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DATA MINING APPROACHES IN MATHEMATICS EDUCATION

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The aim of this study is to introduce data mining, which is a data analysis methodology that has been successfully used in different areas including the educational domain. It has been begun to be used in education recently and is quite new in mathematics education. However, educational data mining (EDM) literature has shown that it can represent new and significant contributions to researches. This paper initiates discussion on the use of data mining in mathematics education.

Keywords: Educational data mining

INTRODUCTION

There are a lot of definitions for data mining in literature such as 'data analysis methodology used to identify hidden patterns in a large data set' (Tiwari & Vimal, 2013), 'the process that analyzes the data from different points of view and summarizes the results as useful information' (Şuşnea, 2009), 'a technology used to describe knowledge discovery and to search for significant relationships such as patterns, association and changes among variables in databases' (Pal, 2012). In brief, data mining can be defined as applications of different algorithms to identify patterns and relationships in a data set.

Data mining is similar to mining to obtain ore from the sand. Namely, it can be considered that sand is data and ore is knowledge. Although it should be defined as 'knowledge mining', it is defined as 'data mining' to emphasize large amounts of data.

Data mining is a process that minimally has four stages (Nisbet, Elder & Miner, 2009): (1) data preparation that may involve 'data cleaning' and 'data transformation', (2) initial preparation of the data, (3) model building or pattern identification, and (4) deployment, which means subjecting new data to the 'model' to predict outcomes of cases found in the new data.

Data mining techniques can be classified as below:

1. Clustering: a process of grouping physical or abstract objects into classes of similar objects (Romero &Ventura, 2007). Clustering is a type of analysis that divides data (cases or variables, depending on how specified) into groups such that members of each groups are as close as possible to each other, while different groups are as far apart from each other as possible (Nisbet et al, 2009).

2. Classification and regression (decision tree, neural network etc.): In classification, the predicted variable is a binary or categorical variable. Some popular classification methods include decision trees, logistic regression and support vector machines. In regression, the predicted variable is a continuous variable. Some popular regression methods within educational data mining include linear regression, neural networks, and support vector machine regression. Classification techniques like decision trees, Bayesian networks etc can be used to predict the student's behavior in an educational environment, his interest towards a subject or his outcome in the examination (Kumar &Vijayalakshmi, 2011). Classification techniques are predictive models. And predictive modelling compares the students behaviour with past similar students behaviours to predict what she will do in order torecommend how to proceed (Lee, 2007)

3. Association rules: associates one or more attributes of a dataset with another attribute, producing an if-then statement concerning attribute values (Romero &Ventura, 2007). Association rules are characteristic rules (it describes current situation), but classification rules are prediction rules for describing future situation (Tiwari, Singh &Vimal, 2013). Association Rule mining can be used in various areas of education data to bring out the interesting rules about the learner's records. It can be used to bring out the hidden facts in understanding the behaviour of the learner in a learning environment, learning style, examination pattern and assessment etc.. These rules can be utilised by the educator to understand the need of the learner and improve the learning skills (Kumar &Vijayalakshmi, 2013).

Data mining has been used in different areas such as Marketing, Banking, Insurance, Telecommunication, Health and Medicine, Industry, Internet, Science and Engineering etc. Recently, one of these areas is educational environment. As a result of application of data mining techniques in education, educational data mining (EDM) field has emerged.

Educational Data Mining is defined as 'an emerging discipline, concerned with developing methods for exploring the unique types of data that come from educational settings, and using those methods to better understand students, and the settings in which they learn' by International Educational Data Mining Society (http://www.educationaldatamining.org).

Data mining has attracted a great deal of attention in the information industry and in society as a whole in recent years, due to the wide availability of huge amounts of data and the imminent need for turning such data into useful information and knowledge (Han & Kamber, 2006). Education sector also has huge amounts of data and needs such techniques. EDM is an emergent discipline on the intersection of data mining and pedagogy. On the one hand, pedagogy contributes to the intrinsic knowledge of learning process. On the other hand, data mining adds the analysis and information modelling techniques (Kumar & Vijayalakshmi, 2011). Many educators

and scholars have begun to pay more attention to applying data mining techniques to educational data.

Three objectives could be identified to use EDM as a technology in the field of education. One of them is pedagogic objectives -to help the students to improve in academics, designing the content of the course in a better way etc. (Kumar & Vijayalakshmi, 2011).

Romero and Ventura (2007:136) summarized a role of data mining in education sector quite understandably:

'The application of knowledge extraction techniques to educational systems in order to improve learning can be viewed as a formative evaluation technique. Formative evaluation (Arruabarrena, Pe´rez, Lo´pez-Cuadrado, & Vadillo, 2002) is the evaluation of an educational program while it is still in development, and for the purpose of continually improving the program. Data mining techniques can discover useful information that can be used in formative evaluation to assist educators establish a pedagogical basis for decisions when designing or modifying an environment or teaching approach'

Compared to traditional statistical studies, data mining can (1) provide a more complete understanding of data by finding patterns previously not seen and (2) make models that predict, thus enabling people to make better decisions, take action, and therefore mold future events (Nisbet, Elder & Miner, 2009).

Data mining performs two functions: one is to identify regularities among data records (e.g., concept cluster, concept comparison, and discrimination), another to find relations among variables in the data that will predict unknown or future values of the variables. Unlike descriptive and inferential statistical analyses that rely on means and standard deviations, data mining uses both logical and mathematical (deterministic, and parametric and nonparametric statistical) reasoning to analyze data records (Liu & Ruiz, 2008)

Following problematic situations and convenience data mining techniques can be example to use data mining in education:

- Determining which factors have effect on misconceptions encountered in especially qualitative researches (classification techniques or association rules)

- Determining which misconceptions or mistakes occurr together. (Association rules)

- Determining factors which are important to form compatible groups for collaborative learning (classification techniques or clustering)

- Determining factors that affect mathematical achievement (classification techniques)

- Predicting students final performance at the beginning of the year and taking precautions (classification techniques)

- To determine characteristics of special needs students (Clustering)

- To investigate relationships among different theoretical perspectives used in education and to link them (association rules or classification techniques)

- Finding out relationships in learners' behaviour patterns (Association rules)

RELATED WORKS

Data mining techniques have been used in two different educational domains such as computer based education and traditional education. Due to the widespread use, some of traditional education studies about EDM are listed in Table 1.

Table 1 Some educational data mining studies

Subject	Studies
Examining and introducing educational data mining techniques, related literature and applications	Peña-Ayala(2014); Romero & Ventura (2007;2010; 2011;2013); Kumar & Vijayalakshmi (2013); Sharma & Singh (2013); Ali (2013);Bhise, Thorat & Supekar (2013); Garg & Sharma (2013); Huebner (2013); Calders & Pechenizkiy (2012);Bala & Ojha (2012);Pandey, Bhardwaj & Pal (2012); Romero C, Ventura S, Pechenizky M, Baker R. (2010);Baker & Yacef (2009); Anyanwu & Shiva (2009)
Determining students' academic performance and the factors that affect it.	Narlı, Aksoy & Ercire (2014); Bilen, Hotaman, Aşkın & Büyüklü (2014);Tiwari, Singh & Vimal (2013); Pal & Pal (2013); Bhise, Thorat & Supekar (2013); Rathee & Mathur (2013); Venkatesan (2013); Turhan, Kurt & Engin (2013); Osmanbegović & Suljić (2012); Yadav & Pal (2012); Yadav, Bharadwa j& Pal (2012); Tair& El-Halees (2012); El Moucary, Khair& Zakhem (2011);Kumar & Vijayalakshmi (2011); Baradwaj & Pal (2011); Lakshmi, Martin, Begum & Oyelade, Oladipupo & Obagbuwa (2010); Liu & Ruiz (2008); Erdoğan &Timor, (2005)
Determining students' features	Kumar& Vijayalakshmi (2013); Gülen & Özdemir (2013); Koç & Karabatak (2012); Im, Kim, Bae & Park (2005)

Some selected studies are summarized:

Peña-Ayala (2014) reviewed educational data mining with two goals; the first is to preserve and enhance the chronicles of recent educational data mining (EDM) advances development; the second is to organize, analyze, and discuss the content of the review based on the outcomes produced by a data mining (DM) approach. Thus, as result of the selection and analysis of 240 EDM works, an EDM work profile was compiled to describe 222 EDM approaches and 18 tools. A profile of the EDM works was organized as a raw data base, which was transformed into an ad-hoc data base suitable to be mined. As result of the execution of statistical and clustering processes, a set of educational functionalities was found, a realistic pattern of EDM approaches was discovered, and two patterns of value-instances to depict EDM approaches based on descriptive and predictive models were identified. One key finding is: most of the EDM approaches are ground on a basic set composed by three kinds of educational systems, disciplines, tasks, methods, and algorithms each.

Nokelainen, Tirri and Merenti-Välimäki (2007), proposed a neural network model for identification of gifted student. With a specially designed questionnaire, they measure implicit capabilities of giftedness and cluster the students with similar characteristics. They also applied data mining techniques to extract a type of giftedness and their characteristics. Data mining techniques such as clustering and classification is applied to extract the type of giftedness and their characteristics. The neural network was used to evaluate the similarity between characteristics of student and type of giftedness. They stated that In the future, they could refine their identification model using various data mining techniques and develop an intelligent learning guide system for "potential" gifted students.

Liu & Ruiz (2007), reported a study on using data mining to predict K–12 students' competence levels on test items related to energy. Data sources were the 1995 Third International Mathematics and Science Study (TIMSS), 1999 TIMSS-Repeat, 2003 Trend in International Mathematics and Science Study (TIMSS), and the National Assessment of Educational Progress (NAEP). Student population performances, that is, percentages correct, were the object of prediction. Two data mining algorithms, C4.5 and M5, were used to construct a decision tree and a linear function to predict students' performance levels. A combination of factors related to content, context, and cognitive demand of items and to students' grade levels were found to predict student population performances on test items. Cognitive demands had the most significant contribution to the prediction.

Narlı, Aksoy and Ercire (2014) aimed to determine the learning styles of prospective elementary mathematics teachers and to explore the relationships between these styles by using data mining techniques. Grasha-Reichmann Learning Styles Inventory was applied to 400 prospective elementary mathematics teachers at Dokuz Eylul University. Results show that more than 50% of female students have "independent''

learning style. At the same time students who have competitive learning style had the least number of students. The male students who have collaborative and dependent learning styles were the majority.. From Class 1 to Class 4, it was observed that the number of students who have individual learning styles was decreasing and the number of students who have cooperative learning styles was increasing. In network graph, it was found that one of the strongest relationships was between the students who have cooperative and independent learning style with high level. On the other hand the relationship between the students who have passive and independent learning style with low level was not seen in graph. The decision tree indicates that the most effective attribute is independent learning style to identify which level of the learning style students have. Besides in the Data mining, learning styles, Mathematics Education association rules model several rules are constructed with %75 confidence.

Bilen, Hotaman, Aşkın and Büyüklü (2014), in their study, 42 different types of high schools in Istanbul from which students took University Placement Exam (LYS) are clustered in terms of their performances. It was also aimed to determine the types of tests that are more efficient among these schools. For this purpose, educational data mining techniques such as clustering and decision tree are used. By deploying the non-hierarchical k-means algorithm, schools are separated into 5 different clusters which have different success level for each of Math-Science (MS), Language and Math (LM) and Language-Social Studies (LS) test scores. It is found that Science High Schools, Private Science High Schools, Anatolian High Schools and Anatolian Teacher Schools found to be in the highest achievement level in all of the test scores. Furthermore, constructed decision tree models with CHAID algorithm show that (1) Chemistry for the score type MS, (2) Math for the score type LM and (3) Turkish Language and Literature for the core type LS were the test types which are primarily effective in the division of schools into clusters.

Gülen ve Özdemir(2013), aimed to predict interest areas of gifted students and discover relationships between these areas by using educational data mining methods. By making use of the Apriori association algorithm, area pairs in which gifted students are frequently interested together are detected. They stated that results obtained from that study will provide many benefits to science and art centers such as giving differentiated instruction by meeting individual needs, organizing course programs more effectively.

Im, Kim, Bae and Park (2005) examined the influence of attribution styles on the development of mathematical talent by using data mining technique. The results of conducted Bayesian classification modeling show that items attributing success to effort and failure to lack of effort are the best predictors for the level of mild mathematical giftedness and gender.

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SUMMARY

This study aimed to introduce educational data mining to mathematics education researchers to discuss its potential applications in this area. Using data mining for educational problems in learning, cognition and assessment, may give opinion to researchers, mathematics educators and parents, besides contributing to the literature.

Educational data mining is a young research area and it is necessary more specialized and oriented work educational domain in order to obtain a similar application success level to other areas, such as medical data mining, mining e-commerce data, etc (Romero & Ventura, 2007).

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PRINCIPLES OF STUDENT CENTERED TEACHING AND IMPLICATIONS FOR MATHEMATICS TEACHING

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This paper aims to present principles of student-centred teaching (SCT) and provide some implications for mathematics teaching. In light of the literature, we have determined six main principles of SCT as i.) Taking students' prior knowledge into consideration, ii.) Handling students' difficulties with appropriate methods, iii.) Developing students' skills (e.g., reasoning) iv.) Providing effective feedback, v.) Creating communicative classroom environment, vi.) Integrating assessment into instruction. We first present the rationale of the study and note the ambiguity regarding student-centred related terms. We then propose that STC approach consists of two main components: mixed teaching methods and principals; and explain each principle. We end the paper with discussions and implications of SCT approach for mathematics learning and teaching and note a need of research for operationalizing the concept of SCT for the practitioners working in the field.

INTRODUCTION

The emphasis on individual learning has paved the way for the emergence of new terminology regarding the learning and teaching both in education as a whole and mathematics education in particular in the last three decades. One such term is that of student-centred teaching. Intuitively albeit it might appear to be a straightforward term, it appears that not only the term is not well defined but also what is attributed to the term is not clear. As teacher educators, our experience with the pre-service and in-service teachers has also revealed that the term student-centred teaching often is attributed to only constructivist approach (e.g., discovery learning) and students' physical activeness in the classroom, whilst cognitive activeness was regarded as secondary if not disregarded at all. The vagueness regarding the meaning of the term has been the rationale for the emergence of this study. With this in mind, this paper attempts to examine the term SCT and aims to propose some principles in order to contribute to its conceptualisation especially for the practitioners working in the field.

Considering that the term SCT is wide-ranging, any attempt to determine its principles requires an examination of multiple theories. To this end, an eclectic literature (e.g., behaviourist, cognitivist, constructivist, sociocultural perspectives on learning and teaching) has been examined. Six main principles have been determined to characterize the SCT. These principals develop from both the relevant literature as detailed below and our interpretation of what the teachers and candidates might need to know for conducting a SCT approach. Although we do not claim that they are sole

principles of SCT, we argue that they provide an overall aspect of what the SCT might include. The determined principles are as follows:

1. Taking students' prior knowledge into consideration

2. Handling students' difficulties with appropriate methods

3. Developing students' process skills

4. Providing effective feedback

5. Creating communicative classroom environment

6. Integrating assessment into instruction

In what follows, we first explain why we chose to examine the term SCT and present our stance on it. We then explain each principle in light of the relevant literature and relate them to SCT. We conclude the paper with discussions of the principals.

THE TERMINOLOGY

Dissatisfaction with teacher-centred approach (often known as traditional teaching) and behaviour-oriented perspective in learning and teaching has directed educators to pay more attention to students and their cognitive needs. This shift in attention has resulted in generating new terms and concepts to capture the new phenomenon. Student-oriented terms that have been commonly used amongst educators are the result of such undertaking. As a result of such endeavours, the terms such as student-centred learning, student-centred pedagogy, child-centred learning, student-centred education, learner-centred learning and student-centred teaching come into use. Common to all these terms is the students and their individual learning.

A close examination of these terms reveals several problematic issues though. First, it appears that student-centred terms have sometimes been reduced to ideas popular to Piaget's constructivist developmental theory and hence "discovery learning". Second, the terms have mainly been associated with students' physical activeness rather than cognitive ones. Third, sometimes a passive role is attributed to the teachers since the students are construed be more active. Fourth, the terms have been loosely used and it is not exactly clear what meaning is actually attributed to them. Lastly, it seems that since the terms have mainly been used by the practitioners for practical reasons and have hence been not the foci of the systematic research, it has been difficult to provide a research-informed operationalization of them for the teaching activities. Given that the terms are commonly being utilized in the field, we as the researchers cannot be incognizant of their uses and need to make contribution into their clarification. In this study, we particularly prefer to use the term SCT for two reasons. First, we think that the term student-centred learning or similar ones have some shortcomings. This is because all learning, passive or active, is student-centred in nature. Besides, whilst examining different approaches to learning and teaching (e.g.,

behaviourism, constructivism), although the quality of learning may show variation, what mainly differs is indeed the teaching or the teaching methods. That is why we prefer to use SCT, not student-centred learning. Second, as the teachers are responsible for the teaching, they need to know how to conduct student-centred teaching and hence we take the teachers as the main addressee. However, that doesn't mean that we don't take the students into account whilst dealing with the principles. On the contrary, we provide the principles of SCT for teachers' use by taking the students' needs in every aspect into account. In what follows, we present our position on SCT.

OUR STANCE ON STUDENT-CENTRED TEACHING

We use the term student-centred in the sense that students and their learning needs should be prioritised in the learning and teaching activity. For instance, if a teacher takes students' difficulty with a concept into account and teaches accordingly, this suggests that students' needs are prioritised and the teaching has a student-centred feature. Determination of students' needs, however, is not a simple endeavour. This, of course, depends on the teacher competency regarding the subject matter they teach. Moreover, the needs of students can show variability. The nature of concepts and the student competency are just only two factors that can cause the variability. For example, in teaching group concept in abstract algebra, to us, what the students need is the definition in the first place as it is almost impossible for them to discover the group concept through such approach as problem-based learning. The concept's nature hence determines what the students' needs are and that affect the teaching. On the other hand, if a teacher values conceptual/meaningful understanding, arousing the need for learning and developing reasoning skills etc., then teaching, for instance, "triangle inequality fact" via problem-based learning method and hence providing the students with opportunity to discover or at least attempt to discover the fact can be more fruitful. Given that all these aspects (e.g., reasoning) are important for the learning, this type of teaching is also considered to have a student-centred feature.

One problematic issue that may arise with "discovering" the inequality fact is that: what happens if a student or students cannot "discover" the fact even though the guidance is provided? If one is concerned with students' needs, it is then possibly acceptable that sharing the formula of "$|a - b| < c < a + b$" with students is more reasonable. That is to say, teachers should (sometimes have to) provide the formula or the fact for the benefit of the students. In teaching, teachers hence may sometimes use a mixed instructional approach (e.g., both traditional and constructivist ones) depending on the concepts and students needs. This is, to us, what makes the teaching student-centred. In fact Godino et al. (2015) also note that there is a need for mixture of construction/inquiry and transmission of knowledge that might optimize learning. They are also critical of basing the instruction solely on "Inquiry-Based Learning" (IBL) or "Problem-Based Learning" (PBL) methods and note that these methods

might be more suitable for only gifted students and that these methods generally disregard heterogeneity of the students and the variety of knowledge to be learnt.

In this paper, even though SCT is often associated with constructivist approach in education, we argue that this view is problematic and student-centred teaching needs reconceptualization. We also think that having a practical method (we name it as mixed teaching method) as we presented above is not sufficient to conduct the SCT either, and that is why we propose its principals as well (see, Figure 1). In practice, there is a need for both principals and the mixed teaching method.
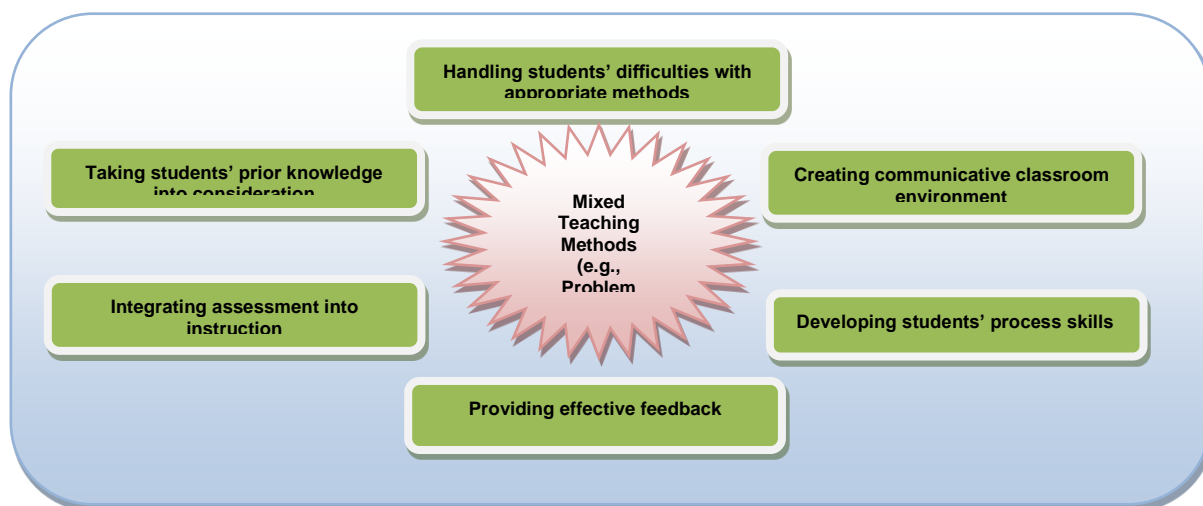
Figure 1: SCT, teaching approach and its principals

As can be inferred from Figure 1, our position is that SCT approach consists of two main components: *mixed teaching methods* and *principals.* In teaching, a teacher might employ mixed methods, that is, the teacher may use both problem-based and expository teaching methods in the same lesson. Yet, to conduct the mixed methods effectively and to take students' needs at the centre, a teacher also needs some principals. The principals guide the methods and enable their implementations. We now turn our attention to principals, their underpinnings and where they stem from.

Taking students' prior knowledge into consideration

Prior knowledge is essential for any learning and teaching activities. Learning theories (e.g., cognitivism, cognitive and social constructivism), particularly the ones shaping the current learning and teaching experiences in many classrooms, emphasise the role of prior knowledge in the learning processes. For instance, as a cognitive learning theorist, David Ausubel put forward the following view on the role of prior knowledge in learning:

If I had to reduce all of educational psychology to just one principle, I would say this: The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly" (Ausubel, 1968, p. 18).

Ausubel's comments can be construed as a radical reaction to behaviourists' view of learning. Prior knowledge draws explicit attention in Piaget's works as well. To Glasersfeld (1995, p.18), one of the two basic principles (radical) constructivism is that "knowledge is not passively received but built up by the cognizing subject". In Piaget's constructivist theory of knowing, since knowledge is actively constructed, not passively received, then the prior knowledge becomes indispensable in the learning process. For instance, in explaining the notion of assimilation, Piaget (1976, p.17, cited in von Glasersfeld, 1995, p.18) notes the importance of prior knowledge:

…no behaviour, even if it is new to the individual, constitutes an absolute beginning. It is always grafted onto previous schemes and therefore amounts to assimilating new elements to already constructed structures (innate, as reflexes are, or previously acquired).

Prior knowledge is not only essential to assimilation but also fundamental to the other two components (accommodation and equilibrium) of Piaget's theory. From a Piagetian perspective, it is thus vital that the teacher takes the prior knowledge into account in teaching. This stance of course requires an examination of students' readiness for the teaching. For instance, in teaching the area of parallelogram, it is important to determine what the students know about the area and the concept itself. The previous experience that the individual brings to learning settings has hence important affects on what he/she is going to learn. We thus take this as a principle of SCT as it is concerned with students' needs. We think that any teaching method with students' needs in mind should begin with determining learners' current knowledge level, types of experience they have and needs analysis.

Handling students' difficulties with appropriate methods

The issues of how students learn and why some have difficulties in learning have always drawn the attention of researchers. Many learning theories (e.g., APOS, Cottrill et. al, 1996) have been put forward for the former. For the latter, it is known that students' learning difficulties, misconceptions and errors are the reality of classrooms. Nesher (1987, p.33) appears to even value the existence errors and notes that "the student's "expertise" is in making errors; that this is his contribution to the process of learning". If students are experts of making errors, then any instructional consideration has to take them into account and teachers need to have an expert approach of handling them. Students' difficulties in learning are also important in the sense that they have been the cause for the emergence of many innovations, including new learning theories, teaching materials and new approaches to teaching etc.

Difficulties generally manifest themselves as errors in the classroom settings. It is critical for teachers to be able to notice the underlying conceptions that cause the errors to emerge. Diagnosing the errors and the causes are hence crucial. Following that, it is essential to have a plan of how to handle the difficulty. This plan might

include selecting the appropriate materials and method of handling. For instance, the relevant literature proposes many different ways of handling the difficulties. Such handling methods as cognitive conflict, giving correction, ignoring are just some examples of teachers' dealing with errors (e.g., Santagata, 2004). Deciding which method to use might depend on the nature of the errors and the teacher's competency. Students' learning difficulties are hence one of the most influential factors that influence the learning and teaching. To us, SCT must take this issue into account and acts accordingly. We think that the teaching concerned with students' difficulties has the characteristic of SCT and has a better chance of getting over students' difficulties.

Developing students' process skills

Traditional teaching has mainly been concerned with the knowledge (e.g., fraction, function, derivative) and its transmission to the students. However, the aim of schooling is not only to transmit the knowledge or teach concepts. One of the essential goals of schooling is to teach students to think (Padilla, 1990) in general and to reason, justify and make connections in particular. As Padilla (1990) notes "all school subjects should share in accomplishing this overall goal."

In addition to teaching concepts, equipping students with basic skills has also become a goal for many curricula. For instance, in science education these skills are named as basic process skills and six such skills are targeted: i.) observation; ii.) communication, iii.) classification; iv.) measurement; v.) inference; vi.) prediction. In mathematics education, NCTM (2000) names the skills as process standards and notes that mathematics instruction should aim to develop such skills as i.) problem solving, ii.) reasoning and proof, iii.) communication, iv.) connections, and v.) representation. In addition to conceptual understanding, procedural fluency and productive disposition, Adding It Up (NRC, 2001) document also propose strategic competence and adaptive reasoning as a part of mathematical proficiency. All these suggest that skills have become an essential goal of the curricula in that the teaching should be concerned not only with concept teaching but also with skills acquisition.

The development of these skills may have many advantages. First, they enable students to think, justify and make connections. Second, skills can help students to have conceptual understanding and therefore meaningful learning (Ausubel, 1968; Skemp, 1978). Without the skills, concepts in mind may stay disconnected and compartmentalised. Third, the skills may help the students to be better problem solvers and hence apply their concepts to real life settings. With all these advantages in mind, we think that the teaching concerned with students' intellectual development must also aim to develop students' process skills. We therefore take the teaching process skills as a main principle of SCT and argue that conceptual and meaningful learning is more plausible through teaching them.

Providing effective feedback

Students' learning is complete with interesting experiences from showing an exemplary performance to making errors, having fundamental misconceptions and not having a sense of direction of what to do under some particular circumstances. An examination of what the students know, where they show good or poor performance and what to do next is sometimes needed for instructional decisions. All these are somehow related to effective feedback and its conduction.

Feedback is regarded as "one of the most powerful influences on learning and achievement" (Hattie & Timperley, 2007, p.81). Feedback is defined as "information provided by an agent (e.g., teacher, peer, book, parent, self, experience) regarding aspects of one's performance or understanding" (ibid., p.81). Winner and Butler's (1994) conceptualisation of the feedback is also helpful.

"feedback is information with which a learner can confirm, add to, overwrite, tune, or restructure information in memory, whether that information is domain knowledge, meta-cognitive knowledge, beliefs about self and tasks, or cognitive tactics and strategies" (p. 5740, cited in Hattie & Timperley, 2007).

As the quotation suggests, feedback can be provided by different agents and in many distinctive forms. A conceptualisation of feedback in the sense of Winner and Butler requires a careful examination of what task to choose, what kind of discourse to create and what method to use to handle students' learning outcomes on the part of the teacher. The teaching concerned with student needs is hence expected to pay attention to the quality of the feedback that the students get and acts accordingly. We therefore argue that one of the basic characteristics of the SCT lies at the quality of feedback provided to the learners.

Creating communicative classroom environment

As students participate in the learning activity as groups and since teaching students in groups is an indispensable reality of the schooling, the teaching cannot solely be reduced to the teaching an individual and that it needs to address the classroom as a whole. In such situations, the issue of how the teaching, which takes students at the centre, can be conducted also needs to be examined and discussed. To us, a communicative classroom environment can be like an open society so that students can freely express their answers, make arguments and explanations. That is to say, a democratic classroom environment is needed so that students express their opinions. In this connection, Yackel and Cobb's (1996) notions of social and socio-mathematical norms can be employed as a guide for creating such a classroom environment. In such classrooms, different solution methods, reasoning, justification can be encouraged for all students. In such an environment, it is then more possible for students to obtain different perspectives and develop a critical habit of mind. We therefore take communicative classroom environment as a principle of SCT to guide

the teacher concerned with student-centred teaching. It should be noted that the application of this principle in the classroom helps the teacher to gain insight into the other principles as well. For instance, a communicative classroom environment may pave the way for the expression of free speech and that might help to diagnose the learners' difficulties. The teacher can hence employ this principle to have an overall picture of the instruction with regard to other principles as well.

Integrating assessment into instruction

Traditionally, assessment follows the instruction. This type of assessment is termed as summative and is concerned with cumulative evaluations. It is currently proposed that assessment needs to be built up into and integral to the instruction. This type of assessment is termed as formative one and is concerned with regular control of students' conceptions and understanding (Van De Walle et. al, 2010). This type of assessment shapes spontaneous decisions regarding the instruction and the findings reveal that effective formative assessment can increase students' speed of learning by giving the effective feedback (Wiliam, 2007).

As far as SCT is concerned, it is proposed that assessment and instruction need be intertwined. Assessment should not be something to be done at the end of instruction. Assessment concerned with students' development, difficulties and learning has to be in time and based on students' needs. In this regard, rather than evaluating students through one method (e.g., test) students' performances need to be assessed through different methods. Assessment also should not only be concerned with concept mastery but also with process skills proficiency. As a result, we think that SCT needs to be student-centred in terms of assessment as well. Moreover, as the Assessment Principle in Principles and Standards stresses: "(1) assessment should enhance students' learning, and (2) assessment is a valuable tool for making instructional decisions" (Van de Walle et. al., 2010, p. 76). When the assessment is carried out in this respect, we think that it can contribute to the development of the SCT instruction.

DISCUSSIONS, LIMITATIONS AND CONCLUSIONS

We have attended to the ambiguity of the term SCT and noted that what is attributed to the term is often not clear. We have also stated that SCT has been mainly associated with constructivist approach and argued that reducing it to this approach is misleading. A functional SCT approach does prioritise the students and their needs rather than a particular instructional approach per se. In the light of the relevant literature, alongside the mixed teaching methods, we have provided six principles that might contribute to conceptualisation of SCT. We are aware that the proposed SCT principles are generic in nature. This is particularly due to both the nature of the term and the teaching itself. Although this is the case, we hold the belief that for the practitioners it is important to have a general perspective of SCT as well. This is because; having a broad perspective can help the teacher to put a particular learning

objective into practice. Therefore, although the proposed principles are generic; they might help the teacher to have a broad perspective on SCT and to put it into practice.

Most of current educational reforms suggest student-centred teaching and the chief addressees are teachers and teacher candidates. Although they are expected to conduct SCT, they generally do not have a guideline of how to do that. We believe that these principles as a totality might act as a guide for teachers and candidates to practice SCT. For instance, the principles can be used to design and implement lesson plans. We also think that these principles can be used to develop or assess in-service and pre-service teachers' competencies and knowledge bases. For instance, a teaching programme addressing methods of handling students' difficulties may contribute to the development of teachers' pedagogical content knowledge base. In addition, the SCT principles can be employed as theoretical framework to analyse the classroom discourse and determine whether the teaching is SCT or not. For example, it can be utilized to determine the extent to which the teaching values the process skills. Similarly, the framework can enable one to see how students' difficulties are handled and to show which the types of feedback are provided in the classroom.

As mentioned above, we are aware that these principles are generic and that is why they cannot be specific to any discipline. The nature of disciplines and their concepts will shape how each principle is put into practice. For instance, whilst handling a difficulty or error, one needs to know the nature of the concept and teach accordingly. More specifically, let's take division of two fractions as an example. If the concept is to be taught in an SCT manner, in the light of SCT principles, the teacher first has to take learners' prior knowledge of fraction and division into account. Knowing students' difficulty with division of fractions can help the teacher make necessary preparation, which would improve the instruction. These all suggest that the nature of concept in a discipline itself can affect how SCT is perceived and conducted.

Process skills can play an important role in making SCT approach specific to a particular discipline or carries its distinctive characteristics. Reasoning, justification or representation of the concepts, for example, can differ from one discipline to another. For instance, the function $f(x) = x^2$ can be represented in many forms (e.g., numeric, graphical, algebraic, verbal). When the teacher teaches this function with its multiple representations alongside with their interconnections to enrich students' understanding, this would suggest that the teaching has a student-centred feature. Therefore, the mathematical instruction that values students' needs should pay attention to mathematics process skills acquisition.

Finally, as this work is still in progress, we suggest that further research needs to be carried out to see how functional the proposed principals are and examine them in the real classroom settings. There is also a need for making each principal more explicit. Further research is also needed to examine practitioners' conceptions (e.g., values, beliefs) of SCT and how they play role in its implementation.

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CONSIDERING THEORETICAL DIVERSITY AND NETWORKING ACTIVITIES IN MATHEMATICS EDUCATION FROM A SOCIOLOGICAL POINT OF VIEW

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The paper focuses on the social dimensions of the issues addressed in this working group, social being considered at different levels, interactions, culture, institutions: what is a theoretical framework? Why are theories so numerous in mathematics education? Is it necessary to reduce this multiplicity? Whether it is or it is not, why? The reflection is based on the anthropological theory of the didactic (ATD) and on Bourdieu's theory of social fields. Assuming that the latter is not necessarily well-known in the mathematics education community, and that it offers an interesting potential to enrich the debate within the networking semiosphere, I devote a substantial part of our text to give an idea about the way Bourdieu applies his theory to science.

Keywords: praxeology, paradigm, institutional determination, symbolic capital

INTRODUCTION

Addressing the topic of theoretical diversity in mathematics education from a social point of view is not something new in the European research community. The central preoccupation in this WG has been, since CERME4, the barrier to effective communication created by the multiplicity of theories, be it communication within the field or with external partners from policy makers to educative professionals. Radford (2008, p. 318) suggests considering the networking practices as located in a semiosphere, e.g. "an uneven multi-cultural space of meaning-making processes and understandings generated by individuals as they come to know and interact with each other". It is quite representative of the interaction dimension in networking activities. Among social aspects I consider in this paper, some have been much more erratically present in the discussions. For instance, the WG11 leaders' introduction (Artigue et al, 2006, p. 1240) refers to a theoretical "more intrinsic diversity linked to the diversity of educational cultures and to the diversity of the institutional characteristics of the development of the field in mathematics education in different countries or global areas." This issue of theoretical multiplicity being linked to cultural diversity has not recently been discussed in CERME. My position is that our reflection about theoretical diversity is obstructed by some self-evidences that should be deconstructed and that, to do so, we need theoretical tools from inside and outside the mathematics education field. In this paper, my objective is to present some tools, borrowed from ATD and from Bourdieu's field theory, I consider as helpful to go forward. I briefly show how I use them to go back on the social dimension of

theoretical multiplicity and to discuss the unifying-theories injunction, thus developing a rational discourse (*logos*) with social concerns about the issues addressed. The adjective "sociological" in the heading must be understood in this etymological meaning, this paper does not avail itself of the sociology scientific field.

