiation Grants for the IEEE Foundation. In the IEEE publications sphere, he served on the Publications Board of IEEE during 1989-91 and 1994-96. He was a member of the IEEE Press Board during 1993-2006 and during 1994-96 was Editor-in-Chief of the Press. Since 1998 he has edited the IEEE Press book Series on Digital and Mobile Communication. He has also served as Associate Editor of the IEEE Transactions on Information Theory (1980-84) and as Guest Editor of the IEEE Transactions on Communications several occasions.

Dr. Anderson is author or coauthor of six textbooks, including most recently Digital Transmission Engineering (IEEE Press, 2nd ed. 2005), Coded Modulation Systems (Plenum/Springer 2003), and Understanding Information Theory (IEEE Press 2005). He is Fellow of the IEEE (1987) and received the Humboldt Research Prize (Germany) in 1991. In 1996 he was elected Swedish National Visiting Chair in Information Technology. He received the IEEE Third Millennium Medal in 2000.