Before turning to the text substance, I will highlight the fact that, in my opinion, a valuable discussion in a group focusing upon connecting theories, relies on the participants knowing a minimum about key points of the theories at stake in the papers. Except for some well-known mathematics education theories, I consider as the author's responsibility to provide the readers with some genuine elements of the involved theories, so that they might build a first understanding. I try to achieve this objective for one of the theories I use as thinking tools, Bourdieu's field theory, assuming that ATD is familiar enough in our research community. Hence, the text second part encompasses large quotes intending to provide the readers with a direct, even though limited, access to the key elements of Bourdieu's analysis of science I draw on later. Other theories appear as examples in the discussion: within the submission format, I can do no better than to give references.

THEORY, RESEARCH PRAXEOLOGY, RESEARCH PARADIGM?

In this part, I recall and connect crucial models elaborated by some of the first participants in this group to address the issue of what is actually a theoretical framework. Then I will propose to encompass into the theory modelling project the contribution of well identified research communities, thus giving an opportunity to consider that networking theories has a social dimension: connecting specific communities associated to the theories.

Networking theories, what will we consider as such in this 2015 session? An eight-years-long joint work in CERME as well as in research projects like Telma and Remath has largely evidenced that what is at stake cannot be reduced to networking of theories considered as "organized networks of concepts (including ideas, notions, distinctions, terms, etc.) and claims about some extensive domain…" (Niss, 2007, p. 1308). Other research aspects are involved in the interconnection activities. Two directions have been proposed to model this complexity. Radford (2008) describes the concept of theory using the triplet (*P*, *M*, *Q)* where *P* is a system of basic principles, which includes implicit views and explicit statements, *M* a methodology, and *Q* a set of paradigmatic research questions. Hence, connecting two theories means connecting two triplets. Artigue, Bosch & Gascón (2011) use the notion of praxeology to model research theories and practices. Introduced by Chevallard (1999) as a general model for all human activities (see Bosch & Gascón, 2014, for an introduction to ATD), a pointwise praxeology is a quadruplet $[T/\tau/\theta/\Theta]$ with only one type of tasks T and one associated technique τ, θ being the technology of τ, i.e. a rational discourse accounting for this technique. "The fourth component is called the

"theory" and its main function is to provide a basis and support of the technological discourse" (*Ibid.,* pp. 67-68). Moreover, ATD considers more complex levels of praxeological organisations gathering pointwise praxeologies which have a common technology (local praxeology) or a common theory (regional praxeology). Hence a regional research praxeology may be described through a set of research questions considered relevant among others that are not, correlated acknowledged techniques, their technologies and at last a theory. Artigue et al. (2011) consider that this is the proper level to address networking issues.

What is a theory in this model? In the case of well-developed research praxeologies, the theory may fit with Niss' definition. However, not all such theories operate as identifier of their associated praxeology, because some are not recognised as "a Theory" in the research field. For instance, let us consider the so called "double approach" (of the teachers' practices) developed by Robert and Rogalski (2002). A regional "double approach" praxeology obviously exists in mathematics education. Its theory, in both ATD and Niss' meaning, is well developed, coordinating elements from several identified theories like Theory of Conceptual Fields and Activity Theory with some more isolated concepts or results from didactics and cognitive ergonomy. Yet, there is no "Double Approach Theory", the praxeology access to social existence in the research field relies on other means, like the publication of a collective book gathering different studies (Vandebrouck 2008, 2013) and its translation into English.

Now, let me emphasise that, within ATD, most praxeologies' theories are not this developed; they may not fit with Niss' definition. It is a strength of this modelling of research activities that it may be used, as Artigue et al. (2011, p. 2382) do, to account for the research praxeological dynamics: "Research praxeologies can appear as different kinds of amalgams, more or less organized depending on the maturity of the field." They emphasise the part played by the technological discourse in such a stage of praxeology, when the theory of the amalgam is underdeveloped and unable to organise through a coherent whole the first results produced by the research practices. I will focus on the social dimension of the development process: the emerging praxeological organisation would not strengthen and access a certain form of social existence in the research field without the setting up of a group of researchers with common concerns, collaborating towards the development of the praxeology. In the case of the double approach, such a group was first created around A. Robert and J. Rogalski within the Parisian laboratory Didirem, especially through the completion of several PhD theses. In 2014, the double approach community still exists; it is disseminated far beyond its original laboratory. This idea that there is no research praxeology recognised in the mathematics education field (or in some subfield) without an associated community of researchers is not accounted for by the praxeological model. Thus, this paper proposes an extended model, called a *research paradigm* [1], composed of a praxeology and a correlated social organisation.

The praxeological model and Radford's model appear as efficient tools to account for the fact that connecting theories is not only connecting conceptual structures. They share several aspects: *Q* is the set of *T*, *M* the set of [τ/θ], the explicit part of *P* belongs to Θ. However, they differ on other points. A praxeology gives a clear place to the '*savoir*' (explicit, socially legitimised knowledge) while in Radford's model it is difficult to locate the theoretical knowledge corpus, crucial in a scientific field. With regard to the techniques, the *M* modelling includes the technological knowledge but it does not provide an appropriate tool to consider what is happening in the case of methodological exchanges between theories (with Radford's meaning of the term), an issue addressed by (Radford, 2008, p. 322). The technique may or may not change, but certainly a new technological discourse will be produced to justify that the imported technique is consistent with the importing theory and its principles. If we consider that a research paradigm is an institution, this is one form of the transpositive process that goes with the inter-institutional movements of praxeologies (Chevallard, 1999, p. 231). Up to that point, the praxeological model appears as more comprehensive and detailed than the (*P*, *M*, *Q*) model. However there is no place in a praxeology for the implicit part of *P*. The paradigm model supplies this lack: according to ATD, a research paradigm, as an institution, exerts some constraints on its subjects. That is to say, within a given paradigm, researchers' actions are regulated by the reference to the research praxeology and through the influence of the associated social organisation.

In summary, the research paradigm model presents three strong points: incorporating the different aspects of the (*P*, *M*, *Q)* and [*T*/τ/θ/Θ] models; including in the modelling project the contribution of the research community that in some cases or times plays a decisive role in the scientific identity of the research praxeology; considering social interactions between communities within the networking issue.

LOOKING AT MATHEMATICS EDUCATION RESEARCH FROM OUTSIDE

I now present tools which I use in the last part of the paper to interpret the paradigm multiplicity in mathematics education and the injunction to unify theories.

Institutional determinations

An ATD important contribution has been to introduce the notion of ecology in mathematics education in order to fight the pedagogical voluntarisms. The mathematical and didactic praxeologies are subjected to a complex system of conditions "that cannot be reduced to those immediately identifiable in the classroom" (Bosch & Gascón, 2014, p. 72). They are constrained by a whole scale of institutional determinations among which ATD considers at the highest generic levels the influence of Civilisation and Society (*Ibid.*, p. 73). This is only one example of the crucial part given to institutions by ATD, it aims to show that this theory always

immerses the addressed questions in the whole anthropological reality, with a special focus on the social organisations and the way they determine human activities. In what follows, I apply this approach to mathematics education research.

Bourdieu's field theory applied to science

A field is a structured social space, relatively autonomous from the wider social space and strongly differentiated from other fields. According to Bourdieu, science is a field. The field theory focuses on the 'closed field' dimension of these spaces, providing analysis of what is going on inside; this is the interesting contribution for our group since ATD provides adequate tools to consider external influences.

A field is characterised by a game that is played only by its agents, according to specific rules. The agents are individuals and structured groups, in science they are isolated scientists, teams or laboratories. The conformity of agents' actions to the game rules is partly controlled by objective visible means, but the key point of the theory, through the concept of *habitus*, is the inculcation of the field social rules into the agents' subjectivity. This individual system of dispositions, partly embodied as unconscious schemes, constitutes an individual's right of entry into the field.

The field game is twofold. Firstly, it is productive of something that is the field legitimised goal in the social space. The rules, and therefore the individual dispositions, are fitted to achieve this goal that every agent considers desirable. In the case of science, the goal is epistemic: accepting tacitly the existence of an objective reality endowed with some meaning and logic, scientists have the common project to understand the world and produce true statements about it. Bourdieu further adds a social dimension to the Bachelardian conception of the scientific fact construction:

> In fact, the process of knowledge validation as *legitimation* (securing the monopoly of legitimate scientific opinion) concerns the relationship between the subject and the object, but also the relationship between subjects regarding the object […]. The fact is won, constructed, observed, in and through […] the process of verification, collective production of truth, in and through negotiation, transaction, and also homologation, ratification by the explicit expressed consensus – *homologein* (Bourdieu, 2004, pp. 72-73).

Despite this social nature, scientific homologation produces objective statements about the world thanks to specific rules of the scientific critical scrutiny, "the reference to the real, [being] constituted as the arbiter of research" (*Ibid.*, p. 69). Bourdieu also emphasises that constructed facts are all the more objective as the field is autonomous and international.

Secondly, the game is a competition between the agents, which results in an unequal distribution of some specific form of capital, source of advantage in the game itself,

source of power on the other agents. Thus, a field, including the scientific one, appears as

> a structured field of forces, and also a field of struggles to conserve or to transform this field of forces. […] It is the agents, […] defined by the volume and structure of the specific capital they possess, that determine the structure of the field […This one] defined by the unequal distribution of capital, bears on all the agents within it, restricting more or less the space of possible that is open to them, depending on how well placed they are within the field… (*Ibid.*, pp. 33-34)

The capital includes several species, for instance, in science, laboratory equipments, funding and journal edition. I focus on the symbolic capital, especially on its scientific modality.

> Scientific capital is a particular kind of symbolic capital, a capital based on knowledge and recognition. (*Ibid.*, p. 34)

> A scientist's symbolic weight tends to vary with the distinctive value of his contributions and the *originality* that the competitor-peers recognize in his distinctive contribution. The notion of *visibility*, used in the American universitary tradition, accurately evokes the differential value of this capital which, concentrated in a known and recognized name, distinguishes its bearer from the undifferentiated background into which the mass of anonymous researchers merges and blurs. (*Ibid.*, pp. 55-56).

This theory of science as a field challenges an idyllic vision of the scientific community, disinterested and consensual; however through the hypothesis of embodied dispositions, it avoids considering the scientists' participation to the capital conquest in terms of personal ambition or cynicism.

In summary, I will focus on the fact that scientific strategies are considered twofold.

> They have a pure – purely scientific- function and a social function within the field, that is to say, in relation to other agents engaged in the field (*Ibid.*, p. 54).

> Every scientific choice is also a strategic strategy of investment oriented towards maximization of the specific, inseparably social and scientific profit offered by the field. (*Ibid.*, p. 59)

One can see a true correspondence between the triplets (institution, subjects, *assujettissements*-subjugation) of ATD (Chevallard, 1992) and (field, agents, *habitus*) of the field theory, with an interesting complementarity of the subject-agent pair. In what follows, I consider mathematics education research as an institution immersed in and determined by a complex system of other institutions, and as a field of forces, subfield of the scientific global field, assuming that Bourdieu's analysis clarifies some aspects of the inner institutional functioning.

EXTERNAL DETERMINATIONS OF THE "THEORIES ISSUE"

Research in didactics as being externally determined in its questions and answers

I now consider the fact that the realm of reality of mathematics education research studies is determined by various economical, political, cultural institutions of different sizes. No one may dispute the vast distance that separates the following two objects of study: on the one hand, the passing down of arithmetic techniques in the Aymara villages of northern Chile, whose culture developed specific calculation praxeologies, and on the other hand, the use of software in the French education system to promote the learning of algebra. Is the epistemic priority of mathematics education research looking for universal regularities when, unlike physics for instance, the studied reality is so diverse? Assuming that such common phenomena exist (the didactical contract is often cited as such), which part of the two aforementioned complex realities are they able to account for? Moreover, given that the research intends to act upon the mathematics education reality, a more crucial question would be: to what extent can these regularities support engineering projects? In this paper, I will consider that adapted tools must be designed to address the problems raised by the diverse educational institutions around the world, in order to understand the dysfunctions and to produce solutions that are acceptable to these institutions and their subjects. The research questions as well as the produced answers are determined by local characteristics. The paradigm multiplicity therefore appears to result from the epistemology of a science intending to act upon the studied reality. To take only one example, the ethnomathematics paradigm has been developed in South America as well as in Africa, as a response to a massive failure in mathematics education within educative systems that are still based on the colonial vision and presenting "mathematics as something "Western" or "European", as an exclusive creation by the white race" (Gerdes, 2009, p. 31, my translation). Ethnomathematics follows as a paradigm from the need to "multiculturalise the curricula of mathematics to improve the quality of education and increase the social and cultural-self-confidence of all students". (*Ibid.*, p. 21)

Research in didactics as being externally determined in its workings

Obviously, research depends on national and inter-national political and economical institutions which provide the material and human resources. From this derives the existence of mathematics education research sub-institutions we partly find in the ICMI structure. But other institutions influence the research activities through less evident ways and means, such as cultures with more or less extended spheres of influence, up to civilisation. In spite of their scientific specific *habitus*, researchers with common culture also build upon this culture to address the research issues. That is one source among others of some tacit principles of a paradigm. In other words, the paradigm multiplicity also results from the cultural multiplicity of the agents within

the mathematics education research field. The researchers' cultural specificity may echo the educative local reality they study, hence resulting in a form of coherence and perhaps of efficiency. At the same time, however, several paradigms may coexist in the same society, in the same country, investigating the same education system with different philosophical, ideological positions. As an example, let me consider ATD and the double approach that are strongly differentiated by their conception of the human being: ATD highlights the multi-institutional building of the framework within which the individual develops and acts (Chevallard, 1992, p. 91), the double approach focuses on the individual variations (Vandebrouck, 2008, p. 20). This second viewpoint is more present in the Western education research paradigms than the first one. I hypothesise that this is deeply correlated with the societies' characteristics and that it is not mere coincidence that ATD emerged in France.

Another example of external determination is the theoretical diversity reducing project itself. This project is epistemologically founded within Bourdieu's theory since, as seen above, communication between researchers at the most international possible level is crucial in the construction of the scientific facts. But it also comes from the requirements of political institutions, the Babel Tower aspect of research in mathematics education affecting its credibility. The proposed solution is unifying theories. Policy makers refer to the exact sciences model, and so does, rather surprisingly, mathematics education research itself, still (over)determined by its alma mater, mathematics. This reference neglects the educative reality diversity. It forgets the exact sciences very long lifetime conducive to the unifying process but also that with the colonial expansion many local paradigms have simply been ignored, the occidental ones being imposed to the defeated countries. So the present homogenous theoretical landscape results as much from domination as from unification.

At this point, I have argued that the paradigm diversity is in some sense epistemologically legitimate in mathematics education and results from some social determinations of research. I have also noted that the unifying injunction might be considered as introduced into the field from outside for questionable reasons.

MATHEMATICS EDUCATION RESEARCH AS A POWER GAME

In this part, I build upon Bourdieu's statement that every scientific strategy has a social function within the field, i.e. has something to do with the distribution of power among the agents. In such a framework, the production of independent theories as well as the call for their integration in new entities is taken as contributing to the contestation and conquest of positions. Clearly, for a researcher, being recognised as the creator of an identified theory increases his scientific capital much more than a less visible participation to the collective development of an existing paradigm would. This "visibility factor" fosters the paradigm multiplication, especially at the theory level; it should certainly be controlled when individual positions are at stake. However, let me now consider an emergent research

community: in this case, developing a specific paradigm is an asset to free from the domination by older communities, generally tending to impose their own paradigms as ready tools which are adapted even for new problems. I have already put forward that the need to unify paradigms could be epistemologically challenged by virtue of the diversity of the didactic reality depending on the societies and countries involved. Now, I question it as an obstacle to an autonomous organisation of didactical research in countries where the latter is just emerging. Besides the ethnomathematics already evoked, I will mention socioepistemology (Cantoral ,2013), deliberately developed by a group of Mexican researchers with the dual intent of creating tools adapted for the educative reality in South America and putting an end to what was taken as an extension of colonisation through the exclusivity of North American and European paradigms in didactics research.

To finish, I reverse this point of view: if developing a paradigm is empowering for a community in the field, the call for reducing the paradigm multiplicity has something to do with relative positions of the paradigm social components and of the more-or-less extended research institutions in which the paradigm has been developed. It is an aspect of the social game in the field, certainly determined by other levels of power struggles outside the scientific field as well.

CONCLUDING REMARKS

In this paper, my intent was not to contest the importance of interactions between mathematics education researchers. I recognise the crucial part of the broadest possible communication in the construction of scientific facts and the major difficulty deriving from the paradigm multiplicity in the field. My aim was bringing to light some aspects, so far nearly unexplored in this WG, of the multidimensional complexity of this well documented phenomenon: multiplicity is an epistemological adaptation to the diversity of educational realities and a social result of symbolic power struggle within a recent research field, somehow less submitted to colonial and capitalistic rules to determine the power repartition than have been (and perhaps are) the oldest basic sciences. Hence, if reducing the number of paradigms appears as a direct solution which favours communication thanks to a common conceptual language, this shortcut may be epistemologically inadequate for mathematics education research. Moreover, from the social point of view, it should be considered as the current hidden form of the exercise of power conquest in the field.

Unifying theories in order to produce a common discourse is not the appropriate way to scientificity for mathematics education research in its present state: that is the opinion I have tried to argue in this text. Building on the Remath project experience among others (Artigue & Mariotti, 2014), I suggest that collaborating which brings together researchers who refer to different paradigms might be more relevant; theory networking will result from working together on the same objects. The challenge is to develop collaboration praxeologies.

NOTES

1. Using the term paradigm may be provisory. Indeed it refers to Kuhn's work on scientific revolutions (1962) where Kuhn defines a scientific paradigm as: "universally recognized scientific achievements that, for a time, provide model problems and solutions for a community of practitioners" (3rd edition, p. 10). If we consider the different components of a paradigm, we find something very close to what Radford and Artigue et al. encompass into their models. In the postscript to the second edition (1970), Kuhn addresses the issue of the community structure of science and writes that: "Paradigms are something shared by the members of such groups [scientific communities]." (p. 178) So it appears that he does not include communities within the paradigm model.

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WHAT IS A THEORY ACCORDING TO THE ANTHROPOLOGICAL THEORY OF THE DIDACTIC?

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The question tackled here centres on the notion—or, more precisely, the many notions—of theory often used in discussing scientific matters. The analysis that we attempt develops within the framework of the anthropological theory of the didactic (ATD). It purports to show that current usage refers mostly to the "emerged parts" of so-called theories and largely ignores their "immersed parts", which are the correlate of their intrinsic implicitness and historical incompleteness. This leads to favour open theorization over entrenched theory.

INTRODUCING THE NOTION OF THEORY

In this study we examine the meaning and scope of a key concept of ATD which, paradoxically, since the inception of this theory, seems to have been consistently overlooked: that of *theory*. A word akin to English "theory" exists in many European languages [1]. According to John Ayto's *Dictionary of Words Origins* (1990), the history of *theory* goes as follows:

> **theory** [16] The etymological notion underlying *theory* is of 'looking'; only secondarily did it develop via 'contemplation' to 'mental conception.' It comes via late Latin *theōria* from Greek *theōríā* 'contemplation, speculation, theory.' This was a derivative of *theōrós* 'spectator,' which was formed from the base *thea-* (source also of *theãsthai* 'watch, look at,'·from which English gets *theatre*). Also derived from *theōrós* was *theōreĩn* 'look at,' which formed the basis of *theórēma* 'speculation, intuition, theory,' acquired by English via late Latin *theōrēma* as *theorem* [16]. From the same source comes *theoretical* [17]. (p. 527)

A paper by a classical scholar, Ian Rutherford, gives more information on the uses of the word *theoria* in Ancient Greece:

> The Greek word *theoria* means "watching," and has two special senses in Greek culture: first, a religious delegation sent by a Greek city, to consult an oracle or take part in a festival at a sanctuary outside its territory, and second, philosophical contemplation. *Theoria* in the first sense is attested from the sixth century bce until the Roman Empire, but the sources are particularly rich in the Hellenistic period. Sacred delegates were called *theoroi*, were often led by a so-called *architheoros*, and if they went by sea, the vehicle was a *theoris*-ship. (Abstract)

The first of these two senses has almost disappeared from modern usage. The second sense opened the way for our common uses of *theory*. In the following, we concentrate on "modern" meanings of this word, which dictionaries usually condense

into a small number of categories, as does for example the English *Wiktionary*. The entry dedicated to *theory* in this dictionary begins classically with the etymology of the word, then passes on to the uses of it that it does retain:

theory (*countable and uncountable*, *plural* **theories**)

1. (obsolete) Mental conception; reflection, consideration. [16th-18th c.]
2. (sciences) A coherent statement or set of ideas that explains observed facts or phenomena, or which sets out the laws and principles of something known or observed; a hypothesis confirmed by observation, experiment etc. [from 17th c.]
3. (uncountable) The underlying principles or methods of a given technical skill, art etc., as opposed to its practice. [from 17th c.]
4. (mathematics) A field of study attempting to exhaustively describe a particular class of constructs. [from 18th c.] *Knot **theory** classifies the mappings of a circle into 3-space.*
5. A hypothesis or conjecture. [from 18th c.]
6. (countable, logic) A set of axioms together with all statements derivable from them. *Equivalently,* a formal language plus a set of axioms (from which can then be derived theorems). *A **theory** is consistent if it has a model.*

In what follows we shall draw upon such semantic summaries in order to suggest that the notion of theory developed in ATD can account for the diversity of usages that exist today.

SOME BASICS OF ATD

In ATD, the basic "entities" are *persons x* and *institutions I*. These notions are close to their ordinary counterparts, although they are more general—in ATD, a newborn infant is a person; and, to take just one easy example, a class, with its students and teachers, is an institution. An institution *I* comprises different *positions p*— in the case of a class, that of student and of teacher. To every person *x* or institutional position *p* is assigned a "praxeological equipment", which is the system of "capacities" that, under appropriate conditions, enables the person *x* or any person *x'* occupying position *p* to act and think through one's actions.

Any praxeological equipment, be it *personal* or *positional*, is made up of, among other things, "notions". Most persons and institutional positions thus have a certain notion of theory—if only through the overused phrase "in theory". The present study could then be said to be partly about the notion of theory in ATD (taken as an institution). However that may be, it is essential to detach oneself from the seemingly undisputed belief that there would exist a unique, shared *notion* of theory of which the meaning would simply vary according to the context of use. In ATD, every person, every institutional position is supposed to be endowed with a peculiar notion of theory, that notion being shaped by the *constraints* to which the person or position is currently subjected. This phenomenon is at the origin of the processes of

institutional transposition, of which didactic transposition is but a particular case (Chevallard, 1992). In order to make headway, we shall now delineate the "anthropological" notion of theory—which, at the start, is only one such notion amongst others.

THE NOTION OF PRAXEOLOGY IN ATD

ATD posits a theory of human activity that hinges on an essential and founding notion: that of *praxeology* (Chevallard, 2006, 2012; see also Bosch & Gascón, 2014). The word *praxeology* has been around for (at least) two centuries in the sense recorded by most dictionaries, in which it is held to refer to the "study of human action and conduct", to the "study of practical or efficient activity", or to the "science of efficient action". The use made here of the word pertains properly to ATD and departs decisively from this old-established, though infrequent, use. A key tenet of ATD is that when a person x acts purposely and knowingly, her doings can be analysed into a (finite) sequence of *tasks* t_1, t_2, ..., t_n. Contrary to the common meaning of the term (which has a ring of unpleasantness about it), *task* is taken here in a very general sense, irrespective of its volume or pettiness: *to open this door* and *to smile to this neighbour* are tasks; *to scratch this person's back*, *to write this sonnet*, *to save this polar bear*, *to prove this theorem*, and *to play this guitar chord* are tasks as well. Any task t is regarded as a "specimen" of a *type* of tasks T. In order to execute the task t of type T, a person x draws on a determined *technique*, denoted τ_T, that is to say a (more or less precise) way of accomplishing (at least some) tasks t of type T. No technique τ can cope with the totality of tasks of a given type T—its range of success is usually called the *scope* of τ. If, for example, it is clear that elementary techniques for factoring numbers all have a limited scope, it is true also, for obvious reasons, that *any* technique whatsoever eventually reaches its limits.

Let us take another example, that of a technique for finding the quotient of number a by number b (with $a, b \in \mathbb{N}^*$), which we make explicit on a specimen. Considering that $12 = 2 \times 2 \times 3$, in order to arrive at the quotient of 417 by 12, we first determine the quotient of 417 by 2, which is the quotient of 416 by 2, i.e. 208. We then calculate the quotient of 208 by 2, which is simply 104; and finally we determine the quotient of 104 by 3, which is the same as the quotient of 102 by 3, or 34. The quotient of 417 divided by 12 is "therefore" 34. (Indeed, $417 = 34 \times 12 + 9$.) The inverted commas that surround *therefore* hint at the fact that many people—including mathematics teachers—will highly doubt the validity of this technique, on the grounds that it leads one to carelessly get rid of successive remainders. This paves the way for another key notion that ATD hinges on: the notion of *technology*. This word is used in ATD with its etymological value: as the suffix *-logy* indicates, a technology is a "discourse" on a given technique τ. This discourse is supposed, at least in the best-case scenario, both to *justify* the technique τ as a valid way of performing tasks t of type T and to throw light on the logic and workings of that technique, making it at least partially

intelligible to the user. As concerns the technique of division shown above, it seems difficult to hit upon a full-fledged technology that would justify it, let alone *explain* it—if the technique is duly valid, *why* is it so? For lack of space, we shall leave these two mathematical tasks—justify and explain the aforementioned technique—to the perplexed reader.

A key point must be stressed. Owing to the presence of the suffix *-logy*, the word *technology* carries with it the idea of a *rational* discourse (about some *tekhne*—a Greek word meaning "a system or method of making or doing", that is, a technique or system of techniques). In the universe of ATD, there is no such thing as *universal* rationality. Every person *x*, every institution *I*, and every position *p* has its own rationality, afforded by the technologies present in its "praxeological equipment." Of course, persons and institutions strive to indulge their "rationality" or even to impose it upon others. The interplay between competing rationalities is a major aspect of what it is the mission of didactics to explore.

We have now arrived at a crossroads. It appears that no technological justification is self-sufficient: it relies on elements of knowledge of a higher level of generality, which, whenever they do not go unnoticed—they often do—, sound more abstract, more ethereal, oftentimes abstruse, as if they expressed the point of view of a far removed, pure spectator—a *theoros*. In ATD, such items of knowledge, sometimes dubbed "principles" (or "postulates", etc.), compose the *theory* that goes with the triple formed by the *type of tasks T*, *technique τ*, and *technology*. This theoretical component is denoted by the letter Θ ("big theta") while the technology is denoted by (small) θ. We thus arrive at a quadruple traditionally denoted by $[T / τ / θ / Θ]$. It is this quadruple that we call *a* praxeology; it is called a *punctual* praxeology because it is organised around the type of tasks *T*, considered as a "point".

It should be clear that, by its very definition, ATD's notion of theory already subsumes case 3 of the English *Wiktionary*'s definition of *theory*: "The underlying principles or methods of a given technical skill, art etc., as opposed to its practice. [from 17th c.]." Let us take a step forward. A central tenet of ATD is that all "knowledge" can be modelled in terms of praxeologies. The "praxeological equipment" of a person *x* or institutional position *p* is defined to be the more or less integrated system of *all* the praxeologies that the person *x* or a person *x'* in position *p* can draw upon to do what this person is led to do. A praxeology can be denoted by the letter ℘ (called "Weierstrass p"). It can be construed as the union of two parts or "blocks": the *praxis* part $Π = [T / τ]$, also called the *practico-technical* block, and the *logos* part $Λ = [θ / Θ]$ or *technologico-theoretical* block. One can write: $℘ = Π ⊕ Λ = [T / τ] ⊕ [θ / Θ] = [T / τ / θ / Θ]$. The operation ⊕ is sometimes called the *amalgamation* of the *praxis* and *logos* parts. The amalgamation of Π and Λ should be interpreted as a *dialectic* process of "sublation" [2] through which the *praxis* and

logos parts are at the same time negated as isolated parts but preserved as partial elements in a synthesis, which is the praxeology $\wp$. Let us for a moment relabel "knowledge part" the *logos* part and "know-how part" the *praxis* part of a praxeology $\wp$. The dialectic sublation of "knowledge" and "know-how" that $\wp$ is supposed to achieve is hardly ever actualized. More often than not, the *praxis* and the *logos* observable in a person's or institutional position's praxeological equipment do not fit well together. The *praxis* block may be poorly developed while the *logos* part seems to be ahead of the game—a state of things often expressed by saying something like "he knows the theory, but can't apply it." Or the *praxis* part seems to be going smoothly but the *logos* part is so poor that it fails to substantially explain or justify the featured technique, which is consequently turned into a mere "recipe." The failure to arrive at a "well-balanced" praxeology is the rule, not the exception—a key phenomenon that we will now dwell upon.

INCOMPLETENESS AND IMPLICITNESS IN PRAXEOLOGIES

When it comes to discussing praxeological matters, people are prone to using metonymies or, more precisely, synecdoches [3]. This synecdochic bent generally selects as a derived name some (supposedly) "noble" part or feature of the thing to name. The widely shared propensity to metonymize shows up in particular in the use of the word *knowledge*—which is the "lofty" part of a praxeology—to name the *whole* praxeology. It is even more manifest in the generalized use of *theory* as including not only what ATD calls *technology*, but also the *praxis* part and, therefore, the whole praxeological matter. In common parlance, *theory* refers usually, though somewhat fuzzily, to a complex of praxeologies sharing a common "theory" (in a sense acknowledged by the naming institutions). Such a "body of knowledge" can be denoted by the formula $[T_{ij} / \tau_{ij} / \theta_i / \Theta]$ with $i = 1, ..., n$ and $j = 1, ..., m_i$, where the theory Θ "governs" all the technologies θ_i, each technology θ_i "governing" in turn the techniques τ_{ij}. Such a praxeology goes by the name of *global* praxeology. It is this generic analysis that ATD offers when one comes to speak of, for instance, "group theory" or "number theory" or "chaos theory" or "knot theory," etc. It is to be observed that, in doing so, the praxeological complex to which one refers is defined "in intension" rather than "in extension." It allows one to identify conceptually the *possible* content of the praxeological complex, while its real "extension" remains somewhat unspecified. Of course, it is risky to be so unmethodical when it comes to describing praxeological organisations. Naming a part to mean the whole leads to forget or neglect other parts: the resulting praxeologies cannot, therefore, be efficient tools for action—just as a car stripped down to the engine is of little avail to travel (even if, again metonymically, "motor" can be used to refer to the whole car).

This is however one aspect only of the problem of incompleteness in praxeologies. Any praxeology whatsoever can be said to be incomplete, be it technically, technologically or theoretically. And it is the fate of all praxeologies to continually go

through a process which can further the development of any of their constituent parts: the technique can be further "technicized", the technology "technologized", and the theory "theorized". Consider the following easy example relating to the century-old "rule of three", that of the so-called "unitary method", which L. C. Pascoe in his *Arithmetic* (1971) introduces as "helpful to those who initially have difficulties with the ideas of ratios" (p. 64). Traditional arithmetical techniques were essentially *oral*: to do mathematics, one had to *say* something, in order to arrive at the sought-for result. For instance, if it is known that 132 tickets cost £165, how much will be paid for 183 tickets? The right "saying" goes somewhat as follows [4]: "If 132 tickets cost £165, then 1 ticket costs 132 times *less*, or £165/132; and 183 tickets cost 183 times *more*, or £(165/132) × 183, that is £228.75." Here the type of tasks T is clearly delineated; and so is the propounded technique τ_0. As is often the case with arithmetic, the technology θ of τ is essentially embodied in the "technical discourse" above, that both activates τ and explains—makes plain—its logic, thereby justifying it. As always, the "justifying efficacy" of θ depends much on the apparent "naturalness" of the supposedly self-evident reasoning conveyed by the technical discourse recited (if n cost p, then 1 costs p/n, etc.). There exist, of course, other techniques. Some centuries ago, people would have said something like "132 is to 165 as 183 is to price p", writing down the "proportion" 132:165::183:p. Using the (technological) assertion that, in such a proportion, the product of the "means" (i.e. 165 and 183) equals the product of the "extremes" (i.e. 132 and p), they would have arrived at the equation 165 × 183 = 132 × p, which gives $p = (165 × 183)/132$ This formula appears to agree with the one found using τ_0, provided one knows the (technological) equality $(a × c)/b = a/b × c$. But this age-old technique τ_{-1} was technologically—not technically—*more demanding*, because the reason why the key technological assertion (about means and extremes) is true remains hidden—which, for most users, turns τ_{-1} into a recipe.

The technique τ_0 can be modified in (at least) two subtly different ways. One consists in introducing an easy technological notion from daily life, that of *unit price*, which leads to a technical variant of τ_0: "If 132 tickets cost £165, then 1 ticket costs 132 times *less*, or £165/132, that is £1.25; and m tickets will cost m times *more*, or £1.25 × m." This technical variant τ_{01} is a little bit more complex technically (by contrast, τ_0 skips the calculation of the unit price, though the technological concept of unit price is already implicitly present); but it provides more *technological comfort* to the layman. Another variant results from a decisive *theoretical* change. While people generally understand the expression "number of times" as referring to a *whole* number of times, as was the case in the tickets problem, a major advance in the history of numbers consisted in regarding *fractions* as true numbers, on a par with what came to be called *natural* numbers—fractional numbers being called by contrast *artificial* numbers. A second step forward, not yet taken by so many people, consists

in extending the scope of the expression "number of times" to include *fractional* numbers, so that, for instance, 183 is 183/132 times 132 (i.e. 183 = 132 × 183/132), from which it follows that the price of 183 tickets is 183/132 times the price of 132 tickets, or 183/132 times £165, that is £165 × 183/132 (which is yet another resolvent). As long as one accepts to think in terms of *fractional* number of *times*, we have a new technique, τ_{02}, much more powerful and comfortable than τ_0 or τ_{01}. Knowing for instance that the price of 2988 tickets is £3735, we can now say that the price of 2012 tickets will be 2012/2988 times the price of 2988 tickets, i.e. £3735 × 2012/2988; etc. While the variation leading to τ_{01} only called for a rather easy modification in the technique's technological environment, here the change affects the *theory* itself, which in turn leads to a new technological concept, that of a fractional number of times.

In mathematics as well as the sciences, praxeologies turn out to be no less incomplete than in other fields of human activity. Many aspects of a praxeology's incompleteness are in fact linked to the impression of "naturalness" that so many people feel when they use (or even observe) this praxeology. Of course, the notion of naturalness undergoes institutional variations—let alone personal interpretations. But it is too often assumed that what is natural is, by definition, an unalterable given that does not have to be "justified." This, of course, runs contrary to the scientific tradition, of which it is the ambition to unveil the figments of institutional or personal imagination. Thus the French mathematician Henri Poincaré (1902, p. 74) regarded the principle of mathematical induction as "imposed upon us with such a force that we could not conceive of a contrary proposition." But almost at the same time, progress in mathematics showed that this supposedly self-existent principle could be derived from the well-ordering principle [4]. The same phenomenon had happened more than two centuries earlier. The leading character was then John Wallis. According to Fauvel, Flood, and Wilson (2013), here is what happened:

> On the evening of 11 July 1663, he lectured in Oxford on Euclid's parallel postulate, and presented a seductive argument purporting to derive it from Euclid's other axioms. As Wallis observed, his argument assumes that similar figures can take different sizes. Wallis found this assumption very plausible, and if it were true then the parallel postulate would be a consequence of the other axioms of Euclid. It does, however, imply a remarkable result: in any geometry in which the parallel postulate does not hold, that similar figures would have to be identical in size as well as in shape, and so scale copies could never be made. (pp. 129-130)

Seventy years later, Girolamo Saccheri was to observe that Wallis "needed only to assume the existence of two triangles, whose angles were equal each to each and sides unequal" (Bonola, 1955, p. 29). Wallis's proof of the parallel postulate [5] opened the way to a major change that we can subsume under a broader historical pattern. By making *explicit* a theoretical property of Euclidean space—"To every

figure there exists a similar figure of arbitrary magnitude" (Bonola, 1955, p. 15)—, Wallis reduced the incompleteness (in ATD's sense) of Euclidean geometry as a praxeological field. But he contributed much more to the mathematical sciences: he discovered a *constraint* that, until then, had been taken for granted (and thus ignored) and which turned out to be crucial in the development of geometry, in that it drew a clear demarcation line between Euclidean geometry and the yet to come non-Euclidean geometries.

At this point we must introduce another key notion of ATD: that of *condition*, stealthily used in the behavioural sciences (through the idea of conditioning or being conditioned) and akin to more widespread notions such as *cause*, *variable*, and *factor*. Didactics is defined in ATD as the science of the conditions of diffusion of knowledge to persons and within institutions. More generally, ATD views any science—including mathematics—as studying a certain kind of conditions with a bearing on human life and its environments. In this respect, given an institutional position p, it is usual (and useful) to distinguish, among the set of conditions considered, those that could be *modified* by the people occupying position p, and those which cannot be altered by these people (though they could be modified by those in some position $p' \neq p$). Any science seeks to accrue knowledge and know-how in order to make the most of prevailing conditions and, in the case of constraints, to create new positions for which these constraints become modifiable conditions. Now, before doing so, it is necessary to *identify* such conditions and constraints, and this is precisely what happens in the Wallis episode, where the Euclidean constraint of invariance by similarity is brought out as a key theoretical property. At the same time, revealing some constraint usually brings forth alternative conditions that had gone unnoticed until then—non-Euclideanism, in the case at hand—and which become new objects of study. It must be stressed here that a science doesn't know in advance the complete set of conditions and constraints it has to cope with: constructing this set is, by nature, a never-ending task. All these considerations extend to any field of activity, whose praxeological equipments are the outcomes of facing sui generis conditions and constraints. We have now arrived at a position where it makes sense to revert to the question from which we started.

WHAT IS A THEORY?

It must be emphasized here that the interrelated notions of technique, technology and theory do not refer so much to "things" as to *functions*. A technique is a construct which, under appropriate conditions, performs a determined function—the technical function. The same may be said about technology and theory, which respectively perform the technological and theoretical functions. Up to a point, these last two functions look weakly distinguishable—indeed, any contrastive definition is sure to be plagued with counterexamples. Obviously, there are some general criteria allowing one to discern the technological from the theoretical: the first of them is

regarded as more concrete, more specific and straightforward, while the second one is approached as being more abstract, more general, more meditative and far-fetched, as if it were reminiscent of its origins. Also, as has been already highlighted, in an intellectual tradition that has persisted to this day, the second one is valued more highly than the other. But these considerations may impede the recognition of an essential phenomenon: the use which is often made of words like *theory* refers to the *explicit* aspects of an entity which we described as definitely subjected to inexplicitness and incompleteness.

From the point of view of ATD, it appears that the technological and theoretical components of a praxeological organisation—that is to say, its *logos* part—are almost always misidentified, because the usual view of them tends to focus on their "explicit" part, which looks generally pretentious and assumptive. This tendency clearly shows through the case 2 of the definition of *theory* given by the English *Wiktionary*: "(sciences) A coherent statement or set of ideas that explains observed facts or phenomena, or which sets out the laws and principles of something known or observed." This of course is representative of a dominant theory about... theories. Moreover, *theory* is often liberally used to label what boils down to a few guidelines or precepts which, taken together, do *not* function as the theory of any clearly identified object; for a theory should always be a theory *of something*, built around the scientific ambition to study this "something".

The metonymic use of *theory* is no problem in itself: when one says that ATD is a theory of "the didactic", *theory* refers, as is usual in mathematics for example, to the whole of a praxeological field. But it is a symptom of our propensity to give the word free rein with the uneasy consequence that the debate on theory is deprived of its object. By contrast, ATD conduces to focus the research effort on examining the implicit, unassuming or even wanting parts of technologies and theories. It then appears that a theory is made up of two main components, that we may call its "emerged part" and "immersed part". To avoid engaging here in a titanic work, we summarize in two points the constant lesson that praxeological analysis consistently teaches us. Firstly, the immersed part of a theory—in mathematics and, as far as we know, elsewhere—is replete with inexplicit tenets that are necessary to keep the emerged part afloat. Secondly, these tenets have surreptitious, far-reaching consequences, which often go unnoticed and usually unexplained at both the technological and the technical levels. What people do and how they do it owes much to "thoughts" unknown to them—unknown, not unknowable.

In ATD a theory is thus a hypothetical reality that assumes the form of a (necessarily fuzzy) set of explicit and implicit statements about the object of the theory. A theory is in truth the current state of a dialectic process of theorisation of which it offers an instantaneous and partial view that may prove delusive. The study and exploration of a theory is tantamount to furthering the very process of theorisation. One main

feature of this process is that it allows for the expansion of too often ad hoc, punctual praxeologies [T / τ / θ / Θ] into deeply-rooted global praxeologies [T_{ij} / τ_{ij} / θ_i / Θ]. The process of theorisation, as well as the networking of theorisations, has thus a liberating effect, in which, by the way, the use of well-chosen terms and symbolic notations helps achieve mental hygiene and theoretical clarity in bringing about what Bachelard once called the asceticism of abstract thought.

NOTES

1. See for instance the list proposed on the page at http://www.collinsdictionary.com/dictionary/english/theory.

2. The word *sublation* is the traditional rendering in English of Hegel's notion of *Aufhebung*. According to *Wikipedia* ("*Aufheben*", n.d.), "in sublation, a term or concept is both preserved and changed through its dialectical interplay with another term or concept. Sublation is the motor by which the dialectic functions."

3. A synecdoche is a phrase in which a part of something is used in order to refer to the whole of it.

4. See at http://en.wikipedia.org/wiki/Mathematical_induction#Equivalence_with_the_well-ordering_principle

5. For Wallis's proof in modern form, see, for example, Martin, 1975, pp. 273-274.

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DISCRIMINATORY NETWORKS IN MATHEMATICS EDUCATION RESEARCH

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This paper is written in an organisational language developed in the context of mathematics education by Dowling (2009, 2013) - social activity method (SAM) - as a commentary on Radford's (2008, 2014) discussion of theoretical networking. An exemplar is given of SAM's approach of recontextualising, and thus learning from, what it finds of interest elsewhere – here, Chevallard's Anthropological Theory of the Didactic (ATD). The approach puts emphasis on the autonomy and emergent quality of well-formed research activity. SAM is not, however, solipsistic: it is designed to recursively self-organise in relation to what it encounters elsewhere but on the explicit basis of its own principles. By biasing a reading of ATD, SAM's organisational language develops in the form of a discriminatory research network.

Keywords: Anthropological Theory of the Didactic, deformance, discriminatory research networks, recontextualisation, Social Activity Method.

INTRODUCTION

Writing about theoretical networking presents a formidable challenge. This paper looks at the connectivity between just two research programmes in the domain of mathematics education research, Social Activity Method (Dowling, 1998, 2009, 2013 – hereafter SAM) and the Anthropological Theory of the Didactic (hereafter ATD; Bosch and Gascón, 2014) together with one meta-theory of theoretical networking (Radford, 2008, 2014). This already involves three specialised assemblages of principles and tacit knowledges: to introduce all three would exceed the space available. This limitation is addressed by considering the other approaches as an illustration of how, *from SAM's point of view,* theoretical networking might be achieved. For this reason it is the principles of SAM that are given most emphasis: these are then used to *select* principles from the other approaches. This means that the principles of ATD and Radford's meta-theory must, fundamentally, be misread – what I shall refer to as a (I hope, productive) *deformance* of them.

SAM has in common with some other research in mathematics education an interest in the specificity of social activity in the context in which it is produced and reproduced (see especially Dreyfus & Kidron, 2014: 87). Its focus is on the strategies that lead to emergent alliance in that action and thus (re)produce the socio-cultural. I first introduce the central *Domains of Action Schema* of SAM. This provides principles for further application of the method in forming a regard on both ATD and Radford's work. One part of this schema – the esoteric domain – is then considered in greater detail to allow a discussion of the continuities and discontinuities between

SAM and ATD. A new schema is then generated to bias a reading of ATD from the regard of SAM. The conclusion places the findings in the context of the recent literature on theoretical networking.

The question I address is: what can a strongly institutionalised research programme in mathematics education (SAM) make of another such strongly institutionalised approach (ATD)? How does this allow *SAM* to learn and thus deform itself? It needs the greatest emphasis that SAM *makes no assumptions at all about what ATD might or might not learn* because SAM assembles only its own principles. A secondary question is: what light does this shed on the need for meta-theories to conceptualise theoretical networking such as the one proposed by Radford?

For the purpose of clarity and to summarise the position and rationale of the paper: well-formed research activities are incommensurable - they are emergent and not graspable as such, even by themselves. The term "continuity" between theories can refer only to those metonymic chains of signifiers that are of interest *to the recontextualising regard of the theory in question* – hence also the possibility of discontinuity. To claim otherwise, I argue, is counter to fundamental socio-semiotic and socio-semantic principles (sense is made locally in the context of an assembled practice not outside of it). It also involves an infinite regress: the claim of similarities or points of contact between theories begs the question of what is the theory that allows such similarity to be discerned.

INTRODUCING THE DOMAIN OF ACTION SCHEMA

1   Public Domain

Radford's (2008, 2014) discussion of "*networking theories*" in mathematics education research recontextualises some aspects of Lotman's (2001) *semiotics* to introduce "the *semiosphere* as a theory networking space". Of particular interest is the resulting delimitation of theoretical work as "*bounded*" by the principles that grant its "*autonomy*". Radford (2008: 319) produces a description of the mathematics research semiosphere that is in "constant motion"; accelerating as information is transmitted and received with new technologies. Autonomy of a theory within the semiosphere is given by a *hierarchical* order of *principles, methodology* and *research questions* in which the *system* (Radford, 2008: 320) of principles is in regulative control. The potential for networking theories is then a question of their *closeness of principle.* Some theories are too far apart to work well together, others may have surprising affinities yet to be articulated. Generally, we may be experiencing a drifting apart: networking might stabilise this, at least for a time.

This paper is written in SAM: the selection of, and extracts from, Radford's paper are motivated by its common interest in the terms given emphasis in the paragraph above such as autonomy and system. But these are expressions not specialised in SAM; and neither is their content - see the axes of Figure 1. My summary of Radford's position is in the *public domain* of SAM – involving weakly institutionalised (I-) expression

and content (Dowling, 2009: 206) *from the regard of SAM*. Radford's language is a highly specialised one in its own terms; but these specialised terms are not recruited in the institutionalisation (I+) of SAM. Figure 1 expresses SAM's *self-reference*: as a research activity it articulates specialised expression and content in its *esoteric* domain, for example "domain of action".

Figure 1: Domains of Action (from Dowling 2009: 206)

Expression (signifiers)	Content (signifieds)	
	I+	I-
I+	*esoteric*	*descriptive*
I-	*expressive*	*public*

2 Esoteric Domain

Radford's "autonomy" is, from SAM's regard, *recontextualised* into the *esoteric domain* of SAM by transforming the semiotic basis of action. Figure 1 schematises this as a socio-semantics rather than a semiosphere – *institutionalisation* (recognisable regularity of practice) occurring as research *activity* where flows of strategic semiosis (gestures, images, words) are *assembled* in more or less stabilised emergent *alliances*. The principles of action in the esoteric domain regulate what can be recognised/realised in the public domain. Weakly institutionalised terms such as autonomy and semiosphere are alienated in favour of I+ terms such as those given emphasis in this section. This is a *deformance*: the "encounter" (Radford, 2008: 317) read through the principles of SAM. Yet the *expressive domain* ensures that self-reference need not become solipsism: the "identity" (Radford, 2008: 319) of the self-reference changes in its engagement with the other.

3 Expressive Domain

The deformance involved in expressive domain action can be illustrated with respect to the expression "networking theories". **(a). Network**. Eco (1984: 81) characterises the semiosphere (in his terms the *global semantic universe*) as a *labyrinthine rhizomatic net.*

> The main feature of a net is that every point can be connected with every other point, and where the connections are not yet designed, they are, however, conceivable and designable. A net is an unlimited territory […] the abstract model of a net has neither a center nor an outside. (Eco, 1984: 81).

A network is not a net (fishing, internet, tuber or any other). The metaphoric expression nonetheless points to potentially productive specialised content. Perhaps its most significant aspect is that a network cannot be described as a whole or from a global point of view; because any attempt at such a description is immediately re-inscribed as new connectivity – an infinite regress. So a different interpretative mode

becomes necessary. (**b). Theory**. At the nodes of the network, Radford has "theories". The discursive bias of this term is ameliorated to some extent by the composition of the "triplet" to include principles (these may be predominantly tacit), methodology and the "template" of research questions. Yet from SAM's regard there is some danger of the term being read as implying potential representational adequacy (the global all-seeing net). For this reason the phrase *institutionalised research activity* or *approach* has been preferred above.

ASSEMBLAGES OF MATHEMATICAL MODES

In its most recent development SAM has considered the esoteric domain of school mathematics to be constituted as an *assemblage* of strategies, a term recontextualised from Deleuze (Deleuze and Parnet, 2007 [1997]: 69; Turnbull, 2000: 44). As a sociology, SAM is concerned with the distributional consequences of the ways alliances emerge through strategic action in the social: these indicate (never quite fix) the norms of who can say, think, or do what (including in school mathematics).

Figure 2: Modalities of the Esoteric Domain Apparatus (Dowling 2013: 333)

Mode of Action	Semiotic Mode	
	Discursive	Non-Discursive
Interpretative	*theorem/enunciation*	*template/graph*
Procedural	*procedure/protocol*	*operational matrix/operation*

An assemblage is specified by SAM as a relational schema – Figure 2 - that can be contingently recruited in the (re)production of school mathematics. The dimension *semiotic mode* distinguishes discursive (explicitly articulated principles, methods and symbols, for example formulae) from non-discursive modes of mathematical engagement (diagrams, equipment such as a compass). The dimension *mode of action* opposes interpretative and procedural activity: in the former case where there is work to be done in making sense of the semiotic mode (formulae, diagrams), in the latter case where there are rules or sequences to be followed (discursively ordered heuristics, non-discursive techniques for manipulating the compass or computer software appropriately). This establishes four *general* strategies: *template, operational matrix, procedure and theorem*. Further, the second term of each strategy in the table denotes *local* rather than generalising action.

The schema suggests competence in that discipline (or anything else) is not *acquired as such* but is constituted by the development of a pragmatic ability to *contingently* deploy a mixture of strategies in local context – upon which action the assemblage

and those whose alliances will be distributed by it will develop or change. Figure 2 is thus an introduction to the technology for generating empirical description in SAM - see the many further schemas in Dowling, 2009. These pin down *modes of action.* This is not a speculative space: it arose from an empirical engagement with a number of mathematical settings (Dowling, 2013).

A RECONTEXTUALISATION OF ATD

Dowling (2014: 528) has noted that Chevallard's Anthropological Theory of the Didactic (ATD) also makes use of a "complementary" concept of recontextualisation – didactic transposition – although with a primary focus on the contextualisation of *cultural* sense-making in pedagogic settings. The schema of the assemblage is potentially in dialogue with ATD's vision of schools as providers of *discoveries* along the way of *research and study paths* (Chevallard, 2012) contingent to the opening up of a body of questions found to be of interest as the research unfolds. In what follows the "amalgam" of the *praxeologique* (Artigue *et al.,* 2011: 2) is recontextualised within the assemblage of SAM – a *deformative re-ordering.*

Consider the praxeological components $[T/\tau/\theta/\Theta]$ of problematic (task), technique, technology and theory (Artigue, Bosch & Gascon, 2011; Chevallard & Bosch, 2014). ATD notices a key dichotomy between *praxis* and *logos*: thus, for example, in the university some action (Bosch, 2014) is seen to hive off $[\theta/\Theta]$ from $[T/\tau]$. This has proved a fruitful distinction: thus, for example, Job & Schneider (2014) use this framework to make a productive separation of the pragmatic praxeology of the development of calculus and the (rather monumentalising) deductive praxeology of analysis imposed on mathematics undergraduates – with school mathematics very much a hotchpotch of elements from both. However, the amalgam $[T/\tau/\theta/\Theta]$ is conceived as containing the "ingredients*"* (Artique *et al.,* 3) of a didactic situation – the elements of a situation to be enumerated.

SAM would recontextualise this (i.e. from the regard of ATD must didactically transpose this) by noticing that the idea of a praxeologique can be schematised. First, it is possible to distinguish what I will call *operationalising* and *orientation.* Orientation concerns what one is about in a specific context: practically as embodied as a *problématique*, logo-centrically as informed by theory. The former involves low discursive saturation (DS-) as it is embedded in the situated interests or (Maussian) habitus of context. The latter is discursively saturated (DS+) i.e. context free. Operationalising involves techniques – in SAM's terminology "DS- skills" or ways of doing – as well as DS+ "technological discourse" (Bosch and Gascón, 2014: 69).

Figure 3: Praxeological Modes

Discursive Saturation
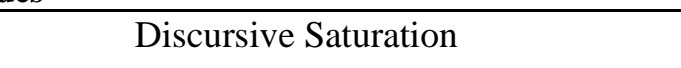
Mode of Action	DS-	DS+
Operationalising	*technique (τ)*	*technology(θ)*
Orientation	*problématique (T)*	*theory (Θ)*
	<u>***skill***</u>	<u>***discourse***</u>

In Figure 3 this produces four *strategies* rather than components. In SAM's research activity the development of schemas such as Figure 3 allows a *particular kind* (without exclusion of others such as ATD) of regulated engagement with the empirical. *One* orienting strategic mode of this is given discursively by the *theory*-logos Θ; self-referentially in SAM's case, particularly the semiotics imbricated in the *raison d'être* of the operationalising *technology*-logos θ of its schemas. Yet much is tacitly acquired: the DS- orientation of SAM's emergent *problématique T* – a concern with emergent alliance - is difficult to explain to novitiates outside a context of apprenticeship. Operationalising - including questions of methodology (Hickman & Monaghan, 2013) – is also composed of strategies of practical *technique τ*. Certainly these can be aggregated in homology with ATD: the DS- modes identified by ATD as [T/τ] can be identified as *skill,* the DS+ strategies of [θ/Θ] as *discourse* (Dowling, 2009: 95); but the recontextualisation now sees each as a strategic mode rather than an element of an amalgam.

The central dichotomy of ATD can then be seen to have been specified in only one dimension. From SAM's regard this is an unnecessary reduction. Yet once relationised in this way, SAM and ATD (from the deforming regard of SAM) have the same objective: the open play of strategies in the assemblage of Figure 2 *and* in the praxeological modes of Figure 3. These point to the principles for a resistance to the closed and syncretic esoteric domains typical of school disciplinary subjects precisely of the kind Job & Schneider (2014) identify. In learning it is then both *operationalising* and the *orientation* of the student to the regularities of practice *in both the DS- and DS+* that would establish apprenticeship

In ATD the theory of didactic transposition acknowledges that school is a specific context of pedagogic relations – thus *of* any practice inaugurated there. As with SAM, ATD thus resists ontologising. In SAM this is expressed as a matter of recontextualising action conceived as a general socio-semantic process of structuration, i.e. in constituting the esoteric domain of a specialised social activity such as school mathematics. In the precursors to ATD this is to be resolved by a "simulated*"* (Brousseau, 1997: 35) reprise of some aspects of phylogeny to constitute ontogeny in the teacher's crafting of appropriate didactic transposition; but from SAM's point of view the principle of recontextualisation makes this an impossible

task as the tacit *skills* of the original problem context are lost. In more recent programmes for ATD (Chevallard & Bosch, 2014) the T of *the current milieu of the child* (in reference to its sociality outside the school) is given appropriate emphasis – this is so often tragically downplayed by policy makers.

As Radford (2008: 322) observes, research questions derive from the principles that allow their articulation. The focus in ATD is on the provision of appropriate activity (and the elimination of the inappropriate) to open to the child the possibilities of what was to become mathematics. To ATD the school may (and often does) block this possibility but this is incidental to the possibility. For SAM, within the research programme identified by Jablonka, Wagner & Walshaw (2013), the *content* of school mathematics is itself always-already recruited in processes of social reproduction – the *particular alliances* (and, of course, oppositions) formed in the schoolroom always *different* to those formed in research (for example, mathematics research).

NETWORKING MODES

In Figure 4 I have recontextualised Lotman's (2001) distinction between what Eco (in Lotman, 2011:xi) has labelled rule-based and repertoire-based texts. The former recite (for example textbooks), the latter transform (e.g. good research). The second dimension distinguishes whether the communication is on the basis of a presumption of the same, or some other, code. Consider first strategies of *equivalence*.

Figure 4: Networking Modes

Code	Message	
	Rule	Repertoire
Same	*equivalence*	*development*
Other	*restriction*	*renewal*
	<u>*literalisation*</u>	<u>*deformance*</u>

For example, many academic papers recite antecedent work – as I have done above in my introduction to SAM. Sometimes the claim to be operating under the same code makes the equivalence seem a little stretched, but these are strategies not truths; such strategies are based on an expectation of *symmetry* – that is, that the literal message will be received as the same by both parties.

Even if there are two humans, one sending, one receiving the text, *equivalence* establishes monologic relations between them. Monologism is also enforced by *restriction* strategies. Some researchers have claimed literal elements of SAM's messages (for example the Domains of Action Schema (DAS) above) for themselves on an explicit rejection of SAM's code. Thus, as I discuss further in Dudley-Smith (2015), for Straehler-Pohl & Gellert (2013: 321) Dowling (1998) is an exercise in

Saussurian structuralism; a code in conflict with functional linguistics on which Bernstein's (2000) sociology of education, the code to which they subscribe, relies. The DAS (Figure 1) is then territorialised by them as a literalised message in the form of a procedure-rule for the analysis of monologic texts but one *restricted* to those circumstances that do not violate the chosen code. Symmetry conditions continue to apply: the implicit assumption is that this text can be transplanted into this other code whilst retaining its *literal* content.

The *development* of research must also involve dialogic reactivation; where sedimented messages inform a repertoire when they are taken *asymmetrically* (Lotman, 2011: 127). Here there is alliance in code (shared principles of a recognisable research activity) but texts are realised (rather than just recognised) as repertoire in ways that change their (con)texts and thus change the network node. This recontextualises Radford's (2008) emphasis on the importance of networking for the evolving "identity" of an autonomous research activity. It is exciting to see researchers making contributions to SAM that put emphasis on the asymmetry of development: see particularly Jablonka & Bergsten (2010), Burke, Jablonka & Olley (2014), and Burke (2015) all of which present potentially productive new relational schemas.

This leaves one further networking strategy identified in Figure 4. In this paper I have taken elements of Lotman's semiotics as expressive (Figure 1) of potential dimensions in the developing (Figure 4) content of SAM's esoteric domain. This is certainly a double deformation. Lotman's work is semiology, SAM's sociology: these are different codes. Lotman's rule and repertoire – appropriated via Eco – are recontextualised as a strategic dimension rather than taken as a fundamental dichotomy. An openness to such otherness *renews*. This formalises Radford's (2014: 285) emphasis on the importance of "unresolved synthesis" to deepen (in SAM's terms) self-reference.

CONCLUSION

The argument considered the way in which SAM might stand in productive relation to other theoretical frameworks and to itself. From the autonomous and self-referential regard of SAM this must be a matter of the principle of recontextualisation, as that is what organises its regard. The self-reference is fundamental; but it is not a solipsism unless foolishly demanding that its categories replace all others to totalise the net. Both development and renewal are possible via an openness to the empirical and to theoretical antecedents. To formalise the situation in the form of a **general argument**, let the operator $\rightarrow$ refer to the recontextualising regard of an approach, ABC, to mathematics education research, and let ES_i be a particular empirical setting.

> If one has SAM $\rightarrow ES_i$ and, elsewhere, ABC $\rightarrow ES_i$ then recognition of commonality would require a general unifying framework, GUF, such that GUF $\rightarrow$ (SAM $\rightarrow ES_i$, ABC $\rightarrow ES_i$) to integrate an

answer to "perspectives *of what*?". This would deny that ES_i is constituted as an artefact of SAM or of ABC (the many observations in **NTRPME** (2014) that the data was not collected appropriately for the theoretical framework concerned). Rather, networking occurs as SAM $\rightarrow$ (ABC $\rightarrow$ ES_i) $\Rightarrow \Delta$SAM with possible answerability of the form ABC $\rightarrow$ (SAM $\rightarrow$(ABC $\rightarrow$ ES_i)) $\Rightarrow \Delta$ABC **& etc**. In each case (Figure 4) the recontextualisation is either misrecognised through literalisation or constituted as a deformative chiasmus (Merleau-Ponty, 1968). For obvious reasons SAM cannot totally catch its own tail: SAM $\rightarrow$ (SAM $\rightarrow$ ES_i) also $\Rightarrow \Delta$SAM; hence the importance of the dialogic (even if with yourself), a potentially infinite recursion.

Certainly "$\rightarrow$" here can be read as "didactic transposition": but in SAM the *content* of that expression, recontextualised as recontextualisation, is generated through schemas such as those introduced above. The general argument rejects the idea that there is a "landscape of strategies for connecting theoretical approaches" (Prediger, Bikner-Ahsbahs & Arzarello, 2008: 170) in favour of the deformative determination of autonomous self-reference. In terms of their key diagram (Prediger, Bikner-Ahsbahs, 2014: 119), there is no role here for understanding, comparison, synthesis or integration, no "relationships between parts of theoretical approaches" (*ibid.,* 118). It is not a question of attempting to find "similarities and differences" *(ibid.*, 119) but to be open to deformative encounters - allowing these to prompt further self-organisation. It is the possibility of *complementarity*, not commonalities, that defeats "isolation", and the principle of *recontextualisation* that annihilates "global unifiers" who put forward GUFs. In Lotman's (2001: 143) semiotics, as in SAM's social-semantics, the principle of *asymmetry* is paramount – information-enriching activity *deforms*. Four modes of connection were discerned (many more are possible with this technology). Two of these use indiscriminate networking to tie literal bonds of alliance. Only those strategies that understand the productivity of deformance in self-reference allow the connection of Discriminatory Research Networks as ways of increasing information.

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SECONDARY MATHEMATICS TEACHER CANDIDATES' PEDAGOGICAL CONTENT KNOWLEDGE AND THE CHALLENGES TO MEASURE

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In this presentation, the authors will discuss pedagogical content knowledge of secondary mathematics teacher candidates in Turkey. The discussion is depend on comparisons between senior students from secondary mathematics education and mathematics departments in terms of their pedagogical content knowledge measured by Teacher Education and Development Study in Mathematics (TEDS-M) released items. In addition to comparison of two groups, there will be discussion on the challenges to measure pedagogical content knowledge.

INTRODUCTION

Teacher knowledge and its components have been described and modeled in different ways by different researchers (Shulman, 1986; Ball *et al.*, 2008; Franke & Fennema, 1992; Tatto *et al.*, 2008). However, it can be said that many teacher knowledge approaches have been influenced by the Shulman's (1986) model of teacher knowledge. Shulman made an important contribution by categorizing teacher content knowledge as Subject Matter Knowledge (SMK) and Pedagogical Content Knowledge (PCK). As Petrou and Goulding (2011) stated, in Shulman model, the most influential categories was the new concept of PCK. Shulman (1986) described PCK as "special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding" (p. 9). According to him as the requirement of PCK, teachers need to know using representations, illustrations, analogies, and demonstrations and also giving examples and explaining concepts in order to make them understandable.

Shulman's conceptualization of teachers' knowledge provided a basis for research field of mathematics education. The knowledge that mathematics teachers need to acquire for teaching was described with the Mathematical Knowledge for Teaching (MKT) model which is the refinement of Shulman's categorization (Ball, Thames & Phelps, 2008). MKT model categorize SMK and PCK into six subcomponents. Ball (2003) defined the subcomponents of PCK by reconsidering Shulman's categorization. The components are Knowledge of Content and Students, Knowledge of Content and Teaching and Knowledge of Curriculum.

Although MKT model has been widely used model, there are some criticism about it. This model was developed considering elementary and middle school mathematics teachers but not secondary. So, it is argued that components of MKT do not meet the mathematical need for secondary mathematics teachers (Zazkis & Leikin, 2010). The claim is that "the higher the level taught, the more the teacher needs to know" (Usiskin, 2001, p. 86) so the nature of mathematics that secondary teachers need to know is at the much higher level than elementary teachers. According to Zazkis and Leikin (2010), Advanced Mathematical Knowledge (AMK) which is defined as knowledge of subject matter acquired during undergraduate studies at universities, is necessary knowledge for teaching mathematics in secondary level. So, it can be said that since generally SMK is prerequisite for PCK (Shulman, 1986), specifically in secondary level AMK is also necessary for PCK. However, it is not sufficient because PCK includes the knowledge of content and teaching and knowledge of content and students and knowledge of curriculum (Ball et al., 2008). Therefore, classroom experiences and practices are also important for the development of PCK. Researchers argue that there is interaction between SMK, PCK, beliefs and practices (Franke & Fennema, 1992; Walshaw, 2012; Türnüklü, 2005). However, PCK has a special importance because it is influenced by all the others; SMK, practice and belief. It can be said that PCK has multidimensional nature. Wilson (2007) claims that this complex nature makes it difficult to investigate PCK by using efficient measures. Even though developing scalable efficient measures for content knowledge for teaching is difficult (Wilson, 2007), researchers tried to develop rigorous, effective and valid instruments to measure mathematics teachers' knowledge (Hill *et al.*, 2004; Krauss *et al.*, 2008; Tatto *et al.*, 2008).

One of the mathematics teacher knowledge instruments is Teacher Education and Development Study in Mathematics (TEDS-M) measure. TEDS-M is a cross-national study in which 17 countries participated but Turkey was not involved. The characteristics that differentiate TEDS-M measure than others are to consider both primary and secondary levels and to be designed for international usage and national adaptations. Differences in students' achievement level in Trends in International Mathematics and Science Study (TIMSS) encouraged researchers to study on teacher education internationally in order to investigate how mathematics teaching quality differs across countries. Therefore, TEDS-M measure was developed to examine future mathematics teachers' mathematical knowledge for teaching based on TIMSS 2007 framework of content areas and cognitive domains. By considering such characteristics of the measure, in this study, TEDS-M secondary released items were used for the investigation of secondary mathematics teacher candidates' mathematical knowledge for teaching.

METHODS

Participants

In Turkey both graduates of secondary mathematics teacher education departments and mathematics departments (after completing teaching certificate program) have chance to be mathematics teachers in secondary schools. Therefore, the participants of the study were senior students from secondary mathematics teacher education departments ($n = 47$) and senior student from mathematics departments ($n = 48$) in two universities in Istanbul. Totally, 32 females and 15 males senior secondary mathematics education students (the mean age is 24) and from mathematics departments 35 females and 13 males students (the mean age is 22) were participated in this study. These two universities were ranked as first and second among the secondary mathematics education departments in national university entrance exam. In the first ranked university, students enroll mathematics and secondary mathematics education programs by getting similar scores from university entrance exam. In the second university, minimum score of secondary mathematics education department is little higher than mathematics department.

These two programs have different curriculum in undergraduate education programs. Secondary mathematics education program includes 50 % content knowledge and skills, 30 % professional teaching knowledge and skills and 20 % general knowledge courses (YÖK, 2007). However, undergraduate program in mathematics department consists of 70 % content knowledge and 30 % general knowledge. Moreover, participants of the study were asked to explain whether they had an informal teaching experience like tutoring or teaching in cram school. As they stated, 76 % of secondary mathematics education students and 70 % of mathematics students had informal teaching experiences.

Instrument

The instrument was designed by TEDS-M researchers considering the framework of Trends in International Mathematics and Science Study (TIMSS) 2007 (Tatto *et al.*, 2008). MCK items comprised of four content areas: number, algebra, geometry and data and three cognitive dimensions: knowing, applying and reasoning. Furthermore, MPCK items consist of two parts: knowledge of curricula planning and interactive knowledge about how to enact mathematics for teaching and learning. These were aligned with PCK domains in literature. Table 1 and Table 2 show the distributions of MCK and MPCK items according to content and cognitive domains and PCK components. (Figures 1 & 2 are examples of MCK items and Figure 3 & 4 are examples of MPCK items in Appendix.)

Cognitive Domain	Content Domain				
	Algebra	Geometry	Number	Data	Total
Knowing	-	2	4	-	6

Applying	5	4	-	1	10
Reasoning	2	1	4	-	7
Total	7	7	8	1	23

Table 1. MCK Secondary Items

	Content Domain				
	Algebra	Geometry	Number	Data	Total
Curriculum and Planning	4	-	-	-	4
Enacting	1	-	3	1	5
Total	5	0	3	1	9

Table 2. MPCK Secondary Items

These items include 23 mathematics content knowledge (MCK) and 9 mathematics pedagogical content knowledge (MPCK) items with three different item formats: multiple choice, complex multiple choice and open constructed response.

In order to compare MKT of participants who were studying in different departments Turkish translated versions of TEDS-M secondary level released items were used. The method which was used while translating the instrument consists of three phases. Firstly, items translated in Turkish by the researcher who is fluent in English. The translated items were reviewed by a mathematics educator who is expert in the content area and fluent in English, a three-year experienced mathematics teacher who is fluent in English and a professional translator. At the second phase, the original tests were administered a group of preservice mathematics teachers who are native in Turkish and fluent in English. The same group took the translated versions of tests 3 weeks apart. At the last phase, the method of back translation was used to check the quality of translation and to investigate linguistic or conceptual errors in translation. Also it was used to consider particular attention to sensitive translation problems across cultural correspondence of the two versions.

Data Collection and Analysis

The data was collected from participants in a single point in different times. Instrument administered to senior students during the last two weeks of spring semester of 2012-2013 academic year just before they graduate.

After data collection all items were scored according to scoring guide of TEDS-M Secondary Items. Participants' scores acquired from 23 MCK items were calculated and called as MCK scores and scores obtained from 9 MPCK items were calculated and called as MPCK scores. Total scores of participants were also calculated by the summation of MCK and MPCK scores.

These two groups of participants' scores were compared by using appropriate statistical methods. For total scores and MCK scores comparisons independent sample t-test was used since the all assumptions were met. For the comparison of MPCK scores non-parametric Mann-Whitney U test was used since the normality assumption was violated.

RESULTS

Participants' scores that obtained from 47 senior students from mathematics teacher education department and 48 senior students from mathematics department were compared. Table 3 shows means and standard deviations of two groups of participants.

		M	*SD*
Total	Math Teacher Education	26.83	3.96
	Math	23.63	4.42
MCK	Math Teacher Education	20.45	3.35
	Math	17.50	3.80
MPCK	Math Teacher Education	6.38	1.19
	Math	6.13	1.35

Table 3. Means and Standard Deviations

The results of t-test indicate that the total mean score of mathematics teacher education students is significantly 3.2 points higher than those from mathematics department, t (93) = 3.72, $p < .001$ and Cohen's $d = .76$ with the marginal large effect size (Cohen, 1988). Also, independent sample t-test results showed that students from mathematics teacher education have significantly higher MCK scores than mathematics students, t (93) = 4.00, $p < .001$, *Cohen's d* = .82 with the large effect size (Cohen, 1988). Moreover, according to Non-parametric Mann-Whitney U test there is no significant difference between them according to MPCK scores, $Z = 1.00$, $p > .05$.

DISCUSSION AND CONCLUSION

The study aimed to compare the mathematical knowledge for teaching of students who will graduate from mathematics teacher education department and who will graduate from mathematics department. In Turkey, graduates of both departments have chance to be mathematics teachers at secondary level but the graduates of mathematics departments need to take teaching certificate before teaching at public

schools. However, the knowledge and skills that graduates are able to acquire through these programs are different than each other. For example, the contents of undergraduate education programs of these departments are notably different. Mathematics departments' program does not consist of any pedagogy or education courses but includes more advanced mathematics courses than mathematics education departments' program. Therefore, the result was an unexpected that mathematics students, who were not required to take any teaching related courses, were not significantly different than students from mathematics teacher education in terms of MPCK scores.

This unexpected result may be explained by discussing the nature of PCK for secondary level mathematics teaching. Even though, teacher education programs are the most influential factors that affect PCK of teacher candidates, there are other factors when the nature of PCK considered. PCK includes knowledge of "the ways of subject that comprehensible to others" (Shulman, 1986, p. 9). It may be conceptualized not only as knowledge of students' thinking and conceptions, but also knowledge of explanations, representations and alternative definitions of mathematical concepts, and knowledge of multiple solutions to mathematical tasks (Shulman, 1986; Ball *et al.*, 2008; Krauss, Baumert, & Blum, 2008). So, teaching experiences play an important role in the development of teachers' PCK (Ball *et al.*, 2008). Because of this, teacher education programs include many teaching experiences opportunities like field experience and practicum. Moreover, both groups of students who were studying mathematics teacher education mathematics department had informal teaching experiences like tutoring and teaching in cram school. Having this kind of teaching experience may explain the result. But this may not be the unique reason. Measuring and assessing PCK is another issue which should be considered by focusing on its nature in order to explain the results of study.

In secondary level, for achieving the specialized knowledge for teaching these kinds of knowledge require Advanced Mathematical Knowledge (AMK) which is defined as the knowledge of the subject matter acquired at universities (Zazkis & Leikin, 2010). Mathematics departments' students take many advanced mathematics courses and they develop their AMK. It should be noted that AMK is necessary but not sufficient condition for achieving the specialized knowledge for teaching in secondary level (Zazkis & Leikin, 2010).

Therefore, as it is seen, in PCK's multidimensional nature, deep mathematical knowledge plays an important role because it can provide teachers to use effective explanations, representations and alternative definitions. These components may contribute to make an explanation for the unexpected result of the study. For example, when MPCK items were examined according to required knowledge and skills needed to give correct answer, the need for AMK might be observed. One of the questions in secondary instrument (Figure 3 in appendix) asks to determine

whether the knowledge is needed to prove the quadratic formula. This question measures knowledge of content and teaching but without knowing how to prove quadratic formula it is not possible to give correct answer. So, it is not easy to differentiate and measure this kind of knowledge and skills. Difficulty in measuring PCK may explain the unexpected result that there is no difference in MPCK scores between two groups of students.

Moreover, in this study PCK was tried to be measured by a few items (4 questions, 8 items). Therefore, only some domains of PCK and some abilities were able to be measured with these items. However, as Shulman (1986) and Ball *et al.* (2008) stated, PCK requires different kinds of knowledge, tasks and skills. This instrument can only address some of them. Table 2 shows the distribution of content and PCK domains of items, and below table (Table 3) shows the intended abilities for each items.

Questions (Items)	Content Domain	PCK Domain	Intended Ability
1 (b)	Algebra	Enacting	Analyze why one word problem is more difficult than the other.
6 (a, b, c)	Number	Enacting	Determine whether student's response is valid proof.
9 (a, b, c, d)	Algebra	Curriculum and Planning	Determine if knowledge is needed to prove the quadratic formula.
12 (b)	Data	Enacting	Explain student's thinking about histogram.

Table 3. TEDS-M Secondary PCK Items' Characteristics

Two different groups of participants' reactions to these PCK items are different. For example, item 9b (Figure 1) were answered correctly by 97 % of mathematics department students and 86 % of students from secondary mathematics education department. On the other hand, 72 % of students from secondary mathematics education department answered item 1b (Figure 4 in appendix) correctly, when 52 % of mathematics department student gave correct response.

APPENDIX

Determine whether each of the following is an irrational number always, sometimes or never. *Check one box in each row.*			
	Always	**Sometimes**	**Never**
A. The result of dividing the circumference of a circle by its diameter.	$\square_1$	$\square_2$	$\square_3$
C. The diagonal of a square with side of length 1.	$\square_1$	$\square_2$	$\square_3$
D. Result of dividing 22 by 7.	$\square_1$	$\square_2$	$\square_3$

Figure 1. An example of TEDS-M Secondary MCK items (Number, Knowing)

Prove the following statement:

If the graphs of linear functions
$$f(x) = ax + b \text{ and } g(x) = cx + d$$
intersect at a point P on the x-axis, the graph of their sum function
$$(f + g)(x)$$
must also go through P.

Figure 2. An example of TEDS-M Secondary MCK items (Algebra, Reasoning)

A mathematics teacher wants to show some <lower secondary school> students how to prove the quadratic formula.

Determine whether each of the following types of knowledge is needed in order to understand a proof of this result.

Check one box in each row.

		Needed	Not needed
A.	How to solve linear equations.	$\square_1$	$\square_2$
B.	How to solve equations of the form $x^2 = k$, where $k > 0$.	$\square_1$	$\square_2$
C.	How to complete the square of a trinomial.	$\square_1$	$\square_2$
D.	How to add and subtract complex numbers.	$\square_1$	$\square_2$

Figure 3. An example of TEDS-M Secondary MPCK items (Algebra, Planning)

(b) Typically Problem 2 is more difficult than Problem 1 for <lower secondary> students. Give one reason that might account for the difference in difficulty level.

Figure 4. An example of TEDS-M Secondary MPCK items (Algebra, Enacting)

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THE EPISTEMOLOGICAL DIMENSION IN DIDACTICS: TWO PROBLEMATIC ISSUES

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This paper presents some theoretical considerations concerning the relationships between epistemology and didactics. We distinguish two big issues that show the mutual enrichment of both fields. On the one hand, considering teaching and learning phenomena as part of the empirical basis of epistemology enables proposing new epistemological models of mathematical bodies of knowledge. On the other hand, these epistemological models provide guidelines for the design and analysis of new teaching proposals, which, in turn, show the constraints coming from the spontaneous epistemologies in school institutions. Some critical open questions derived from these issues draw up the guidelines for a future research programme.

PEDAGOGY, EPISTEMOLOGY AND DIDACTICS

What distinguishes didactics of mathematics (or of any other field of knowledge) from general education or pedagogy is the status given to the knowledge or "content" that is taught and learnt. Pedagogy considers the knowledge to be taught as a given, and focuses on the best, conditions or practices to teach and learn it: the knowledge is not problematic, the relationships of the students to it are (Chevallard 1999). In contraposition, didactics locates the *epistemological problem* at the core of the analysis. A double assumption is meant by this. First, that phenomena underlying teaching and learning processes, at school as well as in other social institutions, are closely dependent on the content that is designed to be taught, actually taught and learnt, and also on how this content is considered by the participants of the teaching and learning process. Second, that the study of these phenomena is also strongly dependent on the way knowledge is considered and modelled by researchers in didactics. In fact, the main point of the paper is to describe a research programme that wants to clarify the problem of how to teach (and learn) mathematics and its relationship with the problem of what is considered as mathematics.

The pedagogical dimension of teaching and learning phenomena refers to generic practices, discourses, strategies and regularities that can be described regardless of the content to be taught. The didactic dimension is reached when the concrete mathematical activities organised by the teacher and carried out by the students, as well as any other fact affecting the delimitation, construction, management, evolution and assessment of these activities are considered.

The need to integrate both the pedagogical and the epistemological dimensions was one of the main motivations for Guy Brousseau to promote the construction of a new field of knowledge called didactics of mathematics and which he contributed to with the first formulations of the theory of didactic situations during the decade 1970-1980 (Brousseau 1998). We are here pointing out two main reasons for this integration that will later on be at the basis of the problematic issues we wish to raise. The first one is the dependence between the dominant *epistemologies of mathematics* (or of any other field of knowledge) at an educational institution and the way teaching is organised in this institution. In other words, the way mathematics and its specific bodies of knowledge are considered in a given institution, usually as implicit assumptions, affects the conditions established for its learning. In this sense, we can say, rephrasing Brousseau (1998), that *teaching organisations* are supported by *spontaneous epistemologies* appearing to the subjects of the school institution as the unquestionable and transparent way to conceive the content to be taught.

The second main reason, also put forward by Brousseau, is related to the implementation of research results wishing to improve teaching and learning. Whatever general strategies or conditions we may find at the pedagogical level, teachers will always have to specify them in terms of what ties them to the students: the knowledge-based learning activities. Of course, it is possible to delimit general pedagogical phenomena affecting any content to be taught and to propose general pedagogical actions in order to improve teaching and learning processes. However, eventually, these actions will need to be concretized and converted into didactic facts and strategies, that is, to specific ways of organizing mathematical contents and designing mathematical activities for the students.

Once the necessity to integrate the "epistemological problem" into the "teaching and learning problem" is assumed, there are different levels to take the epistemic dimension of the teaching and learning process into account. In some cases, the focus can be on a given piece of knowledge ("proportionality", "limits of functions", "linear equations"), or a whole area ("algebra", "calculus", "statistics"), thus considering specific models relying on a more or less explicit conception of what mathematics is and how it can be described. Therefore it can be said that the consideration of the *didactic problem* needs to include, in one way or another, a specific answer to the *epistemological problem*. To properly interpret the deep interrelation between the epistemological and didactical problems, we will now present a historical development of the object of study of each discipline and their respective empirical basis.

THE EVOLUTION OF THE EPISTEMOLOGICAL AND THE TEACHING PROBLEMS

In a previous study, Gascón (2001) describes a rational reconstruction of the evolution of the *epistemological problem* and, in parallel, the evolution of the

didactic problem showing that a certain convergence exists between them. The evolution of the epistemological problem can be interpreted as a successive expansion of what is considered as the object of study and of the consequent empirical basis used to approach it. Briefly speaking, this work shows that the nature of the epistemological problem began as a purely logical problem (EP^1), became a historical problem (EP^2) and ended up being considered, at the end of the last century, as an essentially cognitive problem (EP^3). Its successive formulations together with its corresponding tentative answers can be outlined as follows:

EP^1: How to stop the infinite regress to get a logical justification of mathematical theories?
EPA^1: Euclidean models: logicism (Rusell), formalism (Hilbert) and intuitionism (Brower).
EP^2: What is the logic of the development of mathematical discovery?
EPA^2: Quasi-empirical models (Lakatos)
EP^3: What are the tools and mechanisms found in history and psychogenesis of the development of mathematical discovery?
EPA^3: Constructivist models (Piaget & García, 1982).

This evolution of the epistemological problem can be interpreted as a progressive detachment from logical procedures and an approximation to empirical sciences such as history and psychology. This expansion continues since the 70s, with the inclusion of sociological data. Indeed, sociologists such as Barry Barnes and David Bloor, and later others like Bruno Latour, heavily influenced by the ideas of Thomas Kuhn, tried to highlight the essential social nature of scientific research. Let us notice, however, that, apart from Kuhn's mention of the "textbooks epistemology" (Khun, 1979), none of these approaches seem to consider empirical phenomena related to the teaching, learning and disseminating of mathematics. The division between pedagogy and epistemology appears to be taken for granted in this research domain also.

All of the previously mentioned epistemological models can be related to general teaching models, ranging from *theoricism* (organizing the teaching of mathematics following the logic construction of concepts) and *technicism* (exerciting the main techniques in a given domain without many theoretical tools)*, to *constructivism,* which aims to enable students to construct knowledge according to certain predetermined stages.

Gascón (1993) shows some limitations of the empirical basis used by constructivism to address the epistemological problem. In fact, while taking into account personal psychogenesis data, in some sense completed with those provided by the history of science, it does not integrate *didactic facts* and, can thus hardly explain institutional-depending phenomena as the so-called "personal" construction of knowledge. In other words, and according to Chevallard (1991), the study of the *genesis and*

development of knowledge (traditional object of epistemology) cannot be separated from the study of the *diffusion, use and transposition* of knowledge (object of study of didactics).

It is at this point where both problems, the epistemological and the didactic one, converge, with the consequently significant expansion of the object of study of both disciplines. Historically, this time corresponds to the first formulations of the theory of didactic situations (TDS) proposed by Guy Brousseau in the early 1970s (Brousseau, 1998). It is no coincidence that at this early stage of didactics of mathematics, Brousseau initially considered to name this new discipline "experimental epistemology". In particular, didactics of mathematics accepted the responsibility to elaborate and use epistemological models of mathematical bodies of knowledge as a new way to study didactic phenomena, thus turning the *pedagogical problem* into an *epistemological-didactic* one.

New questions arise from this perspective: What new general epistemological theories, based on which empirical data, may serve to support new teaching organizations in order to overcome the limitations of the current ones? To what extent and by what means can the dominant spontaneous epistemologies in a teaching institution be changed in solidarity with the teaching models based on them?

AN ANSWER TO THE EPISTEMOLOGICAL-DIDACTIC PROBLEM

The anthropological theory of the didactic (ATD), following the research programme initiated by the theory of didactic situations, considers a specific model of mathematical knowledge and its evolution formulated in terms of a dynamical sequence of *praxeologies*. Praxeologies are entities formed by the inseparable combination of a *praxis* or know-how made of types of tasks and techniques, and of a *logos* or knowledge consisting of a discourse aiming at describing, explaining and justifying the *praxis* (Chevallard, 1999). In didactics research, mathematical praxeologies are described using data from the different institutions participating in the didactic transposition process, thus including historical, semiotic and sociological research, assuming the institutionalized and socially articulated nature of praxeologies. Furthermore, a dialogue with the APOS theory shows how data interpreted as the different *levels of development of schemes* by psychogenetic developments, can be reformulated in ATD in terms of the *institutional evolution of praxeologies* (Trigueros, Bosch, & Gascón, 2011).

Reference epistemological models as sequences of praxeologies

To describe and analyse the specific contents that are at the core of teaching and learning processes, the general model in terms of praxeologies is structured in an articulated set of specific models of the different areas of the mathematical activity at stake called *reference epistemological models* (REM) (Barbé, Bosch, Gascón, & Espinoza, 2005; Bosch & Gascón, 2006). The Reference Epistemological Model of a

body of knowledge is an alternative description of that body of knowledge elaborated by researchers in order to question and provide answers to didactic facts and problematic aspects taking place in a given institution. This REM prevent researchers to take for granted how this body of knowledge is conceived in the institution considered. For instance, Ruiz-Munzón (2010) and Ruíz-Munzon, Bosch, & Gascón (2013) present a REM about elementary algebra which is used to analyse the status and role of this area of school mathematics in relation to arithmetic and functional modelling. The model takes into account the processes of didactic transposition to explain what is currently taught as algebra at school and provides a rationale to this area that does not coincide with the official and more limited one assigned by the educational system. Some of the difficulties in the teaching and learning of elementary algebra can then be referred to these limitations and new teaching proposals can be designed to overcome them (Ruiz-Munzón, 2010; Bosch, 2012).

In this REM, algebra is interpreted as a *tool for modelling* any type of (mathematical and extra-mathematical) systems and the process of *algebraization* is divided into three stages. The first one concerns the passage from the execution of computation programmes (sequences of arithmetic operations on numbers like the ones carried out when solving an arithmetic problem) to the written or rhetoric description of their structure; the second stage requires the symbolic manipulation of the global structure of written computation programmes (not only simplifying and developing, but also "cancelling", etc.); at the third stage, the whole manipulation of formulas is reached.

It is important to note that this REM is not a static description of a piece of mathematical knowledge, it also suggests a dynamical process to introduce elementary algebra: starting from the study of arithmetic computation programmes (CP) in order to motivate the entrance into the second stage of algebraization by the limitations of the rhetorical formulation of CPs in the first stage. Encountering problematic questions in this arithmetical work with CP may generate the need to build a written symbolic *formulation* of these CP to globally manipulate their structure, thus promoting the need to establish symbolic codes (hierarchy of operations and bracket rules).

In a similar way, different REM of other specific areas of mathematics have been proposed, all formulated in terms of sequences of related praxeologies: limits of functions (Barbé et al 2005), proportionality (García, 2006); (Hersant, 2001), measure of quantities (Chambris, 2010), real numbers (Bergé, 2008), (Rittaud & Vivier, 2013), among others. In general, the organisation of a teaching process based on the REM of a given mathematical content is called *research and study activities*.

From teaching of contents to enquiry processes: study and research paths

These reference epistemological models correspond to previously established bodies of mathematical knowledge: algebra, limits, proportionality, etc. They provisionally

assume the delimitations of mathematical knowledge provided by the school and the scholarly institutions, which are then often redefined. In order to also take into account enquiry processes that start with the consideration of problematic questions to be solved (instead of pre-established contents to be learnt), REM have been enriched with the proposal of the *Herbartian schema* (Chevallard, 2006) (Chevallard, in press). This scheme is a useful tool to observe, analyse and evaluate existing and potential didactic processes that start with the consideration of a generating question and evolve with the search of partially available answers ("contents" to be learnt) and the construction of new answers through the interaction with a milieu.

The study of a specific *question* leads to a rooted-tree of derived questions and provisional answers, which outlines the generating power of the initial question and the possible paths to follow. We thus obtain new reference epistemological models assigned to problematic questions instead of pre-established praxeological contents. Winsløw, Matheron, & Mercier (2013) provide several examples of this kind of rooted-tree REM, such as the dynamics of a population or the trajectory of a three-point shot in basketball. The enquiry process of a particular generating question materializes in an open didactic organisation called a *study and research path* (SRP). During the development of SRP, the need for new knowledge to solve some of the derived questions found in the path usually leads to the activation of study and research *activities*.

Didactic praxeologies emerging from reference epistemological models

The previous section briefly outlined how the design, implementation and analysis of study and research paths and study and research activities call for the activation of specific didactic techniques and creates new types of didactic tasks. For instance, in the case of elementary algebra illustrated above, the didactic technique proposed by Ruiz-Munzon (2010) consists in introducing the study of "mathemagic" games of the sort "Think of a number, apply these calculations […], you get 73" as generating questions. How do you explain the magician's trick?" These games generate the need to look for new pieces of answers, in the manipulation of the calculation programmes proposed or in their transformation and generalisation through algebraic symbolism. Questions based on "mathemagic" games allow producing an important number of computation programmes economically. They are presented to the students without much artificiality and their first contact with computation programmes is not problematic. Moreover, the limitations of the rhetorical and numerical formulations of computation programmes inevitably appear and they do so soon enough.

PROBLEMATIC ISSUES

The aim of this paper is to formulate some problematic issues at the crossroads of epistemology and didactics. We will initially explain them within the context of the ATD before extending the questioning to other didactic approaches. If we try to

characterize a didactic approach by how "pedagogy" and "mathematics" are integrated, in the case of the ATD such integration can be formulated in terms of two movements. They appear in the design, management and evaluation of teaching and learning processes and can briefly be described as follows:

(1) Starting from the analysis of teaching and learning processes at school and considering an empirical basis of study that is large enough to include the processes of didactic transposition, all this empirical work provides tools to design specific REMs for the main mathematical contents or areas that are designed as knowledge to be taught. We can define this movement as "*using didactic facts and phenomena to produce epistemological models*".

(2) Conversely, the principles and criteria that have guided the construction of a REM for a specific area of school mathematical activity and, in particular, the contrast between the rationale assigned by the REM to this area and its official (explicit or tacit) role in school mathematics, all provide some mathematical and didactic tools to design, manage and evaluate teaching and learning processes based on study and research paths sustained by that REM. This movement can be defined as *"using the epistemological model as the core of didactic tools"*.

This double movement raises different open issues which are at the starting point of the research programme we want to propose in this paper.

New didactic needs

We have seen how previously elaborated REM on mathematical contents or problematic questions (obviously complemented with other methodological design tools) can provide criteria for the design and implementation of teaching and learning processes that are considerably different from the existing ones. In principle, they aim at organising activities that should allow the students to carry out new mathematical tasks and techniques in a more autonomous, functional and justified way. The "mathemagic" games in the case of elementary algebra (Ruiz-Munzón, 2010) or the different enquiry processes described in Winsløw, Matheron, & Mercier (2013) are good examples of this enrichment. Obviously, these new didactic organisations should be made available to the study community and their viability in different school institutions should be tested.

It is important to emphasize that all didactic approaches and theories are also based on general models of mathematical-didactic activities. These general models are a particular way to interpret the mathematical activity and to conceptualize the study process of mathematics (teaching, learning, diffusion and application). Even though these models are not always clearly spelled out, they remain an essential feature of theoretical approaches, as they strongly affect the type of research problems this approach can formulate. Two crucial questions arise:

- In the case of ATD, how to transform the REM into possible didactic organisations that could live in current school institutions? How to take into account the interrelation between the REM and the didactic phenomena appearing in the implementation of these new didactic organisations? How to make this process available to the school institutions, especially to the profession of teachers?
- How is this mutual enrichment between the epistemological and didactic proposals taken into account in other theoretical frameworks?

New epistemological needs

The empirical analysis of the study processes taking place in various institutions (for example, but not exclusively, in schools) clearly shows that the didactic praxeologies are closely related to the epistemological tools available in the institution to describe and manage the mathematical praxeologies. For example, in the institutions where the dominant model is Euclidean, teaching and learning processes are conceived and described in terms of didactic activities around "definitions", "concepts", "theorems", "proofs" and "applications". In addition, these didactic activities tend to be hierarchically structured according to the logical construction of mathematical concepts (real numbers before limits, limits before derivatives, etc.).

If, instead of analysing traditional teaching processes, we look at those based on didactic research, the situation is very similar: how didactic processes and the dynamics of mathematical praxeologies are designed, described and managed also depends on the tools provided by the epistemological model which upholds, more or less explicitly, the didactic approach considered. The further this research-based epistemological model is from the dominant epistemological model at schools and scholarly institutions, the more difficult it becomes for teachers to carry out innovative teaching proposals designed within this frame.

In all these cases, the most remarkable feature is the shortage and inadequacy of tools available in the teaching institution to describe, manage, and evaluate the dynamics of mathematical activity. This lack of tools could in the first place be attributed to the scarcity of spontaneous epistemological models and, in particular, to the shortage of the Euclidean epistemological model of mathematics whose supremacy is still present, to a greater or lesser extent, in most institutions.

- Which new notions or tools are needed to describe and manage the dynamics of the mathematical activity that will take place in study processes? How to describe these tools depending on the role addressed (didactic researcher, teacher and students)? How to make them available in the teaching institution and to the participants of the didactic process?

The evolution of didactic-epistemological models

In order to establish an alternative and rich enough REM of a specific mathematical domain or questioning, it is necessary to take into account the didactic phenomena taking place in teaching institutions. This leads to an enrichment of the spontaneous epistemological model during the first design of the REM. However, it is important to keep the process running during the implementation and the evaluation of teaching proposals based on this REM. The consequent evolution of the REM is a clear example of the dynamic and provisional nature of the epistemological models elaborated by didactics, evolving from its initial proposals through the analysis of empirical facts.

From a mathematical perspective, these continuous evolutions of the REMs can be seen as the incorporation of new notions and organisations into the field of knowledge. This phenomenon can be related to the transformation of some paramathematical notions into mathematical concepts, as happened with concepts (such as "set", "function", "continuity", "graphs", etc.), a transformation which takes place as long as researchers deal with new problems. For instance, in the case of elementary algebra, the notion of "computation programme" is a new and crucial element of the proposed REM. In the experiences described by Ruiz-Munzón (2010), this notion played a very ambiguous role in the management of the teaching and learning processes, given the fact that it did not belong to the official mathematics to be taught and the teacher did not feel at ease with it. A similar phenomenon happened when implementing SRP on population dynamics with notions such as "quantities", "model", "system", "mixed and separated generations", etc.

- Another important and difficult question is the degree of explicitness that should be adopted with the new epistemological models necessary to design, implement and evaluate new teaching and learning processes depending on the participants of the study communities addressed (students, teachers, mathematicians, etc.).
- What kind of similar experiences can be learnt from other approaches? Did they find similar difficulties?

These open questions establish a new research programme where the results of previous investigations carried out within the ATD should be analysed together with analogous research from other perspectives. In all cases, the status given to the epistemological dimension in didactics analysis seems to appear as a crucial question to take into account.

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LINKING INQUIRY AND TRANSMISSION IN TEACHING AND LEARNING MATHEMATICS

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Different theories assume that learning mathematics should be based on constructivist methods where students inquire problem - situations and assign a facilitator role to the teacher. In a contrasting view other theories advocate for a more central role to the teacher, involving explicit transmission of knowledge and students' active reception. In this paper we reason that mathematics learning optimization requires adopting an intermediate position between these two extremes models, in recognizing the complex dialectic between students' inquiry and teacher's transmission of mathematical knowledge. We base our position on a model with anthropological and semiotic assumptions about the nature of mathematical objects, as well as the structure of human cognition.

Key words: mathematical instruction, inquiry learning, knowledge transmission, onto-semiotic approach, mathematical knowledge

INTRODUCTION

The debate between the models of a school that "conveys knowledge" and others in which "knowledge is constructed" currently seems to tend towards the latter. This preference can be seen in the curricular guidelines from different countries, which are based on constructivist and socio-constructive theoretical frameworks:

> "Students learn more and learn better when they can take control of their learning by defining their goals and monitoring their progress. When challenged with appropriately chosen tasks, students become confident in their ability to tackle difficult problems, eager to figure things out on their own, flexible in exploring mathematical ideas and trying alternative solution paths, and willing to persevere" (NCTM, 2000, P. 20).

In the case of mathematics education, problem solving and "mathematical investigations" are considered essential for both students' mathematical learning and teachers' professional development. Constructivist viewpoints of learning shift the focus towards the *processes* of the discipline, practical work, project implementation and problem solving, rather than prioritizing the study of facts, laws, principles and theories that constitute the body of disciplinary *knowledge*.

Nevertheless, this debate is hiding the fact that students differ in skills and knowledge, and most of them need a strong guidance to learn; even when some students with high skills and knowledge can learn advanced ideas with little or no

help. The issue of the type of aid needed, depending on the nature of what is to be built or transmitted is also missed in this debate. Consequently of this situation, the question of the kind of help that a teacher should give to a usually heterogeneous class, when we want students acquire mathematical knowledge, understandings and skills, also arises.

The family of "Inquiry-Based Education" (IBE), "Inquiry-Based Learning" (IBL), and "Problem-Based Learning" (PBL) instructional theories, which postulate the inquiry-based learning with little guidance by the teacher, seem not to take into account the described reality, namely the students' heterogeneity and the variety of knowledge to be studied. These models may be suitable for gifted students, but possibly not for the majority, because the type of help that the teacher can provides could significantly influence the learning, even in talented students.

In this paper we analyse the need to implement instructional models that articulate a mixture of construction/inquiry and transmission of knowledge to achieve a mathematical instruction that locally optimize learning. The basic assumption is that the moments in which transmission and construction of knowledge can take place are *everywhere dense* in the instructional process. Optimization of learning involves a complex dialectic between the roles of teacher as instructor (transmitter) and facilitator (manager), and student's roles as active constructor of knowledge and receivers of meaningful information. "*Because a range of goals might be included in a single lesson, and almost certainly in a multi-lesson unit, the best or most effective teaching method might be a mix of methods, with timely and nimble sifting among them*" (Hiebert & Grouws, 2007, p. 374).

We support this mixed model of mathematical instruction in cognitive (architecture of human cognition) and onto-semiotic (regulative nature of mathematical objects) reasons.

Below we first summarize the main features of instructional models based on inquiry and problem solving and secondly of models that attribute a key role to transmission of knowledge. We then present the case for a mixed model that combines dialectically inquiry and transmission, basing on the epistemological and didactical assumptions of the onto-semiotic approach to mathematical knowledge and instruction (Godino, Batanero & Font, 2007). Finally we include some additional reflections and implications.

INQUIRY AND PROBLEM BASED LEARNING IN MATHEMATICS EDUCATION

As indicated above, the acronyms IBE, IBL, PBL designate instructional theoretical models developed from several disciplines, which have parallel versions for the teaching of experimental sciences (IBSE) and mathematics (IBME). They attributed a key role to solve "real" problems, under a constructivist approach. In some

applications to mathematics education it is proposed that students construct knowledge following the lines of work of professional mathematicians themselves. The mathematician faces non-routine problems, explore, search for information, make conjectures, justify and communicate the results to the scientific community; mathematics learning should follows a similar pattern.

Using problem – situations (mathematics applications to everyday life or other fields of knowledge, or problems within the discipline itself) to enable students making sense of the mathematical conceptual structures is considered essential. These problems are the starting point of mathematical practice, so that problem solving activity, including formulation, communication and justification of solutions are keys to developing mathematical competence, i.e. the ability to cope with not routine problems. This is the main objective of the "problem solving" research tradition (Schoenfeld, 1992), whose focus is on the identification of heuristics and metacognitive strategies. It is also essential to other theoretical models such as the Theory of Didactical Situations (TDS) (Brousseau, 1997), and Realistic Mathematics Education (RME) (Freudenthal, 1973; 1991), whose main features are described below.

Theory of Didactical Situation (TDS)

In TDS, problem - situations should be selected in order to optimize the adaptive dimension of learning and students' autonomy. The intended mathematical knowledge should appear as the optimal solution to the problems; it is expected that, by interacting with an appropriate *milieu*, students progressively and collectively build knowledge rejecting or adapting their initial strategies if necessary.

> The intellectual work of the student must at times be similar to this scientific activity. Knowing mathematics is not simply learning definitions and theorems in order to recognize when to use and apply them. We know very well that doing mathematics properly implies that one is dealing with problems. We do mathematics only when we are dealing with problems—but we forget at times that solving a problem is only a part of the work; finding good questions is just as important as finding their solutions. A faithful reproduction of a scientific activity by the student would require that she produce, formulate, prove, and construct models, languages, concepts and theories; that she exchange them with other people; that she recognize those which conform to the culture; that she borrow those which are useful to her; and so on. (Brousseau, 2002, p. 22).

To allow such activity, the teacher should conceive problem - situations in which they might be interested and ask the students to solve them. The notion of *devolution* is also related to the need for students to consider the problems as if they were their own and take responsibility for solving them. The TDS assumes a strong commitment with mathematical epistemology, as reflected in the meaning attributed to the notion of fundamental situation: *"a situation which makes clear the raison d'être of the mathematical knowledge aimed at"* (Artigue & Blomhoj, 2013, p. 803).

Another important feature of the TDS is the distinction made between different dialectics: action, formulation and validation, which reflect important specificities of mathematical knowledge.

Realistic Mathematics Education (RME)

In RME, principles that clearly correspond to IBME assumptions are assumed. Thus, according to the "activity principle", instead of being receivers of ready-made mathematics, the students, are treated as active participants in the educational process, in which they develop themselves all kinds of mathematical tools and insights. According to Freudenthal (1973), using scientifically structured curricula, in which students are confronted with ready-made mathematics, is an 'anti-didactic inversion.' It is based on the false assumption that the results of mathematical thinking, placed on a subject-matter framework, can be transferred directly to the students. (Van den Heuvel-Panhuizen, 1996).

The principle of reality is oriented in the same direction. As in most approaches to mathematics education, RME aims at enabling students to apply mathematics. The overall goal of mathematics education is making students able to use their mathematical understanding and tools to solve problems. Rather than beginning with specific abstractions or definitions to be applied later, one must start with rich contexts demanding mathematical organization or, in other words, contexts that can be mathematized. Thus, while working on context problems, the students can develop mathematical tools and understanding. The guidance principle stresses also the same ideas. One of Freudenthal's (1991) key principles for mathematics education is that it should give students a 'guided' opportunity to 're-invent' mathematics. This implies that, in RME, both the teachers and the educational programs have a crucial role in how students acquire knowledge.

"*RME is thus a problem-solving approach to teaching and learning which offers important constructs and experience for conceptualizing IBME*" (Artigue & Blomhoj, 2013, p. 804).

TRANSMISSION BASED LEARNING IN EDUCATION

We consider as models based on knowledge transmission various forms of educational intervention in which the direct and explicit instruction is highlighted. A characteristic feature of strongly guided instruction is the use of worked examples, while the discovery of the solution to a problem in an information-rich environment is similarly a compendium of discovery learning minimally guided.

For several decades these models were considered as inferior and undesirable regarding to different combinations of constructivist learning (learning with varying degrees of guidance, support or scaffolding), as shown in the initiatives taken in different international projects to promote the various IBSE and IBME modalities

(Dorier & Garcia, 2013; PRIMAS project). Transmission of knowledge by presenting examples of solved problems and the conceptual structures of the discipline is ruled by didactical theories in mathematics education with strong predicament, as mentioned in the previous section.

The uncritical adoption of constructivist pedagogical models can be motivated by the observation of the large amount of knowledge and skills, in particular everyday life concepts, that individuals learn by discovery or immersion in a context However, Sweller, Kirschner and Clark (2007, p. 121) state that,

> "There is no theoretical reason to suppose or empirical evidence to support the notion that constructivist teaching procedures based on the manner in which humans acquire biologically primary information will be effective in acquiring the biologically secondary information required by the citizens of an intellectually advanced society. That information requires direct, explicit instruction".

This position is consistent with the argument put forward by Vygotsky; scientific concepts do not develop in the same way that everyday concepts (Vygotsky, 1934). These authors believe that the design of appropriate learning tasks should include providing students an example of a completely solved problem or task, and information on the process used to reach the solution. "*We must learn domain-specific solutions to specific problems and the best way to acquire domain-specific problem-solving strategies is to be given the problem with its solution, leaving no role for IL*" (Sweller, Kirschner, & Clark, 2007, p. 118). According to Sweller et al., empirical research of the last half century on this issue provides clear and overwhelming evidence that minimal guidance during instruction is significantly less effective and efficient than a guide specifically designed to support the cognitive process necessary for learning. "*We are skillful in an area because our long-term memory contains huge amounts of information concerning the area. That information permits us to quickly recognize the characteristics of a situation and indicates to us, often unconsciously, what to do and when to do it*" (Kirschner, Sweller, & Clark, 2006, p. 76).

STUDYING MATHEMATICS THROUGH AN INQUIRY AND TRANSMISSION BASED DIDACTICAL MODEL

In the two previous sections we described some basic features of two extreme models for organizing mathematics instruction: discovery learning versus learning based on the reception of knowledge (usually regarded as traditional whole-class expository instruction). In this section we describe the characteristics of an instructional model in which these two models are combined: the students' investigation of problem -

situations with explicit transmission of knowledge by the "teacher system"[1] at critical moments in the mathematical instruction process. We consider that it is necessary to recognize and address the complex dialectic between inquiry and knowledge transmission in learning mathematics. In this dialectic, *dialogue* and *cooperation* between the teacher and the students (and among the students themselves), regarding the situation - problem to solve and the mathematical content involved, can play a key role. In these phases of dialogue and cooperation, moments of transmitting knowledge necessarily happen.

The onto-semiotic complexity of mathematical knowledge and instruction

The semiotic, epistemological and cognitive assumptions of the Onto-semiotic approach to mathematical knowledge and instruction (OSA) (Godino, Batanero & Font, 2007) are the basis for our instructional proposal, which recognizes a key role to both the inquiry and the transmission of knowledge in the teaching and learning of mathematics (and possibly other disciplines). This model takes into account the nature of mathematical objects involved in mathematical practices whose students' competent performance is intended.

The way a person learns something depends on what has to be learned. According to the OSA, students should appropriate (learn) the onto-semiotic institutional configurations involved in solving the proposed problem – situations. The paradigm of "questioning the world" proposed by the Anthropological Theory of Didactics (TAD) (Chevallard, 2013), and, in general, by IBE models is assumed, so that the starting point should be the selection and inquiry of "good problem – situation."

The key notion of the OSA for modelling knowledge is the *onto-semiotic configuration* (of mathematical practices, objects and processes) in its double version, institutional (epistemic) and (cognitive). In a training process, the student's performance of mathematical practices related to solving certain problems, brings into play a conglomerate of objects and processes whose nature, from the institutional point of view is essentially normative (regulative)[2] (Font, Godino, & Gallardo, 2013). When the student makes no relevant practices, the teacher should guide him/her to those expected from the institutional point of view. Thus each object type (concepts, languages, propositions, procedures, argumentations) or process (definition, expression, generalization, ...) requires a focus, a moment, in the study process. In

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[1] This system can be an individual teacher, a virtual expert system, or the intervention of a "leader" student in a team working on a collaborative learning format.

[2] This view of mathematical knowledge is consistent with that taken by Radford's objectification theory. *"Knowledge, I just argued, is crystallized labor - culturally codified forms of doing, thinking and reflecting. Knowing is, I would like to suggest, the instantiation or actualization of knowledge (Radford, 2013, p.16). Precisely ... Objectification is the process of recognition of that which objects us - systems of ideas, the cultural meanings, forms of thinking, etc."* (p. 23). In our case, such crystallized forms of work are conceived as cultural "rules" fixing ways of doing, thinking and saying faced to problem - situations that demand an adaptive response.

particular regulative moments (institutionalization) are *everywhere dense* in the mathematical activity and in the process of study, as well as in the moments of formulation / communication and justification.

Performing mathematical practices involves the intervention of previously known objects to understand the demands of the problem – situation and implementing an initial strategy. Such objects, its rules and conditions of application, must be available in the subject's working memory. Although it is possible that the student him/herself could find such knowledge in the "workspace", there is not always enough time or the student could not succeed; so the teacher and peers can provide invaluable support to avoid frustration and abandonment. These are the moments of remembering and activation of prior knowledge, which are generally required throughout the study process. Remembering moments can be needed not only in the exploratory / investigative phase, but also in the formulation, communication, processing or calculation, and justification of results phases. These moments correspond to acts of knowledge transmission and may be crucial for optimizing learning.

Results of mathematical practices are new emerging objects whose definitions or statements have to be shared and approved within the community at the relevant time of institutionalization carried out by the teacher, which are also acts of knowledge transmission.

Inquiry and transmission didactical moments

Under the OSA framework other theoretical tools to describe and understand the dynamics of mathematics instruction processes have been developed. In particular, the notions of *didactical configuration* and *didactical suitability* (Godino, Contreras & Font, 2006; Godino, 2011). A didactical configuration is any segment of didactical activity (teaching and learning) between the beginning and the end of solving a task or problem – situation. Figure 1 summarizes the components and the internal dynamics of a didactical configuration, including the students' and the teacher's actions, and the resources to face the joint study of the task.

The problem – situation that delimits a didactical configuration can be made of various subtasks, each of which can be considered as a sub-configuration. In every didactical configuration there is an epistemic configuration (system of institutional mathematical practices, objects and processes), an instructional configuration (system of teacher and learners roles and instructional media), and a cognitive configuration (system of personal mathematical practices, objects and processes) which describe learning. Figure 1 shows the relationships between teaching and learning, as well as with the key processes linked to the onto-semiotic modelling of mathematical knowledge (Font, Godino & Gallardo, 2013; Godino, Font, Lurduy, & Wilhelmi, 2011). Such modelling, together with the teachers and learners roles, and their interaction with technological tools, suggest the complexity of the relationships

established within any didactical configuration, which cannot not be reduced to merely inquiry and transmission moments.
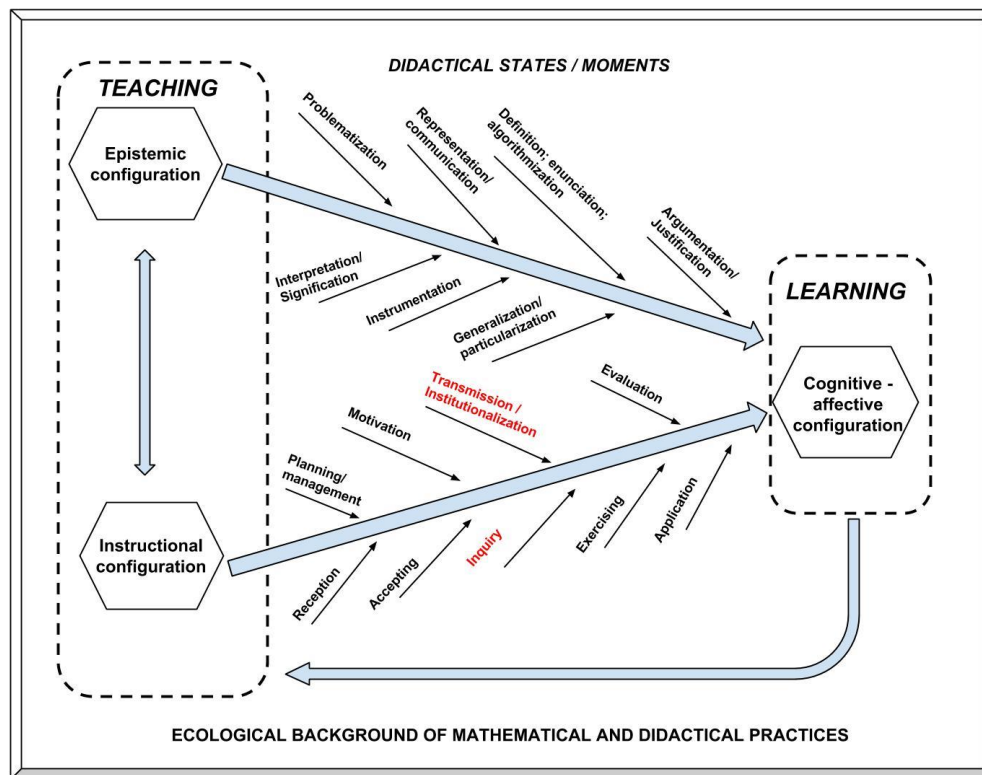

Figure 1: Components and internal dynamic of a didactical configuration

SYNTHESIS AND IMPLICATIONS

In this paper we argued that instructional models based only on inquiry, or only on transmission are simplifications of an extraordinarily complex reality: the teaching and learning processes. "*Classrooms are filled with complex dynamics, and many factors could be responsible for increased student learning. (…) This is a very central and difficult question to answer*." (Hiebert & Grouws, 2007, p. 371)

Although we need to establish instructional designs based on the use of rich problem - situations, which guide the learning and decision-making at the global and intermediate level, local implementation of didactical systems also requires special attention to managing the students' background needed for solving the problems, and to the systematization of emerging knowledge. Decisions about the type of help needed essentially have a local component, and are mainly teacher's responsibility; he /she needs some guide in making these decisions to optimize the didactical suitability of the study process.

We also have supplemented the cognitive arguments of Kirschner, Sweller, and Clark (2006) in favour of models based on the transmission of knowledge in the case of mathematical learning, with reasons of onto-semiotic nature: What students need to

learn are in a great deal, *mathematical rules*, the circumstances of its application and the required conditions for a proper application. The learners start from known rules (concepts, propositions, and procedures) and produce others rules that should be shared and compatible with those already established in the mathematical culture. Such rules (knowledge) must be stored in subject's long term memory and put to work at the right time in the short-term memory.

The scarce dissemination of IBE models in actual classrooms and the persistence of models based on the transmission and reception of knowledge can be explained not only by the teachers' inertia and lack of preparation, but by their perception or experience that the transmission models may be more appropriate to the specific circumstances of their classes. Faced with the dilemma that a majority of students learn nothing, get frustrated and disturb the classroom, it may be reasonable to diminish the learning expectations and prefer that most students learn something, even only routines and algorithms, and some examples to imitate. This may be a reason to support a mixed instructional model that articulates coherently, locally and dialectically inquiry and transmission.

Acknowledgement

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MATHEMATICS COMMUNICATION WITHIN THE FRAME OF SUPPLEMENTAL INSTRUCTION – SOLO & ATD ANALYSIS

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Teaching at Swedish primary and secondary school is often combined with collaborative exercises in a variety of subjects. One such method for learning together is Supplemental instruction (SI). Several studies have been made to evaluate SI in universities throughout the world, while at lower levels, hardly any studies have been made until now. The present study aimed at identifying learning conditions in SI-sessions two Swedish upper secondary schools. Within this study a combination of ATD (Anthropological Theory of Didactics) and the SOLO-taxonomy (Structure of the Observed Learning Outcome) was successfully tried as an analysis strategy.

INTRODUCTION

The teacher's choice of education methods has a high influence on what students learn (Hattie, 2009), and education research has shown to add to a better understanding of the prospects of successful teaching (Good & Grouws, 1979; Hattie, 2009; Hiebert & Grouws, 2007). In spite of previous education research, however, there is no clear answer to the question whether one method has advantages over the other or if whole-class teaching is more successful than "dialogue-teaching".

To strengthen the findings researchers have argued that there is a need for more sophisticated research methods (Jakobsson et al, 2009). There is also a need for more systematic connection between various education research theories—so-called networking (Prediger et al, 2008). According to Prediger et al. (2008) the reasons that theories in mathematics education research have evolved differently are (1) mathematics education is a complex research environment, and (2) various research cultures prioritise different components of this complex field. Different theories and methods have different perspectives and can provide different kinds of knowledge. Thus, different theories and perspectives can connect in different ways.

An educational concept that still needs to be explored, and systematically connected with various theories, is the so-called Supplemental instruction. SI is a method where groups of students are provided peer collaborative learning exercises at meetings led by SI-leaders (Hurley et al., 2006). The method is used worldwide both at the university level and lower levels. To strengthen students' knowledge in mathematics a number of upper secondary schools in Sweden have introduced SI as a complement to regular teaching.

AIM

In this study SI-meetings were analysed in upper secondary school, with the purpose of gaining more insight into the conditions that may facilitate mathematics learning. One aim of the study was to choose a combination of established frameworks that could contribute to deepen the analysis of the students' discussions. With this aim, two frameworks with different focus were chosen and tested: (1) the development of mathematical activities was defined in terms of praxeologies. (Anthropological Theory of Didactics, ATD) (Chevallard, 2012; Winsløw, 2010), and (2) learning outcome quality was defined relative to the SOLO-taxonomy (Structure of the Observed Learning Outcome) (Biggs and Collis, 1982).

The present paper aims at answering the two research questions: *To what extent is a combination of SOLO and ATD a suitable strategy for analysing SI-sessions? Are these two frameworks compatible and complementary*?

THEORY & CONNECTING FRAMEWORKS

Research needs theoretical frameworks. This was stated by Lester (2005), who argued that a theoretical framework provides a structure when designing research studies, and that a framework helps us to transcend common sense when analysing data.

Below frameworks are discussed that have been important for the study. First, the concept Supplemental instruction is presented. Then follows a section about the SOLO-taxonomy – a framework for evaluating learning outcomes. Thereafter, the ATD-praxeology is presented, which is a framework for developing teaching situations and mathematics education. Finally, possibilities and challenges with combining frameworks are discussed.

Supplemental instruction, or SI, is an educational method, used at universities in many countries. Students are asked to discuss and solve problems together, and SI is a complement to regular teaching. No teacher is present at the meetings (Malm et al., 2011a). The groups are instead guided by an older student, who is supposed to provide peer collaborative learning exercises (Hurley et al., 2006). SI has lately been introduced in some upper secondary schools in Sweden. First year students solve mathematical problems together in small groups, while second and third year students serve as SI-leaders (Malm et al., 2012).

In the early 1980s Biggs and Collis (1982) developed the *SOLO-taxonomy* for evaluating learning outcomes among students at tertiary level. SOLO, i.e. Structure of the Observed Learning Outcome, names and distinguishes five different levels according to the cognitive processes required to obtain them. The authors argued that SOLO is useful when categorising test results in closed situations with formulated expectations. They used five dimensions when categorising student responses into the

levels SOLO-1 (*pre-structural*, student misses the point and no knowledge is indicated), to SOLO-5 (Biggs and Collis, 1982, pp. 24 – 31 & 182). They also stated that SOLO describes a hierarchy where each partial construction [level] becomes a foundation on which further learning is built (Biggs 2003; Brabrand & Dahl, 2009).

Later Claus Brabrand and Bettina Dahl (2009) used the SOLO-taxonomy for analysing (1) what curricula focus on and (2) what students actually learn. By using so-called active verbs (see below) the authors state it is possible to understand on which level of knowledge the text/speech is.

SOLO 1 (pre-structural): student misses the point

SOLO-2 (*uni-structural*): paraphrase, define, identify, count, name, recite, follow (simple) instructions

SOLO-3 (*multi-structural*): combine, classify, structure, describe, enumerate, list, do algorithm, apply method

SOLO-4 (*relational*): analyse, compare, contrast, integrate, relate, explain causes, apply theory (to its domain)

SOLO-5 (*extended abstract*): theorize, generalize, hypothesize, predict, judge, reflect, transfer theory (to new domain)

Brabrand & Dahl (2009) discuss whether the SOLO-taxonomy is applicable when analysing progression in competencies in university curricula. They conclude that SOLO can be used when analysing science curricula but they question whether SOLO is a relevant tool when analysing mathematics curricula. They write:

> For mathematics it is usually not until the Ph.D. level that the students reach SOLO 5 and to some extent also SOLO 4. The main reason is that to be able to give a qualified critique of mathematics requires a counter proof or counter example as well as a large overview over mathematics which the students usually do not have before Ph.D. level. […] In fact, the same SOLO verbs can be used for different contents; hence progression in difficulty is not always reflected by the SOLO-progression in verbs. (Brabrand & Dahl, 2009)

Other researchers, however, claim that SOLO is useful in various contexts. John Pegg (2010) has described three studies where SOLO has been used to analyse primary and secondary students' learning mathematics. J. Pegg and David Tall (2005) argue for the use of SOLO in school development. In addition, Pegg (2010) states that SOLO helps to describe observations of students' mathematics performance. John Hattie and Gavin Brown (2004) also describe SOLO as a useful tool in mathematics education. They use a strategy where mathematics exercises are formulated by using SOLO, and they claim it is possible to use SOLO when analysing children's mathematics knowledge and when describing the processes involved in asking and answering a question on a scale of increasing difficulty or complexity.

Anthropological theory of didactics (ATD) is a theoretical framework for analysing and for developing mathematics education, which offers a handful of tools (Chevallard, 2006; Winsløw, 2010). One of these is *the praxeology,* and one of the overarching perspectives is *the paradigm of questioning the world* (Chevallard, 2012). Yves Chevallard (2006), who first developed the theory of ATD, argues that within *the paradigm of questioning the world*, the curriculum is defined in terms of questions. Chevallard also states that "inquiry-based" teaching can end up in some form of "fake inquiries", and he says that this most often is because the generating question of such an inquiry is but a naive trick to get students to study what the teacher will have determined in advance. Chevallard (2012) compares "the paradigm of questioning the world" with "epistemological monumentalism" which he argues is the traditional way of teaching mathematics. Students are there asked to "visit monuments" i.e. "knowledge comes in chunks and bits" without time for background or deeper understanding.

While *the paradigm of questioning the world* defines the perspective of the curriculum, the ATD-praxeology makes a helpful tool for analysing the teaching situation. The praxeology can be described as a four-tuple consisting of: a *type of task* (*T*), a *technique* (τ), a *technology* (θ) and a *theory* (Θ) (Winsløw, 2010). The four constituents – if fully understood and used – can help to construct better education. Mortensen (2011) explains that *task & technique* are called the "practice block" or the "know how", and *technology & theory* are called the "theory block" or the "know why". Hence, a technique is used to solve a special task, while technology justifies the technique and a theory gives a broader understanding of the field.

The ATD-praxeology could be applied at various levels of education. Winsløw (2006), for example, discusses how to use the praxeology when studying advanced mathematics, while Joaquim Barbé et al. (2005) suggest how to use ATD when studying classroom activities at upper secondary school. All together ATD is described as a theory *showing the shortcomings* or even paradoxes of didactic practices. Winsløw (2010) also states that ATD is useful when proposing ambitious ways to *transform* education. Also Bosch and Gascón (2006) argue that ATD has the tools to *analyse* the institutional didactic processes.

No theory can deal with everything. Different theories and methods have different perspectives and can provide different kinds of knowledge. Looking at the same data from different perspectives can give deeper insights (Prediger et al., 2008). In this study, the ATD and SOLO frameworks were combined in order to study the conditions and outcomes of students' learning through SI. The purpose of combining two frameworks was to catch the advantages of each of them, and hence, to contribute to mathematics education research and networking.

METHOD

The focus of the study was to answer the research questions: *To what extent is a combination of SOLO and ATD a suitable strategy for analysing SI-sessions? Are these frameworks compatible and complementary*? The study based its statements on class-room observations. The phenomenon being studied was students' discussions of mathematics. The context was small groups in upper secondary school. The design was flexible as the method was developed step-by-step as the study continued.

Meetings at upper secondary schools in the southern and western region of Sweden were *observed.* There were groups from the humanist, technology and natural science programs. The main criterion for choosing schools was their different experiences of support from the university. Another difference between the two schools was the implementation of SI. The criterion for choosing SI-groups to observe was availability. Not all groups wanted to be observed. Some SI-leaders denied observation of the meetings, while others cancelled already booked observations. Meetings were *videotaped* and the tapes were transcribed. The documents were coded by closed coding, i.e. a deductive *analysis* with codes from theoretical frameworks. During the whole study the analysis strategy was developed and revised.

Two separate SI-groups were observed when discussing the same exercise (table 1). The exercise was part of a former national test, which in 2010 had been intended for all students in the first grade of Swedish upper secondary school (Skolverket, 2011). No help was provided by the SI-leader during the discussion. The observed meetings lasted 40 minutes at one school and 60 minutes at the other. The students were not told anything about the SOLO- and ATD-classification of the exercise.

The exercise was *pre-classified* by SOLO and the ATD-praxeology. The intention was (1) to test if it was possible to do this classification in advance before giving the exercise to the students, (2) to decide whether the two frameworks were a suitable choice when analysing student learning outcome, and finally, (3) if it was possible to correlate every SOLO-level to a specific dimension of the ATD-praxeology.

Three different ways of using the SOLO-taxonomy were found in the literature, and initially all three of them were used when classifying the exercise. One of the three methods was part of the original method defined by Biggs and Collis (1982), with instructions for how to analyse student achievements in elementary mathematics. The authors recommended that the children's solutions were to be analysed by deciding inter alia whether the child can handle several data at the same time and whether the child shows the ability to "hold off actual closures while decisions are made".

A second method was described by Hattie and Brown (2004). They grouped the exercises in advance, so that if a student answered a certain question the student was considered to reach a certain SOLO-level. Finally, Brabrand and Dahl (2009) used the SOLO-taxonomy by the active verbs once formulated by Biggs (2003) and

compared university curricula with the table of verbs. Certain verbs were considered to point at certain "intended learning outcomes" in the curricula. Notice that the verb "calculate" and "do simple procedure" are added to SOLO 2. These verbs are mentioned in Brabrand and Dahl (2009) and in Biggs and Tang (2011). In the result section you will find that not all the three ways of using the SOLO-taxonomy were suitable for the present study.

Although the SOLO-taxonomy is widely used, in different ways, Brabrand and Dahl (2009) argue that SOLO may not be suitable for analyses of mathematics. Furthermore, the work done by Biggs and Collis (1982) was based on closed situations, and not open situations, being one of the main ideas of SI. Thus, it was decided that a complementary framework was needed for this study, specifically designed for mathematics education and also for open situations. Here, ATD was found a suitable complement to SOLO.

ATD is widely used, especially within the French, Spanish and Latin-American mathematics education research traditions (Chevallard, 2012; Bosch and Gascón, 2006). It is developed to fit mathematics education research, and calls for more open situations and open questions at school in general and in school mathematics in particular (Chevallard, 2012). In this study, the analysis and development of open mathematics learning situations was, thus, done by using the ATD-praxeology, while the SOLO-taxonomy was used for the analysis of student learning outcome.

RESULTS

The initial exercise about the volume of a cylinder was coded before it was given to the students (see table 1). The SOLO-coding was based on the three methods described above. First, the "Hattie-Brown-method" was used, as it appeared to be near to practice. It seemed to be easy to decide whether one or two aspects were involved in the question. However, when it came to higher SOLO-levels it was more difficult to judge whether the aspects were "integrated". Here, the "Biggs-Brabrand-Dahl-method" was helpful as it offered additional verbs, alternative to "integrate", e.g. "compare" and "analyse", which could be used for the coding.

An example of the use of active verbs in the coding is the sub-task where students should first calculate two volumes and then compare these two volumes (table 1):

"Starting with rectangular sheets of paper with dimensions 10 cm x 20 cm, two different tubes are made. Find the volumes of the two tubes (cylinders)."

"Compare these two volumes and calculate the ratio between them."

In both sub-exercises several aspects are involved. A volume is calculated by multiple parameters. But the active verbs separate the two sub-tasks, as the first requires only an algorithm – "find" (the volume) – while the second requires that the student goes one step further and makes a comparison – "compare" (these two

volumes). Finally, it was important to compare the coding with the "Biggs-Collis-method", as Biggs and Collis (1982) had formulated the original recommendations for how to use SOLO. In their book, however, the mathematics examples were fetched from elementary mathematics, and it was not obvious how to apply the method in the present study.

To conclude, the active verbs were found to be the most appropriate method when dealing with mathematics exercises. By using SOLO a clear borderline could be drawn between the active verbs "do algorithm" (SOLO 3) and "explain causes" (SOLO 4), and the active verbs made it possible to identify these structural differences between exercises. The initial exercise about the volume of a cylinder was also coded by the ATD- praxeology (table 1). This coding was based on the work done by Mortensen (2011), who has coded museum exhibition exercises – the so-called intended praxeology. In the exercise about the cylinder each sentence was coded. It was for example decided whether the students were supposed to deal with "know how" to solve a problem (the dimensions type of task T & technique τ) or if they were supposed to deal with "know why", i.e. a special technique was to be used (the dimensions technology θ & theory Θ).

Table 1: An exercise was classified by SOLO-taxonomy and the ATD-praxeology.

SOLO level	Exercise: A roll of paper	Praxeology
	A rectangular sheet of paper can be rolled to make a tube (cylinder) as shown in the figure.	
2 (later changed to 3)	Such a tube is made by rolling a square piece of paper with side length 10 cm. ***The diameter of the tube will be about 3.2 cm. Find the volume of this tube (cylinder).**	Technique
2/3 (later 3)	***Show that the diameter of the tube will be about 3.2 cm if the side length of the sheet of paper used is 10 cm**	Technique
3	If the length and width of the paper are different, you can make two different tubes (cylinders) depending on how you roll the paper. ***Starting with rectangular sheets of paper with dimensions 10 cm x 20 cm, two different tubes are made. Find the volumes of the two tubes (cylinders).**	Technique
4	***Compare these two volumes and calculate the ratio between them.**	Technique
4	***Investigate the ratio between the cylinder volumes using sheets of paper with other dimensions. What affects the volume ratio between the tall and the short cylinder?**	Technology
5	***Show that your conclusion is true for all rectangular papers.**	Technology

At first, in the analysis of the described exercise (table 1), SOLO and the ATD-praxeology were laid side by side. The exercise was coded both by SOLO and ATD. However, this caused problems, as ATD and SOLO evaluate different dimensions. While SOLO is a tool for evaluating the quality of students' achievements, the praxeology focuses on the teaching situation, i.e. what is going on in the classroom.

The strategy to try to correlate every SOLO-level to a specific dimension of ATD-praxeology was abandoned at this early stage in the study. During the rest of the study it was discovered that the two frameworks often did not correlate. *Thus, part of the research question was answered: the SOLO-taxonomy and the ATD- praxeology were complementary. It was also concluded that if this had not been the case, the outcome would probably have been that one framework would suffice for the analysis in this study.*

From now on the two frameworks were used for different purposes: SOLO to analyse student learning outcomes, and the ATD-praxeology to analyse teaching situations (or, in the case with SI, the didactic situations) in the classroom. The remaining part of research question 2 was now to be answered: is the combination of SOLO and ATD a suitable strategy and are the two frameworks compatible, i.e. is it possible to use them in the same study?

The next step of the study was to code the group discussions about the cylinder. The sentences of the discussions were coded by the active verbs, and by the praxeology. There were occasions when SOLO and ATD did correlate and there were other occasions when they did not. Table 2 shows part of one discussion and how the discussion can be analysed by SOLO and ATD. The students discussed the volume of the cylinder. They did not remember the formula and therefore they tried different strategies. Finally one student remembered the formula and they managed to solve the first exercise.

Table 2: Quotes from group discussion at school B are analysed by SOLO and by ATD-praxeology. Quotes are translated from Swedish and commented by the observer.

Quotes	SOLO	ATD	Comments
(e) It is the diameter times the length or height ... (a) Is that so? (e) I think so. (a) But no. It does not become square … (a) It is supposed to be CM3. It just gets CM2. It does not work.	1 3/4	Tech-nique/ Tech-nology?	Student (a) and (e) try to find a relevant technique to find the volume of the cylinder. Student (a) notices that their technique doesn't work. (a) tries to discuss "knowing why".
d) How do you count ... We were supposed to have the area of the circle.	3	Tech-nique	A parallel discussion goes on between student (d) and student (b). Student (d) comments what (a) just

(b) Wait what are we supposed to figure out? (reading task) (d) Volume ... then we need the area of the base (b) What?			said.
(a) Yes exactly (b) The area of the base? ... (d) Is not the radius times the radius times pi?	3	Tech-nique	The two groups start to discuss with one another. Student (d) takes the command and finds the technique – the "knowing how".

According to the analysis of the discussions of first this exercise, SOLO was easy to use when classifying students' mathematics discussions. The active verbs clarified the *learning outcome*, and SOLO 4 for example told that students may have "explained" and/or "analysed". If a situation was classified by the ATD-dimension "technology" it showed that the student *dealt with a discussion concerning* "knowing why" a technique was being used. *Thus, it was possible to use the two frameworks within one study – they were compatible.*

DISCUSSION

In this study the ATD-praxeology and the SOLO-taxonomy were combined with the aim to deepen the analysis of students' mathematics discussions. A strategy was suggested and tried when studying discussions at SI-meetings in upper secondary school. Initially, we tried to coordinate SOLO and the ATD-praxeology. The intention was to find out how specific SOLO-levels did correspond to specific ATD-praxeology dimensions. If this had been possible the conclusion must have been that the two frameworks were not complementary. This would have led to the elimination of one of the frameworks from this study.

However, as the two frameworks measure different aspects, this strategy was soon abandoned. Instead, the frameworks were combined. Such networking of frameworks is supported by Lester (2005) and Prediger et al. (2008). Neither of them argue that networking has to imply a total integration or unifying between frameworks. Lester (2005, p. 466) advocates the adaptation of ideas from a range of theoretical sources to suit goals both for research and for developing practice in the classroom in a way that "practitioners care about". Accordingly, the intention of the present study was to adapt the theoretical model for analysing the results of empirical studies, as well as to contribute to the development of strategies for analysing students' learning in teaching practice.

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RE-CONCEPTUALISING CONCEPTUAL UNDERSTANDING IN MATHEMATICS

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In this theoretical paper we explore interrelationships between conceptual and procedural understanding of mathematics in the context of individuals and groups. We question the enterprise of attempting to assess learners' mathematical understanding by inviting them to perform a (perhaps unfamiliar) procedure or offer an explanation. Would it be appropriate to describe a learner in possession of an algorithm for responding satisfactorily to such prompts as displaying conceptual understanding? We relate the discussion to Searle's "Chinese Room" thought experiment and draw on Habermas' Theory of Communicative Action to develop potential implications for addressing the problem of interpreting learners' mathematical understanding.

INTRODUCTION

The quest to help learners develop a deep and meaningful understanding of mathematics has become the holy grail for mathematics educators (Llewellyn, 2012), particularly since Skemp's (1976) seminal division of understanding into "instrumental" and "relational" categories. Relational (or conceptual) understanding is seen as more powerful, authentic and satisfying for the learner, representing true mathematical sense-making. But how can we know whether or not a learner has this relational understanding in any particular area of mathematics? The short, closed questions which dominate traditional paper-based assessments are unlikely to elicit this information. Hewitt (2009, p. 91) comments that "it is perfectly possible for a student to get right answers whilst not knowing about the mathematics within their work", and offers an example in which a learner aged 12–13 was finding the areas of triangles by multiplying the base by the height and dividing by 2, but admitted that he had no idea why he was multiplying or dividing by 2. This same example is used by Skemp (1976) to exemplify his distinction between instrumental and relational understanding of mathematics. Yet inviting learners to go further and explain their mathematics is also problematic. An invitation to "explain" an answer may be experienced as yet another request for "a performance": the "right" explanation that will satisfy a teacher or examiner may be memorised or produced algorithmically, just like the answer itself.

We might ask what it means for learners to have relational understanding of factorising a quadratic expression, for instance (Foster, 2014). If they can perform the procedure fluently (i.e., quickly, accurately, flexibly and confidently) then would we be satisfied (Foster, 2013)? We might argue that relational understanding involves

adapting what is known to novel, non-straightforward problem-solving situations. Yet a robust enough algorithm will dispose of a very wide range of scenarios, including unanticipated ones, and a comprehensive enough set of algorithms might successfully deal with any situation likely to be encountered in any assessment (MacCormick, 2012). If the learner's performance continued to be faultless would we wish to probe their thinking further? To some extent mathematical fluency entails withdrawing attention from the details of why and how the procedure works so as to speed up the process and allow cognitive space for focusing on wider aspects of the problem (Hewitt, 1996; Foster, 2013). A mathematician does not want to have to differentiate $3x^2 - 2x + 4$ from first principles every time, although they are capable of doing so. Perhaps relational understanding involves an ability to deconstruct the procedure *if required* rather than an expectation that this is going on every time it is carried out? But deconstructing a procedure could *itself* be regarded as a procedure, and presumably one that can be prepared for – even memorised, just as proofs can be memorised. So is there something more to relational understanding than expert procedural fluency, and if so how might this be conceptualised? Is there a difference between being able to manipulate syntax and being able to understanding meaning?

PROCEDURAL AND CONCEPTUAL KNOWLEDGE

Skemp's (1976) famous distinction between instrumental and relational understanding characterises relational understanding as "knowing both what to do and why" (p. 20), whereas instrumental understanding is merely "rules without reasons" (p. 20). While acknowledging that "one can often get the right answer more quickly and reliably by instrumental thinking than relational" (p. 23), he nonetheless criticises instrumental learning as a proliferation of little rules to remember rather than fewer general principles with wider application. More recently, the terms procedural and conceptual learning have been widely adopted, and theoretical interpretations of these in mathematics education have increasingly highlighted their interweaving and iterative relationship (Star, 2005; Baroody, Feil & Johnson, 2007; Star, 2007; Kieran, 2013; Star & Stylianides, 2013; Foster, 2014).

The most commonly-used definitions of procedural and conceptual knowledge in the context of mathematics are those due to Hiebert and Lefevre (1986). They see conceptual knowledge as knowledge that is rich in relationships, where the connections between facts are as important as the facts themselves, whereas procedural knowledge is rules for solving mathematical problems. This distinction parallels Skemp's (1976) conclusion that there are really two kinds of *mathematics* – instrumental and relational – dealing with different kinds of knowledge. More recently, Star (2005, 2007) distinguishes between *types* of knowledge (knowledge about procedures or knowledge about concepts) and *qualities* of knowledge (superficial or deep), and complains that these are frequently confounded. He highlights the way in which "procedural" is often equated with "superficial", and

"conceptual" with "deep", and draws attention to the possibility of "deep procedural knowledge" and "superficial conceptual knowledge" as valid categories. Kieran (2013) goes further in declaring the dichotomy between conceptual understanding and procedural skills a fundamentally false one. Other researchers have also explored the interplay between procedural and conceptual knowledge (Sfard, 1991), with Gray and Tall (1994) integrating processes and concepts into what they term "procepts" (Tall, 2013). But there remains the question of what precisely it is that conceptual knowledge consists of beyond confident procedural knowledge.

THE CHINESE ROOM

Searle's (1980) famous thought experiment about a "Chinese Room" was an attack on the "strong" artificial intelligence claim that a computer is a mind, having cognitive states such as "understanding". Searle imagined a native English speaker who knew no Chinese locked in a room with a book of instructions for manipulating Chinese symbols. Messages in Chinese are posted through the door and the English speaker follows the instructions in the book to produce new messages in Chinese, which they post out of the room. Unknown to them, they are having a conversation in Chinese, a language which they do not speak a word of. Searle argued that syntax does not add up to semantics; behaving "as if" you understand is not the same as understanding. But it is very difficult to pinpoint exactly where the difference lies (Gavalas, 2007). Searle does acknowledge that "The rules are in English, and I understand these rules as well as any other native speaker of English" (1980, p. 418), but it remains mysterious exactly what test could distinguish a competently performing machine from a real mathematician. A learner performing a mathematical procedure may be making mathematical sense to an observing mathematician, such as a teacher, without apparently knowing much themselves about what they are doing.

The focus here has now changed from whether the computer (or the mind as a computer) understands mathematics to the question of whether some computer could be such that it is indistinguishable from a real mathematician. It may be that, whether or not you could tell them apart, they would perform the tasks of producing syntactically correct mathematics in importantly different manners. Thus the issue becomes the *sense* in which rules are being followed. If rules are followed in a meaningful sense and their semantic content is well defined and connected within constellations of schemas, then test item responses could be strong evidence of mathematical understanding. But this requires that those items are designed so that they engage procedural knowledge in a sophisticated manner which takes into account all of the aspects of the concept image that is the object of assessment. We could specify an additional requirement that the test be administered to a human being and not a computer. While this may seem flippant, it points to the heart of Searle's argument, which is that humans follow rules through semantic causality that

is more or less part of the "hardware" of our brains; that there is no (or minimal) "software" layer (Searle, 1984). So does this imply that truly instrumental understanding is an impossibility for a human being?

MATHEMATICAL UNDERSTANDING

Searle's later articulation of social theory addresses how language can be used to create a social reality which is iterative and generative (Searle 1995, 2010). Further, Searle articulates an analysis of language that points towards strong connections between the structure of language and the structure of intentional states. In some ways this leads us back to the idea of the mathematician as performing *as though* merely in command of a complex constellation of algorithms that are triggered and brought to bear in a purely syntactical manner. In light of the argument put forth by Searle, we should rather say that the mathematician employs an array of mathematical understandings which have semantic content. While this seems unsatisfying, as though Searle is saying "it is semantic when humans do it", it bears strong connections with Sierpinska's articulation of procedural understanding and its relationship to conceptual understanding. Procedural understandings, according to Sierpinska (1994):

> are representations based on some sort of schema of actions, procedures. There must be a conceptual component in them – these procedures serve to manipulate abstract objects, symbols, and they are sufficiently general to be applied in a variety of cases. Without the conceptual component they would not become procedures. We may only say that the conceptual component is stronger or weaker. (p. 51)

Hence, it is reasonable for a mathematician to see many elements of their understanding as arrays of algorithms that allow them to address wide categories of mathematical problems. Yet this is fundamentally different from how a digital computer would operate in a purely syntactical approach.

Gordon, Achiman and Melman (1981, p. 2) define *rules* as "statements of the logical form 'In type-Y situations one does ... X'". For Wittgenstein (1953), it is not possible to *choose* to follow a rule: "When I obey a rule, I do not choose. I obey the rule *blindly*" (p. 85, original emphasis). Otherwise it is not a rule. It is in this sense that Searle raises a question fundamental to this discussion: Should understanding mathematics be understood as sophisticated algorithmic arrays which are akin to complex computer programs? Searle's (1984) critique of this and related ideas has several facets, the most pertinent of which is that there is an ambiguity in what is meant by rule following and that humans and computers do not follow rules in the same sense. In essence, Searle argues that humans follow rules in as much as they understand the *meaning* of the rules (which is thus semantic and about intentional states), whereas computers are purely syntactical in their rule following; they can be said to "*act in accord with formal procedures*" (ibid. p. 45, original emphasis).

Returning to the question of relational versus instrumental understanding, it seems that if we follow Searle's arguments we can say that mathematical understanding is probably not effective human understanding if it is primarily instrumental (in the sense of syntactical rule following). However, it is clear that procedural, syntactical and algorithmic practices and concepts form an important part of the background to meaningful mathematical understanding. Thus from a perspective of assessment we would expect it to be important to assess algorithmic fluency while also seeking to assess the strength of the conceptual content associated with the procedural performance.

So in contrast to the kinds of digital computers that Searle and Hiebert and Lefevre are talking about, algorithms exist within a semantic framework. Perhaps it is as though a digital computer (syntactical machine environment) is being modelled using a semantic machine environment (the brain). If so, the potential problem for mathematics education relating to instrumental learning in mathematics may be that the seeming simplicity of rule following is made vastly more complicated by its need to run in a sort of virtual syntactic machine running on essentially semantic hardware. On the other hand, the generation of correct syntactical content is a power of certain constellations of semantic knowledge (relational knowledge). It seems that the teaching of algorithms and procedures is crucial for the development of sophisticated mathematical understanding, but also that *how* they are taught is critical to supporting the development in learners of mathematical understanding that goes beyond procedural understandings with weak conceptual content (Foster, 2014).

Habermas' theory of communication, partly based in and complementary to Searle's theories, can point towards models of understanding and how to assess it. In communicative action, as defined by Habermas (1984), action is coordinated intersubjectively through achieving understanding. The theory of communicative action (TCA) analyses communication as having an inherent rationality focused on the goal of achieving understanding. Using speech act theory and argumentation theory, Habermas identifies categories of validity claims that are raised in any communicative interaction and also identifies implicit preconditions for successful communication. The former is referred to by Habermas as 'discourse', but might better be termed 'validity-discourse', in order to differentiate it from other uses of that term in social sciences. The preconditions for communicative action are referred to collectively as the 'Ideal Speech Situation' by Habermas and constitute a set of counterfactual norms identified abductively as necessary for successful communication. These norms are focused on equitable conditions for participation in communication where the 'unforced force of the better argument' has the opportunity to motivate agreement. This is a bit tricky, as Habermas claims that such conditions must be assumed by participants as in operation in order to communicate, despite representing more of an ideal horizon that never completely obtains. Society is

power-laden, and all communication occurs within a social context. Thus the breakdown of communication is all too common, and intersubjective understanding is seen as a fleeting and fallible goal that is ever approached but seldom attained.

The claim that Habermas's TCA and Searle's speech act theory are complementary and can be productively networked is based on the specific arguments made by Habermas in the TCA, his use of speech act theory to develop his ideas of communicative action and also upon analysis of similarities and departures between the principles, methodologies and questions of each author:

> Analytical philosophy, with the theory of meaning at its core, does offer a promising point of departure for a theory of communicative action that places understanding in language, as the medium for coordinating action at the focal point of interest. (Habermas 1984, p. 274)

While it might be possible to argue that Searle's theories depart somewhat from the kinds of analytic theories that Habermas wants to make use of, this is mistaken, since their focus is on incorporating theories of intentionality. Searle beings with the structure of linguistic expressions and then deals with intentionality, and importantly in his later work he introduces the idea of *collective* intentionality, which is focused on the coordination of speakers, and which is closely related to Habermas' ideas about the importance of intersubjectivity in communicative action:

> For a theory of communicative action only those analytic theories of meaning are instructive that start from the structure of linguistic expressions rather than from speakers' intentions. And the theory will need to keep in mind how the actions of several actors are linked to one another by means of the mechanism of reaching understanding. (Habermas 1984, p. 275)

Searle's ideas add rigour and detail at the level of social ontology and may allow for a more sophisticated operationalising of concepts and constructs based in Habermas' TCA. These ideas could be used to further network critical theory, cognitive science, neuroscience and other approaches to the study of mathematics education so that they may inform one another without reducing one to the other. Thus the issue of theoretical incommensurability may be navigated without theoretical insights becoming 'siloed' within various sub-cultures of theory which do not communicate with one another. A common theoretical language might allow researchers to disagree with greater clarity without running the risk of becoming an over-arching 'grand theory'. More broadly, Searle's ideas could serve as tools for building rigorous analysis of particular instances of theoretical networking, allowing productive discussion between theoretical perspectives.

These ideas can be operationalised to analyse small-group problem solving and in this manner interpret the mathematical understanding of participants (Kent, 2013), which could serve as the basis for the development of interactive assessment

techniques, activities and protocols. Understanding from this perspective is about being able to identify what reasons, arguments and evidence could be legitimately raised to justify a claim. This emphasis on the identification of shared bases for validity can serve as a pragmatic approach to the analysis of human understanding in mathematics. Thus when we speak of assessing mathematical understanding we can begin to identify as a community of mathematicians and mathematics educators (with due consideration of developmental and disciplinary appropriateness) the claims and the appropriate reasons that justify these claims. We can consider how to engage participants in communicative actions around mathematical goals that require the articulation of arguments and justifications that show evidence that the participants can explain why certain mathematical claims are true.

Returning to the Chinese Room, this turn to the social does not suggest that there need be two people in the room, but rather that the person in the room must share requisite background knowledge or be able to develop it contextually with the Chinese speakers outside the room. The idea of communicative competence is key: sharing the contextual background knowledge that allows a language to have semantic meaning is the basis for 'understanding'. This is different from quickly and accurately manipulating the symbols in a language in a syntactic fashion: no *shared* understanding entails from such activity. Now it is possible that meaning could be attributed to rules or symbols by the person in the Chinese room, but, without the ability to test these against another person who has semantic understanding of the symbols, no interpersonal communication or shared understanding is achievable. The meaning so developed would be a private language that would not necessarily correspond to that of the interlocutor. Thus the person in the Chinese room might imagine that they were having a discussion about a family's vacation outing when in fact the interlocutor interpreted the exchange of symbols as being a mathematical discourse on the solution to an algebraic problem (or vice versa).

CONCLUSION

These ideas about the nature of the relationship between syntax and semantics, procedure and concept, and instrumental and relational understanding do not undermine the importance of procedural fluency. Pimm (1995) addresses the issue in depth and identifies some of the important features of fluency in mathematics education:

> For me, fluency is about ease of production and mastery of generation – it is used also in relation to a complex system. 'Fluent' may be related to efficient, or just no wasted effort. It is often about working with the *form*. Finally, it can be about not having to pay conscious attention. (ibid. p. 174, original emphasis)

Thus fluency, including syntactical fluency, can serve as partial evidence of understanding in a communicational context. Mathematical fluency, as in non-

mathematical communication, is a sign of communicative competence, which is a prerequisite for interpersonal understanding according to the hermeneutic/communicational tradition (Habermas, 1984; Sierpinska, 1994). Thus when we say that a human being does not follow rules in the same sense as a computer, we mean that the symbolic rule following (or algorithmic manipulation of syntax) is done in the context of mathematical communication, and thus has semantic framing.

Habermas' articulation of rational behaviour in discursive practices has been identified as productive for the analysis of shared cognition in mathematics education (Boero et al., 2010). In communicative action participants achieve shared goals by coordinating action (including speech action) through the development of a shared understanding. Thus, establishing shared goals and coordinating action around an appropriately designed mathematical task could serve as an interpretive basis for the researcher (or other virtual participant) to make a judgement about the understanding of the participants in collaborative learning of mathematics (Kent, 2013).

We suggest that consideration of Searle's (1984) critique of cognitive science allows for ongoing productive insight into what mathematical thinking is and its relation to education. An important problem faced by the mathematics education community is how we can use ideas of relational understanding and instrumental understanding in a sophisticated manner to promote the learning of mathematics. Learners of mathematics should gain genuine experience of real mathematical sense-making rather than engage in a charade of imitating what they think such behavior should look like. The increasing focus on fluency in policy in the UK (DfE, 2013) suggests the need for tools and practices to be developed which coordinate ideas of cognition, mathematical understanding and educational practices of teaching and assessment. Our consideration of Searle's Chinese Room argument has sought to highlight the nuance involved in these issues and the kinds of practices and theoretical frameworks that could be leveraged to address the problem of interpreting learners' mathematical understanding.

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THE EPISTEMOLOGICAL DIMENSION REVISITED

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Epistemology and networking was discussed in the last CERME working group on theory. This paper aims to continue the discussion. I reflect on epistemological analysis and the cultural dimension of knowing and present examples which demonstrate how the changes in the cultural context influence the epistemological analysis. Then, I reconsider the epistemological dimension and the networking of theories. In some cases, the epistemological dimension permits the networking. In other cases, we notice how by means of networking, strong epistemological concerns in one theory might be integrated in another theory in a way that reinforces the underlying assumptions of this other theory. I end the paper with an example of networking that demonstrates how the social dimension might influence the epistemological analysis.

Keywords: cultural dimension; epistemological analysis; networking theories; social dimension;

EPISTEMOLOGY AND NETWORKING THEORIES IN THE PREVIOUS CERME WORKING GROUPS ON THEORIES

The present paper aims to continue the work done at the previous CERMEs in relation to the epistemological dimension in theories. At CERME 8, the focus on networking and epistemology was stronger than in the previous working groups on theory. For example, the role of epistemology in the networking of theories was an explicit focus in the paper by Ruiz-Munzón, Bosch and Gascón (2013). The idea of a "reference epistemological model" (REM) was introduced for networking Chevallard's Anthropological Theory of the Didactic (ATD) and Radford's Theory of Knowledge Objectification (TKO). The authors analyzed how each approach addresses the nature of algebraic thinking. The point of view of the ATD was presented with its own REM about elementary algebra as well as the kind of questions addressed by this approach, in relation to the TKO.

In their paper, presented at CERME 8, Godino et al. (2013) analyzed two approaches to research in mathematics education: "Design-based research" (DBR) and "Didactic engineering" (DE), in order to study their possible networking. DE (closely linked to Brousseau's theory of didactical situations) focuses on epistemological questions; DBR does not adopt a specific theoretical framework, nor does it explicitly raise epistemological questions. In the working group (Kidron et al., 2013) interesting

questions arose like the following one:"is the epistemological focus only a question of "cultural and intellectual context" or is an epistemological reference necessary for each theoretical approach used in design based research in math education?".

Artigue (2002) wrote that the anthropological approach shares with the socio-cultural approaches the view that mathematical objects are not absolute objects, but are entities which arise from the practices of given institutions. These practices are described in terms of tasks in which the mathematical object is embedded, in terms of techniques used to solve these tasks and in terms of discourse which both explains and justifies the techniques. It is interesting to note that the nature of mathematical objects was a theme that appears at CERME 4 in the context of the need to be aware of the underlying assumptions of each theory and that underlying assumptions also concern ontological or epistemological questions such as the nature of mathematical objects. This theme reappears in the next CERMEs especially at CERME 7 while networking was needed in order to analyze the emergence and nature of mathematical objects. This was well demonstrated, for example, in the paper presented by Font et al. (2011). The authors asked "What is the nature of the mathematical objects?" They explored this question by the use of a synthesis between the onto-semiotic approach (OSA), APOS theory and the cognitive science of mathematics (CSM) as regards their use of the concept "mathematical object". APOS theory and CSM highlight partial aspects of the complex process through which, according to OSA, mathematical objects emerge. OSA extends APOS theory by addressing the role of semiotic representations; it improves the genetic decomposition by incorporating ideas of semiotic complexity, networks of semiotic functions and semiotic conflicts; it offers a refined analysis due to the way in which it considers the nature of such objects and their emergence out of mathematical practices. Considering mathematical objects not as absolute objects, but as entities which arise from the practices of given institutions, leads us to analyze the role of both, the epistemological dimension and the socio cultural dimension, in theories.

EISTEMOLOGICAL ANALYSIS AND SOCIO CULTURAL DIMENSION

The following question was asked by Luis Radford at the colloquium at Paris in honour of Artigue (2012):

"How can epistemological analysis take into account the social and cultural dimension of knowing?"

In the last decades the increasing influence of socio cultural approaches towards learning processes is well recognized. Therefore, the question is essentially *how* the social and cultural dimensions are taken into account in the epistemological analysis.

In this section I will consider this question in relation to the cultural dimension of knowing, I analyse the changes in the cultural context and their influences on the epistemological analysis. In the section about epistemological dimension and networking theories, I will reconsider Radford's question in relation to the social dimension of knowing.

Changes in the cultural context and their influences on the epistemological analysis

In the last decades we face the changes of our cultural environment as well as the changes of the context in which our theory emerged. I will give an example from my own research on students' conceptual understanding of central notions in calculus like the notion of limit in the definition of the derivative. In my previous research, using essentially theories that privilege epistemological and cognitive dimensions, I was aware of the cognitive difficulties relating to the understanding of the definition of the derivative as the "limit of the quotient $\Delta y/\Delta x$ as Δx approaches 0". In my epistemological analysis, my first thinking was that these cognitive difficulties are inherent to the epistemological nature of the mathematics domain. I realized that students viewed the limit concept as a potential infinite process and I understood that this was a possible source of difficulties. Moreover, previous researches (Tall, 1992) expressed students' belief that any property common to all terms of a sequence also holds of the limit. I therefore realized that this natural way in which the limit concept is viewed might be an obstacle to the conceptual understanding of the limit notion in the definition of the derivative function $f'(x)$ as $\lim_{\Delta x \to 0} \Delta y / \Delta x$. In particular, the derivative might be viewed as a potentially infinite process of $\Delta y / \Delta x$ approaching $f'(x)$ for decreasing Δx. As a result of the belief that any property common to all terms of a sequence also holds of the limit, the limit might be viewed as an element of the potentially infinite process. In other words, $\lim_{\Delta x \to 0} \Delta y / \Delta x$ might be conceived as $\Delta y / \Delta x$ for a small Δx. I therefore looked for a counterexample that demonstrates that one cannot replace the limit " $\lim_{\Delta x \to 0} \Delta y / \Delta x$ " by $\Delta y/\Delta x$ for Δx very small. "Finding such a counterexample .. was crucial to my research focus. Such a counterexample demonstrates that the passage to the limit leads to a new *entity* and that therefore omitting the limit will change significantly the nature of the concept. It demonstrates that the limit could not be viewed as an element of the potentially infinite process" (Kidron 2008, page 202). In Kidron (2008) I explain that such counterexample exists in the field of dynamical systems which is considered as a new field in mathematics. In the counterexample (the logistic equation), the analytical solution obtained by means of continuous calculus is totally different from the numerical solution obtained by means of discrete numerical methods. The essential point is that using the

analytical solution, the students use the concept of the derivative as a limit $\lim\limits_{\Delta x \to 0} \Delta y / \Delta x$ but, using the discrete approximation by means of the numerical method, the students omit the limit and use $\Delta y / \Delta x$ for small Δx. Students reactions are analyzed in Kidron (2008), in particular how students reach the conclusion that passing to limits may change the nature of a problem significantly. The essential point is that the changes in the cultural context permit the new settings for the learning experience. More precisely, the changes in the cultural context permit modern results in research Mathematics which influenced my own research in mathematics education by means of changes in the didactical designs. The didactical design described in Kidron (2008) was possible by means of the epistemic status of the new artifacts used in the research study. The way the students interacted with the software demonstrates that the artifact used in this study should not be considered only as an aid for the students. It had a deep cognitive role while learners interacted with it. The artifact was conceived as co-extensive of thinking: the students act and think *with* and *through* the artifact as described by Radford (2008). In another study (Kidron and Dreyfus, 2010) we also notice this specific epistemic status of the artifact as co-extensive of thinking while the computer is considered as a dynamic partner. Kidron and Dreyfus consider the influence of a CAS context on a learner's process of constructing a justification for the bifurcations in a logistic dynamical process. The authors describe how instrumentation led to cognitive constructions and how the roles of the learner and the computer intertwined during the process of constructing the justification.

Another example describing how epistemological analysis takes into account the cultural dimension of knowing is described in Artigue (1995, page 16) in which the author describes her mathematical research in differential equations and the way she notes the epistemological inadequacy of teaching in this area, for students in their first two years at university. By means of epistemological analysis, Artigue described how historically the differential equations field had developed in three settings: the algebraic, the numerical and the geometric settings. For many years, teaching was focused on the first setting due to epistemological and cognitive constraints. Reflecting on these constraints was a starting point towards building new teaching strategies which better respect the current fields' epistemology. By means of the epistemological analysis, Artigue could see the epistemological evolution of the field towards new approaches, the geometrical and numerical approaches. The essential point in this example is that the epistemological evolution is a consequence of the changes in the mathematical culture and the epistemological analysis highlights the crucial role of the cultural dimension.

THE EPISTEMOLOGICAL DIMENSION AND THE NETWORKING OF THEORIES

Epistemological sensitivity

A new view on the epistemological dimension is offered in Kidron et al. (2014) by means of the networking between three theories, TDS - the Theory of Didactic situations (Artigue, Haspekian, & Corblin-Lenfant 2014), ATD - the Anthropological Theory of the Didactic (Bosch and Gascón, 2014), and AiC - the theory of Abstraction in Context (Hershkowitz et al. 2001; Schwarz et al. 2009; Dreyfus and Kidron, 2014). The foci of the three theoretical approaches are different. In particular, AiC focuses on the learner and his or her cognitive development, while TDS and ATD focus on didactical systems. The three theoretical approaches are sensitive to issues of context but, due to these differences in focus, context is not theorized and treated in the same way. The authors expected some complexity in the effort of creating a dialogue between the three theories in relation to constructs such as context, milieu, and media-milieus dialectic. However, they observed how the dialogue between the three theories appears as a progressive enlargement of the focus, showing the complementarity of the approaches and the reciprocal enrichment. A new term was introduced in this research study: *epistemological sensitivity.*

The authors explain the meanings of the terms context (for AiC), milieu (for TDS) and media-milieus dialectic (for ATD), each of them being a cornerstone for the theory while all of them try to theorize specific contextual elements. The three theories share the aim to understand the epistemological nature of the episode described in the paper but in each of the three theories different questions were asked. Questions for analyses in AiC stressed the epistemic process itself, whereas researchers in TDS and ATD asked how this process is made possible. Nevertheless, these questions indicated that the researchers were able to build on the other analyses in a complementary way. The dialogue between the different approaches was possible because a point of contact was found. In this case, we may talk about a common *epistemological sensitivity* of AiC, TDS, and ATD, which can be noticed in the a priori analyses provided by each frame. This initial proximity was essential for the dialogue to start and become productive, showing the complementarity of the approaches and the reciprocal enrichment, without losing what is specific to each one. The three concepts, context, milieu and media- milieus dialectic were accessed by different data or different foci on data in a complementary way sharing *epistemological sensitivity* , which facilitated establishing connections and reflecting on them.

Epistemological concerns as a consequence of networking

It is not by chance that the common *epistemological sensitivity* of AiC, TDS, and ATD, was noticed in the a priori analyses provided by each frame: the reason is that the a priori analyses take into account the mathematical epistemology of the given domain. In the last years, the AiC researchers decided to implement the idea of a priori analysis in an explicit way. This happened as a consequence of the networking experience with the TDS researchers. An example of such a networking experience is described in Kidron et al. (2008). Three theories were involved in this case of networking: TDS, AiC and IDS - the theory of Interest-Dense Situations (Bikner-Ahsbahs and Halverscheid, 2014). Kidron et al. (2008) focus on how each of these frameworks is taking into account social interactions in learning processes. The authors wrote that "In a more general way, the different views the three theoretical approaches have in relation to social interactions force us to reconsider these approaches in all their details. The reason for this is that the social interactions, as seen by the different frameworks, intertwine with the other characteristics of the frameworks" (Kidron et al. 2008, page 253). The authors identified not only connections and contrasts between the frameworks but also additional insights, which each of these frameworks can provide to each of the others. In this paper, we only focus on a specific kind of insights: the epistemological concerns which were highlighted as a consequence of the networking of theories. We first characterize the epistemological dimension in each of the three theories before the networking experience:

TDS provides a frame for developing and investigating didactical situations in mathematics from an epistemological and systemic perspective. TDS combines epistemological, cognitive, and didactical perspectives. TDS focuses on the epistemological potential of didactical situations

IDS- the theory of interest-dense situations is "a social constructivist theory that cannot say much about cognitive processes of individuals and does not provide tools for epistemological analyses" (Bikner-Ahsbahs & Halverscheid, 2014, page 102).

AiC analysis focuses on the students' reasoning; mathematical meaning resides in the verticality of the knowledge constructing process and the added depth of the resulting constructs. An epistemological stance is underlying this idea of vertical reorganization but AiC analysis is essentially cognitive.

Focusing on epistemological concerns as mentioned earlier, we will only characterize the insights offered by TDS to AiC as described by Kidron et al. (2008):

> According to Hershkowitz et al. (2001), the genesis of an abstraction originates in the need for a new structure. In order to initiate an abstraction, it is thus necessary (though not sufficient) to cause students' need for a new structure. We may attain this aim by

building situations that reflect in depth the mathematical epistemology of the given domain. This kind of epistemological concern is very strong in the TDS, and the notion of fundamental situation has been introduced for taking it in charge at the theoretical level. It could be helpful for AiC.

This was an invitation for AiC researchers to build an a priori analysis that reflects in depth the mathematical epistemology of the given domain. In the same vein the a priori analysis of TDS offers another perspective to IDS to think about the building of situations reflecting in-depth the mathematical epistemology of a given domain and the consequence of such reflection on the analysis of the social interactions.

The social dimension and its influence on the epistemological analysis

In the following, I analyze a case of networking between AiC and IDS which demonstrates mutual insights in the process of networking. In particular, we will observe how the epistemological analysis carried by the AiC researchers is influenced by the social dimension of knowing which characterizes IDS. This case of networking illustrates how the epistemological analysis might take into account the social dimension of knowing.

Kidron et al. (2010) focus on the idea of networking and on two theoretical concepts: the need for a new knowledge construct, and interest. IDS considers social interactions as basis which constitutes learning mathematics. Interest-dense situations provide motivation for processes of in-depth knowledge construction. AiC is a theoretical tool to investigate such processes. As already mentioned, in the AiC analysis, the first stage of the genesis of an abstraction is the learner's need for a new construct. Such a need might arise when the learner's existing knowledge is insufficient to solve a task or to understand a new concept. This individual need is related to the specific mathematical situation at hand. Analyzing this need is a part of AiC epistemological analysis. For IDS the situation is different: interest constitutes a psychological source to gain more knowledge. This need is nested in the situational interest rather than shaped by the epistemic nature of the topic. The aim of the networking was to relate these two concepts: need and interest. As mentioned earlier, the AiC researchers implemented the idea of a priori analysis. Their analysis was based on an a priori analysis of the knowledge elements intended by the design. The AiC analysis focused on the students' reasoning and mathematical meaning resided in the verticality of the knowledge constructing process. The AiC researchers identified students' constructs of the intended knowledge elements. They expected to identify students' need for the new constructs before or during the process of knowledge construction. However, the researchers found it difficult to identify a need for a specific new construct. Networking the two approaches was helpful: The IDS analysis focuses and reconstructs the whole situation sequentially on the basis of utterances that show intense social interactions, whereas the AiC analysis focuses on segments that appear relevant to the constructing process. In fact, the excerpts

ignored at first by the AiC researchers did contribute to the constructing process thanks to the social interaction analysis provided by IDS which allowed the AiC researchers to focus on and incorporate these seeds of construction in their analysis. The networking helps AiC researchers realize that there are situations in which constructing actions can occur on the basis of a general epistemic need rather than on the basis of specific needs for new constructs. The benefit of networking was mutual thanks to the epistemological nature of AiC a priori analysis which makes the researchers sensitive for the mathematics at stake and implicit mathematical ideas were identified very early. This was very helpful towards IDS re-analyzing of the epistemic actions in the research study.

CONCLUDING REMARKS

In the last CERME we discussed cases in which the epistemological dimension permitted the networking. This was done, for example, by means of the idea of "reference epistemological model". In this paper, we notice how by means of networking, strong epistemological concerns in one theory might be integrated in another theory in a way that reinforces the underlying assumptions of this other theory. This was illustrated by the insights offered by means of a priori analysis. We also analysed examples that demonstrate the influence of the cultural context as well as the influence of the social dimension on the epistemological analysis. The cultural context in which the different theories emerged is changing all the time. As a result of these changes, a new view on the epistemological dimension is offered. This new view should be further discussed.

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TOWARDS A CONFLUENCE FRAMEWORK OF PROBLEM SOLVING IN EDUCATIONAL CONTEXTS

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An exploratory confluence framework for analysing mathematical problem solving in socially different educational contexts is introduced. The central premise of the framework is that a key solution idea to a problem can be constructed by a solver as a result of shifts of attention that come from individual effort, interaction with peer problem solvers or interaction with a source of knowledge about the solution. The framework consolidates some existing theoretical developments and aims at addressing the perennial educational challenge of helping students become more effective problem solvers.

RATIONALE

It has been repeatedly asserted that problem solving is an activity at heart of doing and studying mathematics. Its central feature is that problem solving requires the engaged person(s) to invent a solution method rather than to recall and implement a previously practiced method (e.g., Kilpatrick, 1982; Schoenfeld, 1985; NCTM, 2000). Accordingly, one of the central challenges associated with the use of mathematical problems in educational contexts – how to help learners to become more effective or successful problem solvers – can be worded as the challenge of supporting learners' mathematical inventiveness in ways that preserve their problem-solving autonomy and self-efficacy.

For the last 50 years this challenge has been approached through various conceptual frameworks and models (see Carlson & Bloom, 2005; Schoenfeld, 2012; Törner, Schoenfeld & Reiss, 2008, for comprehensive accounts of the state of the art). Each framework has aimed at addressing specific queries of pragmatic and theoretical importance. Some of the queries were:

- How do mathematicians solve problems? What phases and cycles are they going through while solving problems? (Carlson & Bloom, 2005; Pólya, 1945/1973). Can, and if yes, how problem-solving heuristics be taught? (Schoenfeld, 1979, 1985; Koichu, Berman & Moore, 2007).

- What are the attributes of mathematical problem solving besides heuristics? (Schoenfeld, 1985). What is the role of affect in problem solving? (DeBellis & Goldin, 2006).

- How do the problem-solving attributes come to cohere? (Schoenfeld, 1992). How does decision making occur when an individual solves a problem? (Schoenfeld, 2012).

- What sociomathematical norms should be promoted for supporting learners' intellectual autonomy in problem solving? (Yackel & Cobb, 1996).

Some of these and such queries have been addressed. For instance, we know a lot about phases and cycles involved in problem solving by experts and by some categories of students. The use of problem-solving attributes and phases as a research tool has proven to be particularly helpful for analysing the phenomenon of unsuccessful problem solving. For example, if there is evidence that a particular belief about mathematics is depriving an individual from persisting when solving a problem, then that belief might provide a sufficient explanation for the problem-solving failure (see Furingetti & Morselli, 2009, for an elaborated example). When, however, one problem is solved and another is not by an individual who possesses all needed mathematical, cognitive and affective resources for solving both problems, the explanation of the success and the failure can sometimes be sought outside of the existing problem-solving models and frameworks (Koichu, 2010).

Some of the queries about problem solving have proven to be hard nuts to crack by means of mathematics education research (e.g., Schoenfeld, 1992, 2012). For example, Schoenfeld's (1992) question about how the problem-solving attributes – knowledge, heuristics, control and beliefs – come to cohere has been under research scrutiny for more than two decades. Furthermore, different problem-solving frameworks and models have emerged from different contexts and situations. As a result, it is sometimes difficult to use one model outside its original context. A recent example of extending the scope of a particular problem-solving model to another context is given by Clark, James and Montelle (2014) (their work is discussed in more detail below), but it is rather an exception than a trend (cf. Koichu, 2014, for a collection of views on the recent trends in research on problem solving). As a rule, problem-solving frameworks and models co-exist with little coordination. This is one of the reasons for which, in terms of Mamona-Downs and Downs (2005), a clear identity for problem solving in mathematics education has not yet been developed.

The aim of this article is to present an exploratory problem-solving framework that has the potential to consolidate some of the previous frameworks and can serve as a research and pedagogical tool in different educational contexts. The framework is, in a way, a tool for better understanding the process that Pólya (1945/1973) might term as a heuristic search embedded in the planning phase of problem solving. The central query of the framework is, simply stated, "Where can a solution to a problem come from?" A more precise formulation of the query is as follows: "Through which activities and resources can a chain of shifts of attention towards an invention of a key solution idea to a mathematical problem be constructed by a problem solver in socially different educational contexts?"

CONFLUENCE FRAMEWORK

The confluence framework is schematically presented in Figure 1. A *key solution idea* notion is in its core of the framework. It is a solver-centered notion. Along the lines defined by Ramon (2003), *a key solution idea* is a heuristic idea[1] which is invented by the solver and evokes the conviction that the idea can be mapped to a full solution to the problem. The full solution is a solution, which, to the solver's knowledge, would be acceptable in the educational context in which problem solving occurs.
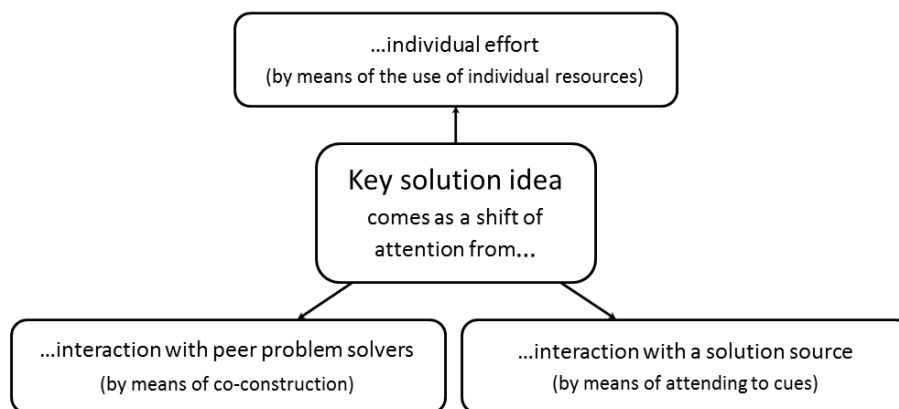

Fig. 1: Confluence model of mathematical problem solving

Examples of key solution ideas include: an auxiliary construction that enables the solver to see a chain of deductions connecting the givens of a geometry problem with the claim to be proved, a way of reassembling the terms of a sophisticated trigonometric equation so that the solver begins to see the equation as a quadratic one, a way of representing a word problem (e.g., Euler Seven Bridge Problem) as a graph that makes the solution to the problem transparent. One can see connections between the notions of a key solution idea and of an illuminating or insightful idea. An insight, however, is frequently defined as restructuring the initial representation of the problem followed by so-called aha-experience. A key solution idea does not necessarily emerge at once and accordingly its invention is not necessarily accompanied by an aha-moment.

The framework relies on three premises. ***First premise:*** Even when a problem is solved in collaboration, it has a *situational solver*, an individual who invents and eventually shares its key solution idea. ***Second premise:*** A key solution idea can be invented by a situational solver as a shift of attention in a sequence of his or her shifts of attention when coping with the problem. ***Third premise:*** Generally speaking, a solver's pathway of the shifts of attention can be stipulated by: (i) individual effort and resources, (ii) interaction with peer solvers who do not know the solution and struggle in their own ways with the problem or attempt to solve it together, (iii) interaction with a source of knowledge about the solution or its parts, such as a

textbook, an internet resource, a teacher or a classmate who has already found the solution but is not yet disclosing it. The possibilities (i)-(iii) are intended to embrace all frequent situations of problem-solving. These possibilities can be employed in separation or complement each other in one's problem solving.

The framework is a *confluence* framework because it consolidates ideas taken from several frameworks and theories by means of a strategy that has been introduced at CERME-8 as *networking theories by iterative unpacking* (Koichu, 2013). Mason's theory of shifts of attention (Mason 1989, 2008, 2010) serves as the overarching theory of the framework. Additional theories are embedded. Each of the next four sub-sections begins with a brief introduction of a particular theory and proceeds to show how the theory contributes to the confluence framework.

Invention of a key solution idea as a shift of attention

Mason's theory of shifts of attention had initially been formulated as a conceptual tool to dismantle constructing abstractions (Mason, 1989) and then extended to the phenomena of mathematical thinking and learning (Mason, 2008, 2010). Palatnik and Koichu (2014, submitted) adopted the theory as a tool for analysing insight problem solving.[2] Mason (2010) defines learning as a transformation of attention that involves both "shifts in the form as well as in the focus of attention" (p. 24). To characterize attention, he considers not only *what* is attended to by an individual but also *how* the objects are attended to. To address the how-question, Mason (2008) distinguishes five different *ways of attending* or *structures of attention*.

According to Mason (2008), *holding the wholes* is the structure of attention, where the person is gazing at the whole without focusing on particular. *Discerning details* is a structure of attention, in which one's attention is caught by a particular detail that becomes distinguished from the rest of the elements of the attended object. Mason (2008) asserts that "discerning details is neither algorithmic nor logically sequential" (p. 37). *Recognizing relationships* between the discerned elements is a development from discerned details that often occurs automatically; it refers to specific connection between specific elements. *Perceiving properties* structure of attention is different from recognizing relationships structure in a subtle but essential way. In words of Mason (2008), "When you are aware of a possible relationship and you are looking for elements to fit it, you are perceiving a property" (p. 38). Finally, *reasoning on the basis of perceived properties* is a structure of attention, in which selected properties are attended to as the only basis for further reasoning. Palatnik and Koichu (2014, submitted) added a *why*-question to Mason's *what*- and *how*- questions: Why does an individual make shifts from one object of attention to another in the way that he or she does? Possible ways of addressing this query are related to the obstacles embedded for the solver in attending to a particular object and to continuous evaluating potential "gains and losses" of the decision to keep attending to the object or shift the attention to another one (Metcalfe & Kornell, 2005).

The process of inventing a key solution idea is seen as a pathway of the solver's shifts of attention, in which objects embedded in the problem formulation or *problem situation image* (this notion is used in the meaning assigned to it by Selden, Selden, Hauk & Mason, 2000) are attended to and mentally manipulated by applying available schemata. The process at large is goal-directed, but particular shifts can be sporadic. A pathway of the shifts of attention depends on various factors, including: the solver's traits, his or her mathematical, cognitive and affective resources and a context in which problem solving takes place. I now turn to discussing the specificity of the process in three socially different educational contexts.

Shifts of attention in individual problem solving

The lion's share of the data corpus that underlies the development of the foremost problem-solving frameworks (e.g., Schoenfeld, 1985; Carlson & Bloom, 2005) consists of cases of individual problem solving. Carlson and Bloom (2005) consider four phases in individual problem solving by an expert mathematician: orientation, planning, executing and checking. The model also includes a sub-cycle "conjecture—test—evaluate" and operates with various problem-solving attributes, such as conceptual knowledge, heuristics, metacognition, control and affect. Generally speaking, Carlson and Bloom's framework offers a kit of conceptual tools that can be used for producing thick descriptions of individual problem-solving effort. These conceptual tools enter the suggested confluence framework as tools for addressing *how-* and *why*-questions about the shifts of attention.

For example, when solving a challenging geometry problem, a solver can direct her attention to proving similarity of a particular pair of triangles, and then shift her attention to another pair of triangles. The pre- and post-stages of the shift can be described as two "conjecture—test—evaluate" sub-cycles within the *planning* phase. The shift itself can be viewed in terms of the mathematical, heuristic and affective resources of the solver (see Palatnik & Koichu, 2014, for an elaborated example).

Shifts of attention when interaction with peers is available

While studying problem-solving behaviours in small groups of undergraduate students, Clark, James and Montelle (2014) extended Carlson and Bloom's (2005) taxonomy of problem-solving attributes by introducing two new categories/codes. They termed them *questioning* and *group synergy*. The former category was introduced in order to give room in the data analysis to various questions (for assistance, for clarification, for status, for direction) that the participants had asked. The latter category appeared to be necessary in order "to capture the combination and confluence of two or more group members' problem-solving moves that could only occur when solving problems as a member of a group… A key characteristic of this group synergy code is that it leads to increased group interaction and activity, sometimes in unanticipated and very productive ways." (p. 10-11).

Indeed, when a possibility to collaborate with peers is available to a solver, his or her shifts of attention can be stipulated also by inputs of the group members, especially when the inputs are shared in some common problem-solving space (e.g., a small-group discussion or an internet forum) in a non-tiresome way. Here I would like to stop on the word "sometimes" in the above quotation. The possibility to collaborate can increase one's chances to produce a key solution idea, but can also be overwhelming or distracting. When nobody in a group knows how to solve the problem, the other members' inputs of potential value are frequently undistinguishable for the solver from the inputs of no value. Consequently, it can become too effortful for the solver to follow and evaluate the inputs of the others.

Schwartz, Neuman and Biezuner (2000) deeply explored, in laboratory setting, the cognitive gains of two children, who fail to solve a task individually, but who improve when working in peer interaction. They characterized the situations, in which (in their words) *two-wrongs-make-a-right* vs. *two-wrongs-make-a-wrong*. The mechanisms of co-construction behind *two-wrongs-make-a-right* phenomenon were: the mechanism of disagreement, the mechanism of hypothesis testing, and the mechanism of inferring new knowledge through challenging and conceding. These mechanisms might be involved in those cases of collaborative problem solving, in which group synergy led the participants in Clark, James and Montelle's (2014) study to "very productive ways" (ibid) of solving the given problems.

The confluence framework seeks to consolidate the theoretical insights of the aforementioned studies. In particular, to further explore the phenomenon of group synergy, it seems me necessary to acknowledge that the above mechanisms can become active on condition that at least sometimes a solver shifts his or her attention from an object that he or she is being exploring to an object attended to by the peer. I plan to present at the conference an elaborated illustration of the process of co-constructing a key solution idea as a pathway of one solver's shifts of attention. I am going to show that the shifts are stipulated either by individual or by shared problem-solving resources and show how the aforementioned mechanisms enter the process. An example concerns a situation, in which a group of 16 10th grade students were engaged in solving the following problem:

> Two circles with centres M and N are given. Tangent lines are drawn from the centre of each circle to another circle. The points of intersection of the tangent lines with the circles define two chords, EF and GH (see Figure 2). Prove that segments EF and GH are equal.
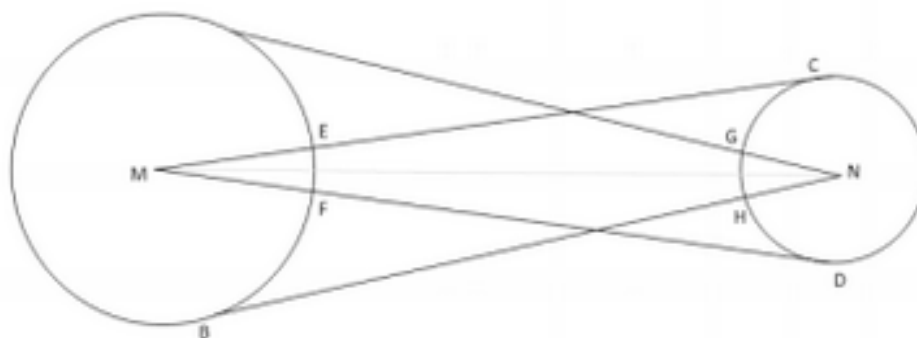
Figure 2: A problem about two chords

The students could solve the problem individually, but also share their ideas in a closed forum in one of the social networks. Eventually, three different solutions were invented. Interestingly, all the students indicated in the reflective questionnaires that they had worked collaboratively for about 40% of time that had been devoted to the problem. (On average, the students worked on the problem for 3 hours). As a rule, the students chose to collaboratively work in the forum when they were stuck and sought for new ideas or for the feedback on their incomplete ideas.

Shifts of attention when interaction with a solution source is available

The option to interact with a source of knowledge about a key solution idea to a problem can drastically change a pathway of one's shifts of attention, up to the point that the entire process can stop being a problem-solving process and become a solution-comprehending process. The suggested framework seeks to encompass only the situations in which a solution source is present as a provider of cues to the solution or as a convenient storage of potentially useful facts, but not as a source of *telling* the solution. Such situations are common, for instance, when a teacher orchestrates a classroom problem-solving discussion.

When a source of knowledge about the solution is present but does not tell the solution, the solvers may attempt to extract the solution from the source (e.g., see *questions for assistance* and *questions for direction* in Clark, James & Montelle, 2014; see also Koichu & Harel, 2007). In some cases, the solver's shifts of attention may occur as a result of a conflict that emerges when more knowledgeable and less knowledgeable interlocutors assign different meanings to the same assertions (cf. Sfard, 2007, for commognitive conflict).

For example, the assertion "Triangle similarity is a good idea" can either pass unnoticed in the group discourse or be a trigger for the solver to shift his or her structure of attention. The effect of the assertion would depend on who it has come from, a regular member of the group or a teacher or a peer who acts as if she has already solved the problem. The occurrence of the shift in one's attention as a result of another person's assertion depends not only on that person's status in the group. It

is mainly the matter of different meanings that can be assigned to the assertion by different individuals. In one case, the assertion about triangle similarity may be perceived as, "It is possible that similarity helps," in another, "I've tried it and it helps," and in yet another, "This is the direction approved by the authority." Stimulated by Sfard (2007), I suggest that such a conflict of meanings can first be unnoticed, then it can hinder the communication, and then (when the assigned meanings are explicated), it can help the less knowledgeable solvers to progress.

SUMMARY AND FURTHER DEVELOPMENT

Developing a confluence framework of mathematical problem solving that would be applicable to different educational contexts is motivated by several causes. First, with few exceptions, the existing problem-solving frameworks utilize *different* conceptual tools for exploring problem solving in socially *different* educational contexts. Second, the foremost frameworks are comprehensive within the problem-solving contexts from which they have emerged but it is sometimes difficult to apply them to additional contexts. Third, in spite of the comprehensive nature of the existing frameworks, the central problem-solving issue of inventing (as opposed to recalling) a solution method is still not sufficiently understood. At the same time, theoretical tools that can help to progress the state of the art are available from the other sub-fields of mathematics education research. Hence, a *confluence* framework.

In this article a particular way of constructing a confluence framework is presented. The confluence effect is pursued by considering common roots of problem solving in three socially different contexts. This is done in terms of Mason's theory of shifts of attention, which initially had been constructed for other reasons. Simultaneously, the specificity of the attention shifts in different problem-solving contexts is considered by means of additional theoretical constructs. The use of the model as a research tool for understanding heuristic aspects of problem solving is stipulated by availability of research methodologies for identifying and characterizing shifts of attention in socially different problem-solving contexts. In part, such methodologies are available from past research (e.g., Mason, 1989, 2008, 2010; Palatnik & Koichu, 2014) but they should be further developed. Our research group currently works in this direction and explores long-term geometry problem solving supported by online discussion forums.

As mentioned, the framework is only *exploratory*. The outlined mechanisms of attending to, proceeding of and shifting between objects of attention should be further unpacked. At this stage, it seems that further unpacking would require the adaptive use of selected theories that have been developed outside of the field of mathematics education. For instance, research on learners' decisions about how to allocate study-time (e.g., Metcalfe & Kornell, 2005) can be a source of insights about why some objects of attention are short-living, and the others are long-living. Research on hypothetical thinking and cognitive decoupling (Stanovich, 2009) can be useful for

understanding how the attended objects are mentally manipulated. The hope is that, eventually, the confluence framework would have power not only to usefully describe, but also explain the emergence of problem-solving ideas in different educational contexts.

NOTES

1. Ramon (2003) explains what a heuristic idea of a proof is as follows: "This is an idea based on informal understandings, e.g. grounded in empirical data or represented by a picture, which may be suggestive but does not necessarily lead directly to a formal proof." (p. 322). Note that not any heuristic idea is a key idea.

2. The next 10 sentences consist of an abridged version of the description that appears in Palatnik and Koichu (2014).

ACKNOWLEDGEMENT

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THEORIES THAT DO AND DON'T CONNECT: DOES THE CONTEXT MAKE A DIFFERENCE? AN EARLY INTERVENTION PROGRAMME AS A CASE

Lena Lindenskov, Pia Beck Tonnesen, Peter Weng, Camilla Hellsten Østergaard

The paper investigates possible contextual influences on networking theories. The paper draws its examples from early mathematics intervention programmes. The paper presents our view on how theories of students' cognitive development, theories of school mathematics, and theories of searching for effects in Realistic Evaluation tradition are chosen and networked in the Mathematics Recovery Programme in Australia, the UK, the USA and Canada. The paper discusses how the same kind of networking theories fit into Danish context for early mathematics interventions? Although Danish Programme for early mathematics intervention is inspired by Mathematics Recovery programme, the differences are substantially. The different practices reflect differences in networking theories.

CONTEXT AND OUR CONTEXT

The paper presents some general considerations on networking theories as contextually influenced. It is shown that culture does matter for mathematics education. As an example, teachers in London and in Beijing hold different views on mathematical learning and teaching. Teachers in London see syllabus and textbooks as less important in determining the content taught than interest and meaningfulness, which is again opposite to teachers in Beijing (Leung, 2006, p.33-4). Similarly, teachers in London see students' ability as more important for their learning than effort, which is the opposite of teachers in Beijing (p.34-5). . We claim that it is by now a well-known fact that culture matters for mathematics education, but how culture matters in networking theories, is in need for more exploration.

Our own context is a local, national context, where we are engaged in developing and researching interventions for marginal student groups in normal mathematics school classes. The first project, which started in 2009, focused on offering second graders considered to be at risk for mathematics difficulties one-to-one tutoring, 30 minutes a day, four days a week over 12 weeks. The most recent project started in 2014 and is called TMTM. TMTM means **T**idlig **M**atematikindsats **T**il **M**arginalgrupper [Early Intervention for Marginal Groups], meaning intervention using same materials for second graders considered to be at risk for mathematics difficulties and for highest performing second graders. Encouragement for starting the projects came from local and international sources, and from research interest as well as from educational policy.

We were encouraged to introduce early mathematics interventions by Nordic and international colleagues in mathematics education research and by national and municipal educational policy:

- International colleagues: Through reports and articles by and communication with colleagues, for instance in 2008 at ICME-11: Topic Study Group 7 on Activities and programs for students with special needs.

- Nordic colleagues: At conferences of the Nordic Research network on Special Needs Education in Mathematics (NORSMA) we presented new constructs.

- National educational policy: The 2004 OECD Review of the Danish primary and lower secondary school emphasizing the need to support failing pupils in mathematics in the first school years (Mortimore, David-Evans, Laukkanen & Valijarvi, 2004).

- National educational policy: Students failing at mathematics are described in the official guidelines to the 2003 national mathematics curriculum (UVM, 2003) and are described in detail in the official guidelines to the revised 2009 national curriculum (UVM, 2009).

- National and municipal educational policy: Implementation on a regular basis of early reading interventions in many schools in Denmark from the first grade. The implementation is seen as a successful support to individual children, who show signs of reading difficulties.

- Municipal educational policy: Politicians and school authorities at Frederiksberg, a municipality in the metropolitan area, focused on Mathematics in schools 2007–2013, including starting early intervention in Mathematics in all public schools.

We are inspired from the Mathematics Recovery Programme (MRP) as implemented in Australia, the UK, Ireland and the USA (Wright, Martland & Stafford, 2000; Wright, Ellemor-Collins & Tabor 2011). As a starting point for our own first tutor training we invited the national coordinator for Ireland, Noreen O'Loughlin, University of Limerick to give a lecture. Too, we are inspired from Dowker (2008), who underlines the importance of beliefs, feelings and motivation, and from Gervasoni and Sullivan (1997) and Gervasoni (2004), who underline the diversity of mathematical understandings amongst children identified as at risk, and who developed the theory of mathematical growth points.

One main element of MRP is an intensive one-to-one tutoring offered to first graders falling behind and at risk of mathematics difficulties. We appreciated, among other elements, that tutor training was a mandatory part of the programme. We appreciated that a learning framework as well as an instructional framework were included, both based on research. We also appreciated the diagnostic tutoring with adaptation of

instruction according to students' reactions. All these elements are copied in the Danish Model.

What worried us was the idea of tutors' formative assessment to identify the student's specific level of competence in order to appropriately adapt the instruction. We are sceptical about the idea of 'level' of competence as the main guiding principle for adaptation to individual students. This implies that while we value MRP's focus on tutor training, we prefer higher tutor autonomy on deciding how to adapt material to an individual student.

From the very beginning we realised that taking MRP as one starting point would be an inspiring endeavour for developers, researchers, tutors and teachers in Denmark. We explored, through research and theoretical reflection, the extent to which MRP aspects could be used directly, revised or rejected in Danish contexts, concerning among other aspects, the aspect of guiding principles for adaptation to individual students. Soon we realised that adaptation would not be an easy, straightforward process. Although Denmark is a western country like Australia, Ireland, UK and the USA, there still are differences in the labour market culture, in policy, and in school mathematics culture that may need to be taken into account.

Work and policy culture

The Danish labour market for teachers is highly regulated. Teachers must have nationally approved teacher training to work at public primary and lower secondary schools and teachers' wages and working conditions are regulated by each municipality in accordance with national regulations negotiated with teacher unions. Education is free and publicly financed. Even private schools are primarily public financed. School mathematics is regulated by national guidelines, by national tests in Grades 3 and 6, and by national written and oral final examinations after Grade 9 or 10. As mentioned above, the problem of students failing is described in national guidelines. Despite national regulations (or because of them?), at the school level mathematics teachers have a very high level of self-confidence and a very high level of influence on their daily teaching practice.

It is our impression that Danish teachers have a relatively high self-confidence and a relatively strong wish to influence. (We build on various international comparisons of labour culture.) This led us to include a group of teachers, in a decisive way, in design cycles in the development towards a Danish model, and it led us to make the material much open for teachers' adaptation.

School mathematics culture: aim and subject matter choice

School cultures in Denmark are influenced by the German/continental tradition in educational philosophy. Danish language has, like German language, two main concepts for 'Education': one is 'Dannelse' (German: Bildung), another is

'Uddannelse' (German: 'Erziehung'). Wolfgang Klafki's 'Kategoriale Bildung' is a frequently used concept in teacher training and in educational research in Denmark. In mathematics education scholars such as Wagenschein, Freudenthal, Wittmann, Mellin-Olsen, Skovsmose and Niss, are studied by many. At the level of school mathematics this tradition emerges in the common aim for primary and lower secondary school mathematics by pointing at everyday life, citizen life, creativity, problem solving, and democratic responsibility and impact:

> The aim is that students develop mathematical competences and acquire skills and knowledge in order to appropriately engage in math-related situations in their current and future everyday, leisure, education, work and citizen life.
>
> Subsection 2. Students' learning should be based upon that they independently and through dialogue and cooperation with others can experience, that mathematics requires and promotes creative activity, and that mathematics provides tools for problem solving, reasoning and communication.
>
> Subsection 3. Mathematics as a subject should help the students experience and recognize the role of mathematics in a historical, cultural and social context, and that students can reflect and evaluate application of mathematics in order to take responsibility for and have an impact in a democratic community. (UNI-C, 2014) (Our translation)

We concluded that despite the young age of the intervention students, according to the national aims and applications in many life spheres, creativity and problem solving should be included in a Danish model for early intervention, not as an appendix, but in the core of the frameworks. We also concluded that not only cognitive, but also affective and identity aspects have to be included, again not as an appendix, but in the core of the frameworks.

According to the choice of subject matter, the school mathematics tradition emerges in broad scope throughout all school grades and for all students. Danish students are not streamed before Grade 10. Mathematical competences, numbers and algebra, geometry and measurement, and statistics are included for all students from the very start of primary school. We concluded that therefore the focus in MRP on numbers and arithmetic had to be expanded with other content strands, such as geometry, measuring, and data handling, resulting in the areas A-J below:

A. Knowing numbers, names and symbols
B. Numbers as cardinal, ordinal and nominal [in Danish: identifikation] numbers
C. Basic strategies in addition and subtraction
D. Understanding basic number connections
E. Basic strategies in multiplication and division

| F. Basic descriptions and terms related to geometrical forms |
| G. Basic strategies and understanding related to geometrical forms |
| H. Strategies for recognising and producing numerical and geometrical patterns |
| I. Basic understanding of part-whole concept |
| J. Basic understanding of measuring |

The RMP covers A-E according to Wright et al (2006) and A-E and I according to Wright et al (2011).

THE VIEW OF MATHEMATICS AND OF DIFFICULTIES

We are inspired by the Danish scholar Ole Skovsmose and the Norwegian scholar Stieg Mellin-Olsen. The paradigm on learning mathematics as landscapes of investigation suggested by Skovsmose (2001), as opposed to the exercise paradigm inspired us. The metaphor of travel identified by Mellin-Olsen (1991) in teachers' thinking about instruction as a common teacher-student journey inspired us.

Since 2003 (Bøttger, Kvist-Andersen, Lindenskov & Weng, 2004) we have been involved in demonstrating mathematics learning as a journey in landscapes, which evolve with hills and holes as you travel, and where many routes can be appropriate for the teachers and students involved. This implies a new view on students in difficulties in learning mathematics, condensed in the construct 'math holes'.

Within the construct 'math holes' students in difficulties are not students lacking behind or students with special neurological characteristics as implied by some definitions of dyscalculia.

Within the construct 'math holes' students in difficulties are students who stopped progressing learning, and this is visualised as 'getting stuck in a hole'. Gervasoni and Lindenskov (2011) describe the construct with the following:

When mathematics is seen as a landscape it means that whenever students stop learning and feel stuck it is as if they 'fall into a hole'. There are several ways for a teacher to cope with a student's 'fall'. First, a teacher can invite the student to move to another type of landscape, maybe far away from the hole in which the student was stuck; this means that even when students fail to thrive in one area of mathematics there are still many other mathematics landscapes to experience and learn. Second, teachers can help students 'fill up' the hole from beneath with mathematical building stones; or third, teachers can 'lay out boards over the hole' in order to let the student experience new and smart mathematical approaches.

This contradicts the MRP's sole focus on numbers and arithmetic, and it contradicts the MRP idea of prescribed stages, tailored instruction building up precisely according to 'level' of student's competence.

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CONNECTING THEORIES - ADAPTATION AND EXTENSION?

Thus, the Danish developmental and research projects on early intervention are based on original construct, the Math Holes Construct. Also theories upon which international frameworks like MRP are built are utilised, but when necessary they are adapted according to the Math Holes Construct.

With Skott (in press), in our view networking theories is very much depending on what we mean by theory, constructs and conceptual framework. Skott points to decisive elements as a) preliminary understanding of concepts involved, b) theoretical stance on interpretation of these concepts, and c) the overall rationale for engaging in the field of inquiry.

The Math Holes Construct implies differences compared to RMP concerning all three elements. In the evaluation of RMP by Smith, Cobb, Farran, Cordray and Munter (2013) in twenty schools in two states in USA, the theories built into MRP are explained. Two kinds of theories are included. The first kind is cognitive models of children's numerical reasoning to delineate developmental progressions for four early numerical skills and concepts. This includes work from

> a number of researchers (Baroody, 1987; Baroody & Ginsburg, 1986; Carpenter & Moser, 1982, 1984; Clements, 1999; Fuson, 1988, 1992; Steffe, Cobb, & von Glasersfeld, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983). (pp.400-402)

The second group of theories draws on research on early number instruction for RMP's Instructional Framework with its developed

> instructional tasks appropriate for children at different levels of the Learning Framework (…) (Baroody, 1990; Beishuizen, 1993; Carpenter, Franke, Jacobs, Fennema, & Empson, 1997; Clements, 1999; Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Fuson, 1990; Fuson, Wearne, et al., 1997; Hiebert & Wearne, 1992). (p.402)

It is, in our view of the Math Holes Construct, crucial to draw attention to the individual pupil: Much still needs to be done, following Ginsburg's (1997) more than 15-years-old call for teaching experiments focusing on pupils with learning difficulties, as children today are exposed to physical and social environments that are rich in mathematical opportunities. In Denmark as in many other countries, children today are exposed for instance to even very big numerals in computer and board games and in family activities. This questions the generality of the existing cognitive models of children's reasoning and of the theories of instruction. For instance it questions the motivating effect as well as the learning effect of splitting instruction on numbers into first 1-20, and then 1–100, as it is done in MRP.

Children's exposure rapidly develops, and no one can predict what will happen just a few years from now. This means that the generalisation of identified mathematics

learning trajectories in research studies may not be generalizable to all children in the world. On the other hand, the learning trajectories, which are found in research, and upon which MRP have built their stages, do show valuable insight into children's learning, which is not visible for teachers even with profound training. We do advocate for teachers to obtain extended knowledge on researched learning trajectories.

The adaptation of theories from MRP on learning and instruction trajectories is in the Danish model for Early Intervention in Mathematics, devised according to Math Holes Construct. We see described trajectories as a valuable tool for teachers' formal assessment and adaptation of early intervention ideas to individual students. But not used 'automatically': we agree with Simon (1995) that described learning and instructional trajectories - as well as constructs like five levels of geometrical reasoning from van Hiele (1985/1959) - may serve as teachers' a priori hypotheses on their students' competence and needs. Besides adaptation of theories according to Math Holes Construct, the Danish Model extends MRP into the mentioned ten key mathematical aspects (A through J below) and five psychological, sociological and organisational aspects, which are

Beliefs about and attitudes towards mathematics,

Pupils with a need for early intervention and identification of them,

Integration of pupils' learning from early intervention into the mathematics classroom,

Communication with and integration of parents in early intervention,

Integration of early intervention into the mathematics profile of the school.

Tutor material is published in a 191 pages book (Lindenskov & Weng, 2013).

## THEORIES FOR REALISTIC EVALUATION OF EARLY INTERVENTION PROGRAMS

Realistic evaluation (Pawson & Tilley, 1997) is to our understanding more and more used in evaluation of complex social programmes and policies. Early mathematics intervention programmes are indeed complex social programmes. The intention by using realistic evaluation is to provide broad results by looking at effect not only as outcomes but also to theoretically consider mechanisms that cause outcomes and lack of outcomes. This can be visualised as what Smith et al (2013) term as The program logic model.

The expected mechanisms that cause outcomes and lack of outcomes have tremendous implications for methodological issues of effect studies. In the evaluation of MRP presented in Smith et al (2013) it is seen as data pollution and lack of fidelity if tutors and teachers collaborate, because the collaboration is not included in the

program logic model. The presentation underlines that lack of collaboration between tutors and teachers may be a cause behind the relatively weak long-term effects revealed by the evaluation, despite strong short time effects.

The program logic model for the Danish Model TMTM shown below, includes all elements from the MRP model. But the TMTM extends MRP by including 'tutor-teacher collaboration', 'continued tutor professional development' and 'Improvement of the school's mathematical profile'.

TMTM Program Logic Model
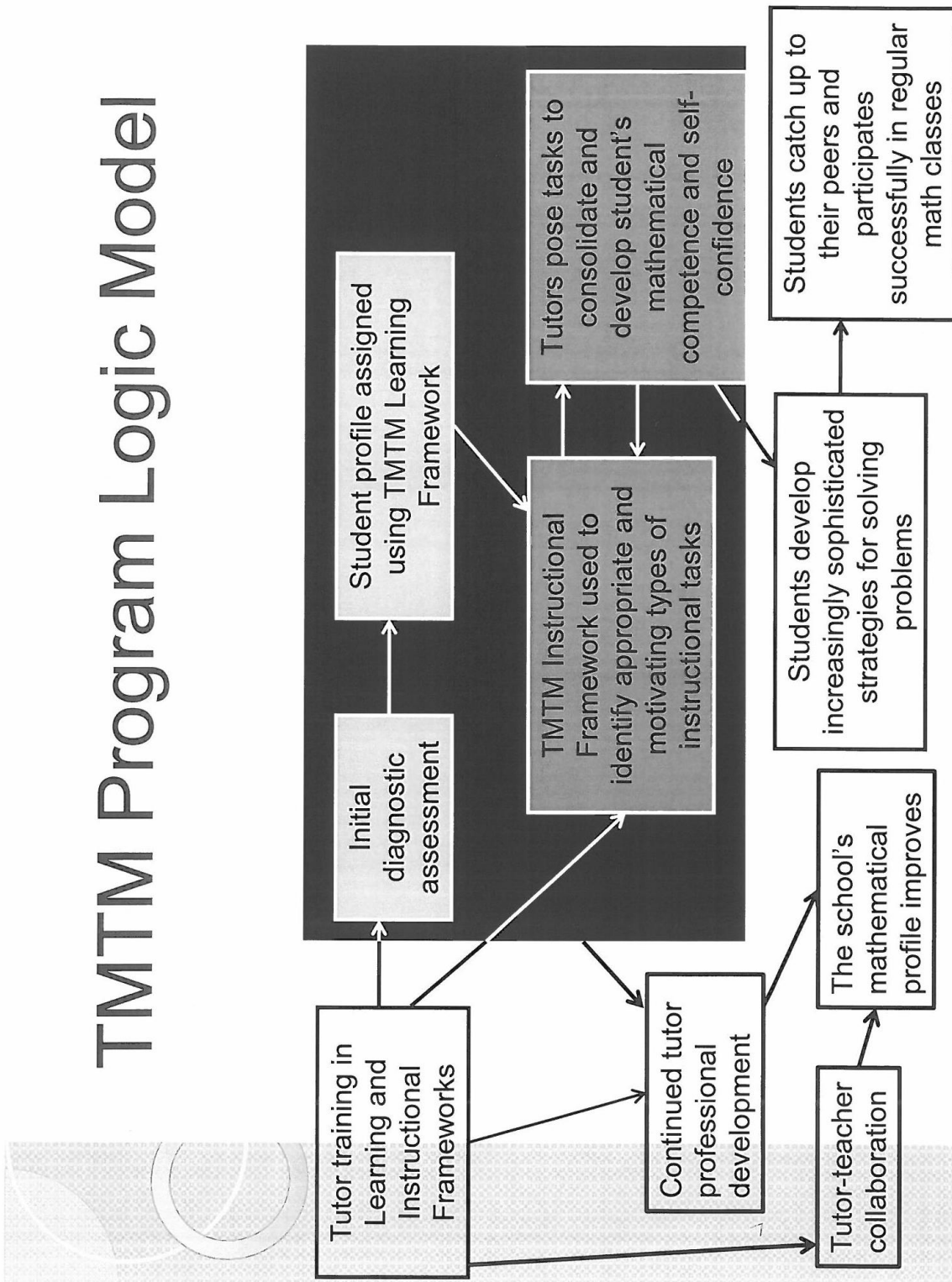
Although we in an ongoing effect study of TMTM ask tutors and teachers not to collaborate in the period when the one-to-one intervention is going on, as we have chosen to measure short time effects of isolated tutor intervention, we find it meaningless on a long time basis to exclude any part of the TMTM program logic model in long term effect studies.

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TOOL USE IN MATHEMATICS: A FRAMEWORK

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In the course of research into the interpretation of tools in the didactics of mathematics I found both voids and conflicts. This paper presents the results of my research and a resultant statement on tool use in mathematics education. The statement incorporates constructs from several theoretical frameworks and I consider the consistency of my statement on tool use with regard to activity theory.

Key-words: activity theory, agency, artefacts, mediation, tools

INTRODUCTION

In the course of work on tool use in mathematics I examined literature which I summarise in this paper. The literature sit in various theoretical frameworks and this paper, in the language of Prediger, Bikner-Ahsbahs & Arzarello's (2008), can be considered as an attempt to 'synthesise' frameworks with regard to a statement on tool use in the didactics of mathematics. This synthesis, however, does not aim at synthesising complete theories but synthesising activity theory with principles from other theories. This paper has the following structure:a definition of tools; a survey of theoretical frameworks with regard to tools; an exposition of activity theory with regard to tools; actor network theory ideas that augment an activity theoretic account of tools; an activity theoretic statement on tool use in mathematics education which incorporates ideas from outside of activity theory, and a consideration of the consistency of this statement with regard to networking theories.

TOOLS: A DEFINITION

I define a tool via four action-related distinctions, the first of which is between an artefact and a tool. An artefact is a material object which becomes a tool when it is used by an agent to do something; a compass becomes a tool when it is used to draw a circle (its intended purpose) or to stab someone. This establishes that tool use cannot be separated from the animal using the tool and the purpose of use. After being used as a tool (for whatever purpose), the compass returns to being an artefact. The materiality of an artefact is not just that open to touch. An algorithm, e.g. for adding two natural numbers, is an artefact. It is material in as much as it can be written down or programmed into a computer. My second distinction is between an artefact/tool and ways of using the artefact/tool. For example, I could use a calculator to perform 45+67 by typing in '45+67=' or I could imitate the standard written algorithm (adding the units, storing the result) and adding the tens and adding on my stored results. My third distinction is between the mental representation of a tool and material actions in tool use. This distinction comes with an interrelationship: to carry out material actions with an artefact you need some form of mental representation,

which may be quite crude, of how to act with the artefact-tool, but actions with the artefact-tool will provide feedback to the user which may change the mental representation. My fourth distinction is between signs and tools. Signs, like tools, are artefacts but a sign points to something whereas a tool does something. Having said this, signs or systems of signs, can function as tools. Similarly representations can function as tools but they may also have non-tool functions, e.g. to point to an object.

Is there such a thing as a 'mathematical tool'? – only in use, a compass is a mathematical tool when it is used to draw a circle but not when it is used to stab someone. When artefacts are used for mathematical purposes they generally incorporate mathematical features, e.g. a compass encapsulates the equidistant relationship between the centre and points on the circumference of a circle.

A SURVEY OF FRAMEWORKS WITH REGARD TO TOOLS

I conduct an historical tour of theoretical frameworks employed in Western mathematics education. I select papers from the 1960s to the present which reflect dominant 'grand theories' over this time that address or ignore tools. Behaviourism regarded artefacts as a means of stimulating a response in a subject. Suppes (1969), for example, considers computers as tutorial systems that can provide:

> individualized instruction [where the] intention is to approximate the interaction a patient tutor would have with an individual student … as soon as the student manifests a clear understanding … he is moved on to a new concept and new exercises." (ibid., p.43).

Suppes does not consider the environment in which the tool is used. During the period when behaviourism ruled two psychologists, E. & J. Gibson, ventured on a non-behaviourist route to the theory construct of affordances (and constraints):

> The *affordances* of the environment are what it *offers* the animal, what it *provides* … If a terrestrial surface is nearly horizontal … nearly flat … and sufficiently extended (relative to the size of the animal) and if its substance is rigid (relative to the weight of the animal), then the surface *affords support*. (Gibson, 1979, p.127)

There is no mention of tools in this quote but mathematics educators have learnt that the construct 'affordances' is useful in considerations of the relevance of artefacts and digital software environments to students' mathematical learning.

The demise of behaviourism in mathematics education saw the rise of cognitive studies and Piaget was the Guru. The interesting thing about Piaget's extensive output with regard to tool use is that he says nothing at all about the role of tools in cognitive development. Piaget's work inspired several 'local theories' in mathematics education: Brousseau's theory of didactical situations (TDS), constructionism and constructivism. TDS was developed over decades starting in the 1960s. The influence of Piaget in Brousseau's work is explicit. An important construct of TDS came to be called the 'milieu' which includes the teacher, the materials and the designed learning

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strategies. I know of no explicit consideration of mathematical tools in 20th century TDS but tools are a part of the milieu. Papert, who spent several years with Piaget, experimented with children using the computer language *Logo*. Constructionism views that learning occurs through the construction of meaningful products. *Logo* is integral to constructionism but, despite statements that these languages equip students with tools to think with, there is no clear statement as to what a tool is in Papert (1980) and a clearer constructionist view of tools did not emerge until Noss & Hoyles (1996) – by which time constructionism had relinquished its Piagetian roots and embraced socio-cultural viewpoints.

Piagetian ideas were the inspiration for the constructivism, which focused on the ontogenic development of the individual child but developed to include a focus on microgenetic (child-environment) development (social constructivism). Yackel & Cobb (1996) is a developed form of social constructivism which examines teacher-student discussions and argumentation in a classroom context. This paper introduced the construct 'sociomathematical norms'. The classroom considered in the paper had various resources (centicubes and an overhead projector) but the paper does not mention tools. This neglect has been noticed by others, e.g. Hershkowitz & Schwarz (1999, p.149) "... socio-mathematical norms do not arise from verbal actions only, but also from computer manipulations as communicative non-verbal actions."

In summary, 20th century mathematics educator frameworks influenced by Piagetian ideas had little to say on tools in learning and teaching but outside of mathematics education, deep ideas, published in the 1970s, on tools were in circulation.

Wartofsky (1979) includes an essay on perception, "an *historically* evolved faculty … based on the development of historical human practice" (ibid., p.189). Practice is "the fundamental activity of producing and reproducing the conditions of species existence … human beings do this by means of the creation of artefacts … the 'tool' may be *any* artefact created for the purpose" (ibid., p.200). Wartofsky extends the concepts of artefacts to the skills required to use artefacts as tools:

> *Primary* artefacts are those directly used in this production; *secondary* artifacts are those used in the preservation and transmission of the acquired skills or modes of action or praxis by which this production is carried out. Secondary artefacts are therefore *representations* of such modes of action (ibid., 202)

Vygotsky (1978), published posthumously, was to have a profound influence on mathematics education. Vygotsky was interested in language, signs and mediation. His interest in tools was in their mediating qualities, "the basic analogy between sign and tool rests on their mediating function that characterizes each of them" (1978, p.54). The difference between signs and tools rests on:

> The tool's function is to serve as the conductor of human influence on the object of activity; it is *externally* oriented; it must lead to a change in objects … The sign, on the

other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself; the sign is *internally* oriented. (ibid., p.55)

The reader, however, may note the influence of both Wartofsky and Vygotsky in my definition of tools above. I now move on to the work of B. Latour and P. Rabadel.[1] Latour is a sociologist and, around 1980 and with others, established what is now called actor network theory (ANT); Latour (2005) is a fairly recent exposition. ANT is a theory about how to study social phenomena – by following the actors, where an actor is "*any thing* that does modify a state of affairs by making a difference" (ibid., 71). ANT symmetrically views both society and nature as being in a state of flux and looks to the performance of the actors in situations. Objects (artefacts/tools) can make difference in performance and so can be actors, exerting agency, in the playing out of social situations. Pickering (1995), who is 'almost ANT' in my opinion, examines practices of 20th century elementary particles physics. He accepts ANT's human and material agencies and adds 'disciplinary agency' (in our discipline $a+a=2a$ regardless of what we might want it to be). He proposes a 'dance of agency' where, in the performance of scientific inquiry, human, material and disciplinary agencies "emerge in the temporality of practice and are definitional of and sustain one another" (ibid., p.21). I see this in 'dance' in techno-mathematics lessons – a myriad of influences between students, teachers, computers and mathematics.

Rabadel introduced the 'instrumental approach' which distinguishes between an artefact, as a material object, and an instrument as a psychological construct. An instrument is an emergent entity that begins its existence when a person appropriates an artefact to do something; this has influenced my distinction, above, between an artefact and a tool. The instrumental approach has been well known in mathematics education since Guin & Trouche (1999). This views an instrument as a composite entity composed of the artefact and knowledge (knowledge of the artefact and of the task constructed in using the artefact). Artefact and the agent(s) are interrelated: the artefact shapes the actions of the agent, *instrumentation*; the user shapes the use of the artefact, *instrumentalisation*. The process of turning an artefact into an instrument is called 'instrumental genesis'. The agent brings her/his knowledge and the artefact brings its potentialities and constraints to the artefact agent interaction.

I leave my historical tour at this point with the observation that a lot of the frameworks used in mathematics education pay scant regard to the nature of the tools used in doing mathematics but frameworks initiated by Wartofsky, Vygotsky, Latour, Pickering and Rabardel provide interesting, though diverse, insights into the role of tools in activity. I now turn to the focus framework of this paper, activity theory.

ACTIVITY THEORETIC CONSIDERATION OF TOOLS

I briefly outline activity theory (AT), trace its genesis into mathematics education research (MER) and consider differences in approaches.

AT is an approach to the study of human practices. It sees constant change (flux) in practice. Activity became a focus for Vygotsky in his conviction that consciousness originated in socially meaningful activity. In AT 'object orientated activity' is the *unit of analysis*, that which preserves the essence of concrete practice. 'Object' here refers to *raison d'etre* of the activity. Educators employing AT must take care that they do not merely employ the word 'activity' without considering the object and the unit of analysis. Vygotsky's AT is often presented via a triangle with 'subject', 'object' and 'mediating artefacts' at its vertices. Leont'ev (1978) developed Vygotsky's work by considering individual and collective *actions* (usually with tools) and *operations* (things to be performed or modes of using tools) involved in socially organized *activity*. Engeström (1987) extends Vygotsky's and Leont'ev's frameworks to 'activity systems' and extends the focus on mediation through signs and tools to multiple forms of mediation including the community and social rules underlying activity. Activity systems research often examines interactive activity systems with a focus on the objects of activity in the two systems; the place of tools in such research usually emphasises tool use in the context of the whole system. I now turn to the influence of activity theory in MER.

I was curious of AT's introduction into Western MER literature and I traced its introduction into the journal *Educational Studies in Mathematics* (ESM). Two AT papers appeared in ESM in 1996. Crawford (1996) is an exposition of Vygotskian AT and asks "What difference does the use of tools such as computers and calculators make to the quality of human activity?" (ibid., p.47) but does not explore the nature of tools further. Bartolini Bussi (1996) reports on a teaching experiment on geometric perspective. The word 'tool' has two uses in the paper: Leont'ev's theory as a tool for analysis; 'semiotic tools', which are defined via examples. In 1998 two ESM AT papers considered tool use in different ways to Bartolini Bussi (1996). Chassapis (1998) focuses on the processes by which children develop a formal mathematical concept of the circle by using various instruments to draw circles: by hand; using circle tracers and templates; and using a compass. "The process of learning to use a tool … involves the construction of an experiential reality that is consensual with that of others who know how to use [the tool]" (ibid., p.276). Pozzi, Noss & Hoyles (1998) focuses on nursing and ask "how do resources enter into professional situations, and how do they mediate the relationship between mathematical tools and professional know–how?" (ibid., p.110) The paper states that AT provides evidence that "acts of problem solving are contingent upon structuring resources, including a range of artefacts such as notational systems, physical and computational tools" (ibid., p.105). Radford (2000) focuses on early algebraic thinking "considered as a sign-mediated cognitive *praxis*" (ibid., p.237):

to accomplish actions as required by the contextual activities … The sign-tools with which the individual thinks appear then as framed by social meanings and rules of use and provide the individual with social means of semiotic objectification (ibid., p.241).

The first mention of Engeström in ESM is in Jaworski (2003, p.249). This outlines "*insider* and *outsider* research and *co-learning* between teachers and educators in promoting classroom inquiry" and is not concerned with tool use in mathematics.

Thus, although AT is quite an old theory, it is a fairly recent theory in terms of Western MER and there is wide variation with regard to the meaning of tools in ESM AT papers from 1996 to 2003. After 2003 a considerable number of ESM papers used AT as a theoretical papers but I do not have room to summarise. To get a handle on contemporary AT conceptions of tools in MER I go to a special edition of *The International Journal for Technology in Mathematics Education* devoted to AT approaches to mathematics classroom practices with technology. For reasons of space I focus on three (of 11) papers which illustrate a range of approaches.

Chiappini (2012) focuses on the learning and teaching of algebra with software with a visual 'algebraic line' and conventional algebraic notation, to draw students' attention to the culture of mathematics. Chiappini is interested in 'cultural affordances', which, "allow students to master the meanings, values and principles of the cultural domain" (ibid., p.138). With regard to tools, Chiappini's focus is the evaluation of software designed to exploit visuo-spatial and deictic affordances and allow teachers to consolidate student learning. Ladel & Kortenkamp (2013) focuses on the design and use of a multi-touch-digital-table to engage young children in meaningful work with whole number operations, "We want to restrict the students' externalizing actions to support the internalization of specific properties … mediation through the artefact is characterized by restriction and focussing." Artefacts are the focus of attention and the word 'tool' is not mentioned in the paper. They hold that "the artefact itself does not have agency and is only mediating … [but] the artefact changes the way children act drastically and in non-obvious ways" (ibid., p.3). Mariotti & Maracci (2012) outline the Theory of Semiotic Mediation (TSM) with regard to "the use of artefacts to enhance mathematics learning and teaching, with a particular focus on technological artefacts" (ibid., p.21); like Ladel & Kortenkamp (2013) above, the word 'artefact' is favoured over the word 'tool'. This paper continues the work of Bartolini Bussi (1996) considered in the previous section and is critical of research where "the mediating function of the artefact is often limited to the study of its role in relation to the accomplishment of tasks" (ibid.). TSM views that "teaching-learning … originates from an intricate interplay of signs… mathematical meanings can be crystallized, embedded in artefacts and signs" (ibid.) The paper presents a rather strange (to me) take on mediation, "The mediator is not the artefact itself but it is the person who takes the initiative and the responsibility for the use of the artefact to mediate a specific content" (ibid. p.22). To mediate the learning of mathematics the

teacher has to design specific circumstances, a didactical cycle, aimed at fostering specific semiotic mediation processes.

Differences in the papers outlined above include the unit of analysis, cognition, the words used, mediation and agency. Some papers explicitly state the unit of analysis, e.g. Chassapis (1998), but many do not. Chassapis' unit of analysis is 'quite small' compared to Engeström's, the activity system itself. I think the 'size' of the unit of analysis impacts on the extent to which the AT analysis permits a study of microgenetic learner development with tools (i.e. Chassapis' unit of analysis allows a focus on cognition and tool use but details of cognitive development are easily 'lost' when the focus is on activity systems). With regard to the words used it is clear that some scholars use 'artefact' for what I refer to as a tool. This seems unimportant but the difference between sign and tool is important and the fact that this difference is sometimes blurred does not downplay this importance; some of the papers do not consider signs vs tools. With regard to mediation the biggest difference is between Ladel & Kortenkamp, where artefact mediation is central, and Maracci & Mariotti, which holds that people and not artefacts mediates. My final consideration concerns agency. Only Ladel & Kortenkamp comments on this, to claim that artefacts do not have agency. The differences noted above show that AT in MER is a collection of approaches, not a unified theory, and there are many ways to view tools within AT.

ANT IDEAS THAT AUGMENT AN AT ACCOUNT OF TOOLS

I am drawn to AT as a framework because it mirrors my view that tools are important but tool use is not an activity in itself though tool use and activity are interrelated. But I detect an anthropocentric position in AT – even though AT recognises that people think through/with tools, people are at the centre, they appear as 'the' agents. This anthropocentrism is explicit in Maracci & Mariotti's view that artefacts are not mediators and Ladel & Kortenkamp's statement that artefacts do not have agency. I think tools can be powerful things and I am drawn to an ANT view on material agency, but can ANT ideas be brought into AT? I first look at a potential major obstacle to networking these theories and a difference between Latour and Pickering.

Miettinen (1999) considers ANT and AT as approaches to studying innovations and locates the main division between these approaches as ANT's generalised principle of symmetry which states that the same "vocabulary must be used in the description and explanation of the natural and the social … no change of register is permissible when we move from the technical to the social aspects of the problem studied" (ibid., pp.172-173). This is a problem for AT because the object (of activity) is generated from human needs. OK, humans do generate the object but once the object is established the agency which follows in the activity can be distributed. Indeed, Latour (2005) states that he abandoned most of the symmetry metaphor because what he had in mind was a "joint dissolution of both collectors" (ibid., p.76). Pickering (1999, p.15) also considers the generalised principle of symmetry to be problematic,

"As agents, we humans seem to be importantly different from nonhuman agents". With the generalised principle of symmetry 'put in to perspective' I now look to two commonalities in principles between Latour and Pickering: focus on performance; don't restrict agency to animals (humans) alone.

Latour (2005) mentions performance with to regard groups, social aggregates. Classical sociologists are accused of making *ostensive* definitions of groups – there's a group of teachers –and focusing on stability but, from an ANT point of view, "the rule is performance and what has to be explained, the troubling exceptions, are any type of stability over the long term [and this cannot be explained] without looking for vehicles, tools, instruments, and materials able to provide such a stability" (ibid., p.35). This focus on performance is akin to flux in AT. A sketch of a performative view of science is presented early in Pickering (1995, p.6), instead of a world where scientists only generate knowledge from facts, he sees a world filled with agency:

> The world … is continually doing things, things that bear upon us not as observation statements upon disembodied intellects but as forces upon material beings … Much of everyday life … character of coping with material agency, agency that comes at us from outside the human realm and that cannot be reduced to anything within that realm.

Later, in Pickering (1995), 'disciplinary agency' and the 'dance of agency', as described above, are introduced. Neither Latour nor Pickering are concerned with mathematics education but their 'multi-agent' stance resonates with my experience of mathematics classrooms. When a teacher uses a tool in a mathematics class, then s/he is only one of the agents in the activity, other potential agents are: other teachers; students; the curriculum; the institution; other available artefacts; and the tool itself.

I now turn my attention to mediation and what it is that mediates. There are at least four contenders: language, signs, artefacts and people. I think the problem here can be viewed via the ostensive-performative distinction. Scholars have different interests and tend to point to something and say "that (those) is (are) the mediator(s)" whereas the mediator in a specific situation exists in relation to what is actually done (the activity/performance). I am, for instance, interested in artefact/tool-mediation but two learners may be involved in ostensibly similar activities with a mathematical tool but one learner may be heavily reliant on the tool whereas the use of this tool to the other learner may be peripheral; mediation by the tool comes down to the actual use of the tool. Similarly Mariotti & Maracci (2012) may expect the mediator to be the teacher but I doubt if this is always the case. Latour (2005, p.39) appears to present a similar idea in distinguishing between mediators and intermediaries, "An intermediary … is what transports meaning or force without transformation … Mediators transform, translate, distort, and modify".

A STATEMENT ON TOOL USE IN MATHEMATICS EDUCATION

The considerations above, together with those in the previous three sections, provide a basis for the following statement (in italics) on tool use in mathematics education.

AT provides a framework to interpret tool use in practice but the level of detail on tool use will depend on the 'size' of the unit of analysis. An AT account of tools would benefit from being augmented by constructs from instrumentation theory and the theory of affordances. Activity is mediated by human and non-human mediators but this mediation cannot be stipulated in advance of the performance of the activity. Human and non-human agents impact the activity; as with mediation, the impact of these agents cannot be stipulated in advance of the performance of the activity.

I now state my networking argument. The theories of affordances and of instrumentation have few assumptions and a lot of application. Recognition of the relationship between learners and their environments is important in AT as is the process by which an artefact becomes a tool for learners. Both theories can be used in MER to shed light on the action and operation aspects of AT without compromising any tenets of AT. With regard to taking ideas from Latour and Pickering I focus on the two principles outlined above. The 'focus on performance' principle is entirely consistent with the concept of flux in AT. AT focuses on describing practice and tools (and, I add, other things) are used as they are used (or not) – there is no pre-ordained plan. As for not restricting agency to humans alone, well, this is a problem for many activity theorists because the object of an activity is generated by humans. But if the principle of non-human agency is weakened to restrict non-humans from initiating activity, then I don't think there is a problem.

NOTES

1. My initial draft considered the use of his anthropological theory of didactics (ATD) by M. Artigue and J-b. Lagrange, but my draft was too long. I look forward to discussing this at CERME.

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ADAPTIVE CONCEPTUAL FRAMEWORKS FOR PROFESSIONAL DEVELOPMENT

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In this paper, I present the notion of adaptive conceptual frameworks that I have used to conduct design-based research with the aim of developing ICT supported mathematics instruction. In this approach, empirical data is connected with various theories in an adaptive and iterative process. I differentiate between Conceptual Framework for Development (CFD) and Conceptual Framework for Understanding (CFU) depending on how the frameworks are used in the design process. Using adaptive conceptual frameworks contribute to the transparency in the design process by making explicit the levels at which different theories operate.

INTRODUCTION

During the last decades, several similar methodologies have emerged that address the desire to conduct educational research with relevance for school practices. For example, design-based research aims explicitly at developing theories that could do "real work" by providing theoretically underpinned guidance on how to create educational improvement in authentic settings (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003; McKenney & Reeves, 2012). A common feature of these approaches is the design of teaching activities in an iterative design process that shares many similarities with teachers' daily work.

This paper contributes to research by describing how the design process may be co-determined by the interaction between different stakeholders such as researchers, teachers, disciplinary knowledge, theoretical frameworks, and other resources. At the core is the development of adaptive conceptual frameworks that were used to guide and justify an intervention in a lower secondary school with the overall aim of developing ICT supported mathematics instruction. These efforts have been inspired by co-design, as a design methodology that highlights the importance of involving different stakeholders such as teachers in the design research process in order to address the issue of ownership of innovation (Penuel, Roschelle, & Shechtman, 2007). Furthermore, working in close collaboration with teachers deepens our knowledge about pragmatic issues and promotes development of "*innovations that fit into real classroom contexts*" (ibid. p.52). Following the conceptualization of knowledge proposed by Chevallard (2007) in the Anthropological Theory of the Didactic (ATD), the two different perspectives of understanding and development could be viewed as two inseparable aspects of knowledge, integrating a practice that includes the things teachers do to solve different educational tasks (*Praxis*) with a

discursive environment that is used to describe, explain, and justify that practice (*Logos*). The adaptive conceptual frameworks explicitly address both perspectives.

The case study presented in this paper involves two lower secondary mathematics teachers. Empirical data is only used to motivate the development of the adaptive conceptual frameworks. Thus, a full analysis of the empirical data with respect to the intended learning objectives is not provided. The purpose of this paper is to describe how the use of adaptive conceptual frameworks has contributed to meet the emerging needs in a design process of ICT supported mathematics instruction during one design cycle.

ADAPTIVE CONCEPTUAL FRAMEWORKS

In this approach the researcher connects empirical data with various existing theories that are chosen in retrospect and that are used to generate additional empirical data in an iterative, incremental and adaptive process. Thus, theory is not applied onto practice, it is more about a "progressive interaction between theory and practice, by means of appropriating existing theoretical tools" (Bartolini Bussi, 1994, p. 127). Furthermore, the adaptive conceptual frameworks are considered in a state of flux and changeable according to the different challenges that might emerge when conducting design-based research. Thus, the adaptive conceptual frameworks should be regarded as tentative and a result of a research work that has similarities with research that sometimes is portrayed by the "bricolage" metaphor (Kincheloe, 2001), particularly regarding the efforts of embracing methodological flexibility and plurality of theories. From this perspective, this research approach aligns with the Singerian inquiry system (Churchman, 1971; Lester, 2005).

The workflow of the formal stages of a design cycle is illustrated in Figure 1. Each design cycle starts with a planning phase, followed by an implementation phase involving the teachers. The cycle is completed with an evaluation of outcomes. Three different frameworks are distinguished depending on their role in the different phases:

- methodological framework for professional development (MFPD),
- conceptual framework for development (CFD),
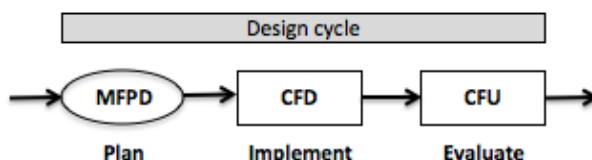- conceptual framework for understanding (CFU).

Figure 1: The adaptive frameworks for research and professional development

The researcher uses the methodological framework for professional development (MFPD) to plan the interventions involving the teachers and to operationalize his

current understandings before engaging in a new design cycle. The conceptual framework for development (CFD) is used to describe and justify the different activities that the researcher engages in together with the teachers. Finally, the conceptual framework for understanding (CFU) is used to understand the outcomes of an intervention and to plan the next design cycle. While the CFD and CFU naturally share similarities, since they both put focus on the design process, the MFPD should be regarded as a separate framework for organizing and supporting the teachers' professional development.

The different frameworks consist of multiple components, which need to be considered carefully how they interact. For this purpose, the categorization presented by Prediger, Bikner-Ahsbahs, and Arzarello (2008) of different levels of connected theoretical approaches was used. In their landscape of different levels of integration, the authors present a scale ranging from one extreme of ignoring other theories to the other extreme of unifying theories globally. Those strategies that are intermediate are called networking strategies. Networking strategies include strategies such as *comparing*, *combining*, *coordinating* and *integrating locally*. According to Prediger et al. (2008) the strategies of *coordinating* and *combining* are mostly used for a networked understanding of an empirical phenomenon or a piece of data and are typical for conceptual frameworks that, as in our case, not necessarily aim for a coherent theory. While comparing and contrasting always are possible the strategies of *coordinating* and *combining* can be a more difficult task especially if the theories are not compatible relative a specific purpose. The *coordinating* strategy is in turn used when a conceptual framework is built on well-fitting theoretical elements (ibid.). The networking strategies used in this study were *comparing* and *coordinating*.

THE BACKGROUND OF THE CASE STUDY

The participating teachers were involved in a developmental project in their school on how ICT could enhance their students' learning of mathematics. The teachers participated in a one-day event with lectures and hands-on learning activities developed by researchers from media technology and mathematics education. One specific learning activity was designed to stimulate students to communicate, collaborate and generate general problem solving strategies (Sollervall & Milrad, 2012). Mobile phones were used in this activity to bridge between formal and informal learning spaces. During the discussions about the activity the teachers seemed to be more worried about the practical issues rather than the didactical issues. This made the researcher aware of a possible misunderstanding. My concern was that connecting between the students' actions outdoors and a mathematical content is not necessarily a straightforward task. A successful orchestration would depend on the quality of the student-generated artifacts as well as the teachers' ability to orchestrate this remaining part of the activity performed indoors.

Later on, two of the mathematics teachers from the school and I met to discuss the prospects of developing new activities supported by ICT. The teachers expressed their concerns about their students' inability to use the distributive law and we all agreed on that it would be interesting to focus on algebra. The teachers had themselves completed the above-mentioned activity, which also could be used to address students' conception of the distributive law by connecting multiple representations (ibid.). Using the activity with this particular focus towards the distributive law would not require any modifications of the activity itself but would require the teacher to orchestrate the activity towards this goal. None of the teachers seemed to perceive this opportunity and the continued discussions revealed that they did not know about possible geometrical representations of the distributive law.

These circumstances influenced the design process in a very straightforward manner. For the planning phase of the design, the researcher decided to address the teachers' ability to adapt ICT to different situations and towards different goals. At the moment, perhaps this was more important than developing new activities with the teachers. With this pre-understanding the planning phase of the design was initiated.

METHODOLOGICAL FRAMEWORK FOR PROFESSIONAL DEVELOPMENT

The methodology of collaborative design based research is at the same time a process of professional development for the teachers (Penuel et al., 2007) and any change in teachers' knowledge base, attitudes and beliefs that this process may require should be regarded as a gradual and difficult (Guskey, 2002). The teachers' insufficient understanding of mathematical representations was taken as a constraining factor for the teachers' participation in the design process. To address this issue, two complementary theories were used to guide and plan for the teachers' professional development. One of the frameworks specifically focuses on knowledge for teaching mathematics: Mathematical knowledge for Teaching (Loewenberg Ball, Thames, & Phelps, 2008) and the other framework focus on the affordances provided by ICT and on the integration of ICT in different subject areas: Technological Pedagogical Content Knowledge (Koehler & Mischra, 2008).

The strategy of *comparing* (Prediger et al., 2008) was used to identify common principles in these two theories related to the use of ICT to support students learning mathematics. Based on this comparison, the researcher decided to specifically recognize and support teachers' understanding of the affordances for representation and communication provided by ICT.

CONCEPTUAL FRAMEWORK FOR DEVELOPMENT

The idea was to use the dynamic geometry software GeoGebra (www.geogebra.org) to develop an application, with focus put on providing affordances for representation, that the teachers could use in a learning activity to address their students' conception

of the distributive law. The researcher was interested in understanding how the teachers would perceive and make use of the specific affordances for representation and communication provided by the application in "live" settings. Thus, the software was an instrument for the researcher to provide competence development as well a didactical tool for the teachers to use with their students. The teachers were not familiar the software so the application was designed for them as end-users to operate only by using "click and drag" features.

Inspired by the work of Duval (2006), the dynamics of GeoGebra is used to illustrate how numerical expression can be interpreted and represented geometrically. Although figures and expressions are organized in a determined order in the application (see Fig. 2), the teacher still needs to consider how to use the application and create a hypothetical learning trajectory (HLT), i.e. "the consideration of the learning goal, the learning activities, and the thinking and learning in which students might engage" (Simon, 1995, p. 133). In other words, the researcher made the didactical design but the pedagogical design was intended for the teachers to decide.
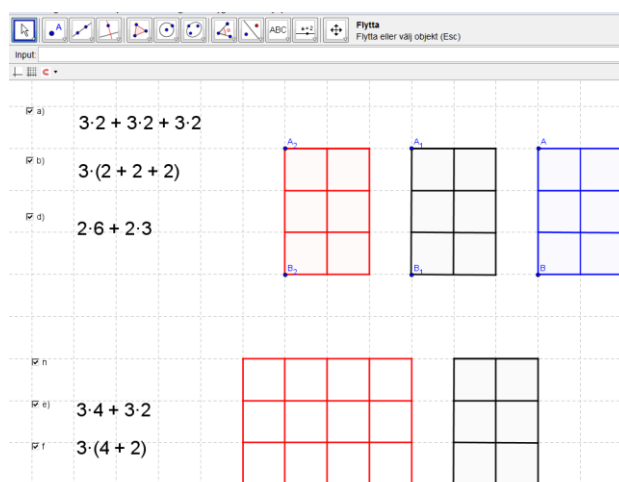

Figure 2: Snapshot of the application, implemented in GeoGebra

When the application was presented to the teachers they wanted immediate access to it. They seemed to recognize the limitations of the explanations that they normally used that were exclusively based on instructions on how to manipulate different variables. The teachers were provided with the application and they agreed on using it but they never did. Therefore, there was an additional meeting, where the researcher demonstrated a possible way to use the application in a learning activity. The demonstration was followed by a discussion about possible ways to orchestrate the interplay between different representations and the dynamical affordances (dragging mode, show/hide figures) supported by the application. By discussing related pedagogical issues and offering the teachers opportunities to adapt the application according to their needs, the researcher wanted to challenge the teachers to create their own hypothetical learning trajectory (HLT).

The CFD was developed by using the networking strategy of coordinating (Prediger et al., 2008) theoretical components (i.e. representation, GeoGebra and HLT) for practical reasons without aiming for a deeper integration. In contrast to the other components, the notion of hypothetical learning trajectory (HLT) was not presented explicitly to the teachers. In the next section we continue by presenting the crosscutting features of the enacted lessons.

Teachers orchestrating the application in a learning activity

The teachers used different interpretations of multiplication simultaneously and alternately without making explicit why and when an interpretation was preferable in some situations and not in others. This lack of explicitness resulted in vague connections between the numerical and geometrical representations. Justifications were based on computations or algebraic manipulations instead of referring to the available geometrical representations in the application. When the teachers became uncertain on how to proceed with the activity they tended to rely more on the numerical and algebraic representations to maintain the flow of the lesson. A significant part of the lessons was also dedicated to what seemed to be other more familiar activities such as formulating expressions for area and perimeter.

Furthermore, the teacher-initiated communication with the students did not seem to support a discussion on how and why things work the way they do. Occasional misinterpretations of students' responses, not acknowledging their responses as correct, and not connecting their responses to the available representations, further contributed to the activity not proceeding as intended.

CONCEPTUAL FRAMEWORK FOR UNDERSTANDING

The enacted lessons were also different compared to the suggestions the teachers had themselves when discussing different ways to orchestrate a lesson supported by the application (when discussing the HLT). During the first two phases of this design cycle, the focus was on teacher knowledge but the crosscutting features of the lessons revealed another dimension. How does teacher knowledge come into play in the moment of teaching? In order to understand why the teachers did not make use of the ICT-supported affordances for connecting representations, the researcher decided to go beyond the theories of representation and teacher knowledge used previously. In other words, a different representation was chosen to address this emerging challenge and to evaluate the design process so far.

Developing the CFU

The Anthropological Theory of the Didactic (ATD) provides a different conceptualization of knowledge. In this theory a body of knowledge (a praxeology) consist of two inseparable blocks, the *praxis* and the *logos*. The *praxis* block refers to the kind of given *tasks* that you aim to study and the different *techniques* used to face

these problematic tasks. In this sense the *praxis* block represents the "know-how" of the praxeology and is the minimal unit of human activity. The *logos* block provides a discourse that is structured in two levels with the purpose to justify the *praxis*. The first level of the *logos* is *technology*, which provides a discourse about the *technique*. The second level of the *logos* is *theory*, which provides a more general discourse that serves as explanation and justification of the *technology* itself (Chevallard, 2007) by providing a framework of notions, properties and relations to organize and generate technologies, techniques and problems (Barbé, Bosch, Espinoza, & Gascón, 2005).

The ATD includes the study of didactic transpositions processes, which concerns the transformation of knowledge through different institutions. The transposition is a process of de-constructing knowledge and rebuilding different elements of knowledge into a more or less integrated whole with the aim of establishing it as "teachable knowledge" while trying to keep its character and function (Bosch & Gascón, 2006). It consists of the four following steps; scholarly knowledge, knowledge to be taught, taught knowledge and learned knowledge. The different steps provided a new way to describe the intervention. In this case, the focus of the intervention was on the connection between intended and enacted knowledge, that is, between the second and third step of the transposition of knowledge (see Fig. 3).
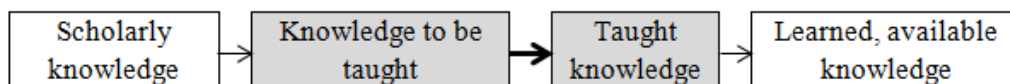

Figure 3: The transposition of knowledge

Furthermore, teaching is a didactic *type of task* that teachers can solve in a complex process of didactical transposition by using a set of available resources (*didactical techniques*), both external resources (curriculum, textbooks, tests, ICT-tools, colleagues, manipulatives, etc.) and teachers' internal resources that in our case of ICT-supported instruction could be related to technological-pedagogical content knowledge (Koehler & Mischra, 2008). The logos block of a didactical praxeology then serves as means to describe and justify teaching and learning practices in the considered institution (Rodríguez, Bosch, & Gascón, 2008).

The notion of HLT was replaced by the notion of *routines* (Berliner, 2001) with focus on the *IRE* sequence (Initiate, Response, Evaluate). The IRE sequence is a three-part pattern where the teachers ask a question, students reply, and teachers evaluate the response or gives feedback (Mehan, 1979; Schoenfeld, 2010). In its most basic form the teacher initiates the sequence by posing a question to a student to which the teacher already knows the answer. The student then replies and the teacher evaluates by using phrases such as "yes" or "that's fine" and continues with the next question or next problem. This adaptation was made in order to better describe the teachers' overt orchestration of the lessons and especially the communication patterns between teachers and students. Furthermore, communicational exchange patterns, such as the

IRE sequence, can be regarded a didactical technique that teachers use in the creation of a mathematical praxeology. This theoretical component was further developed in a second design cycle into a didactical resource (Perez, 2014).

Moreover, *representations* were placed within the notion of *praxeologies* instead of being treated as a separate theoretical component as in the CFD. The role of representation is multifaceted. From one perspective it is a generic property of many ICT tools (Koehler & Mischra, 2008). From a second perspective, mathematical representations have important didactical affordances (Ainsworth, 1999), and finally representations are essential to mathematics as a discipline (Duval, 2006). Thus, mathematical representation is closely related both to *praxis* and *logos* of a mathematical praxeology. Furthermore, instructional strategies that systematically focus on knowledge about representations could be conceptualized as an element of a didactical technique and consequently a part of a didactical praxeology. Thus, depending on the purpose in which representations are used, the role of representation for a discipline as mathematics could be attributed to both a mathematical and a didactical praxeology. These adaptations allowed the researcher to provide a more comprehensive description of the crosscutting features in the enacted lessons and to evaluate the efforts of providing competence development.

In summary, the conceptual framework for understanding (CFU) consists of several theoretical components where the ATD is used as the dominant theory. The purpose of the conceptual framework for understanding (CFU) was to better understand an emerging empirical phenomenon (the crosscutting features). The CFU was developed by the researcher by using the strategy of *coordinating* different theoretical components (Prediger et al., 2008). To achieve this, the theoretical components of representations and routines (the IRE sequence) were interpreted as knowledge resources in accordance with the ATD and its focus on the epistemic dimension of teaching and learning processes in different institutions.

Evaluating the design process

The theoretical notions provided by the CFU allowed the researcher to capture the essence of this part of the design process. In summary, the intention to introduce the geometrical representation as a technological element in a mathematical praxeology was instead treated by the teachers as a didactical technique to allow the students to work with more open-ended tasks. Thus, the affordances of the embedded geometrical representations as a technological element were not used as intended. Furthermore, the communicational patterns (IRE sequence) used by the teachers did in many cases not support the creation of a mathematical praxeology including a well-developed *logos* discourse. In summary, the underlying principle-based learning objectives did not survive the transposition from how the researcher intended the application to be used and how it was actually used by the teachers. The transposition of knowledge between "knowledge to be taught" and "taught knowledge" proved to

be of greater difficulty requiring more scaffolding than the researcher had anticipated and planed for. With this understanding, a new design cycle could be initiated.

SUMMARY

In this case study the possibility of viewing a design process as incremental and adaptive has been considered. This should not be interpreted as a matter of searching for whatever works in the current situation. Instead, it is about the problematic task of assuring that the activity of inquiry is meaningful relative to the research objectives, i.e. the problem of developing systems guarantors (Churchman, 1971). This is a basic problem for any researcher but in this case, the problem of guarantors were not settled a priory and once and for all. By questioning the assumptions of the inquiry system, the design problem of knowing when and how to revise becomes difficult because there is no a priory authority to rely on. Instead, the question of why revise would depend on the measure of the performance of the system relative to the purpose (Churchman, 1971). Furthermore, in order to make tactical decisions that require an authority, the researcher must be prepared to consider a "whole breadth of inquiry in its attempt to authorize and control its procedures" (ibid. p. 196). In this case study, the choice to change theoretical perspective during the design process was considered necessary in order to adapt to an unforeseen and, from the researchers perspective, problematic situation. Thus, the question of why revise was motivated by an emerging phenomenon that questioned the performance of the design process. This resulted in a more comprehensive conceptual framework for understanding (CFU) where the Anthropological Theory of the Didactic served as an overarching theoretical perspective. The development of adaptive conceptual frameworks could be understood as a modeling process that aims at developing system guarantors. But as any model it only provides different affordances and constraints that may be used with varying levels of success to justify the choices we make and explain different phenomena that we seek to understand. The use of adaptive conceptual frameworks specifically affords transparency in the design by making explicit the levels at which different theories operate and the measures that are used to evaluate the system performance.

Finally, allowing the design process to be co-determined by the interaction between different stakeholders and resources is far from an straightforward task, but I believe that it allows the researcher to make use of available resources to address authentic educational needs as expressed by practicing teachers.

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COMMUNITIES OF PRACTICE: EXPLORING THE DIVERSE USE OF A THEORY

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The social learning theory of communities of practice is frequently used in mathematics education research. However, we have come to recognise that the theory is used in diverse ways, regarding both the parts that are used and the ways in which those parts are used. This paper presents an overview of this diverse use of the theory based on three themes: Are communities of practice viewed as pre-existing or are they designed within the study? Are individuals or groups foregrounded in the study? Which parts of the theory are mainly used? The aim of the paper is twofold: to make visible the diverse possibilities within one single theory, and to make visible how, even though we might think we know what a theory implies in research, if we look beneath the surface we may find that "the same" theory can imply many different things.

Keywords: communities of practice, theory, social, learning, Wenger

INTRODUCTION

Since Etienne Wenger published his book *Communities of Practice: Learning, Meaning, and Identity* in 1998 the notion of communities of practice has become common in mathematics education research as well as in other areas of educational research. Both authors of this paper have been using Wenger's social theory of learning in research within mathematics education. In reading other researchers' work we have discovered that the theory of communities of practice is frequently used in mathematics education, but there are many differences regarding both *which* parts are used and *how* those parts are used. In this paper we will explore some of the ways in which the theory of community of practice is used in different mathematics education studies. The aim of this is twofold: to make visible the diverse possibilities and uses of one single theory, and to make visible how we in research may think we know what using a specific theory in a study implies, but when we look beneath the surface we may find that "the same" theory can imply different things to different researchers.

The notion of communities of practice has been investigated and discussed before, for example by Kanes and Lerman (2008). They investigated similarities and differences in how the notion is used by Lave and Wenger (1991) and by Wenger (1998), respectively. (However, we find Kanes and Lerman's (2008) description of Wenger's communities of practice very different from our own interpretation and the interpretations we found when preparing this paper.) In this paper we focus only on research referring to Wenger's 1998 book, in which he writes that his aim is to

present a conceptual framework where learning is placed "in the context of our lived experience of participation in the world" (p.3). In this paper we will not present Wenger's theory more than that, in order to avoid imposing our own interpretations of which concepts are the main ones in his theory. Instead, the use of communities of practice will be explored according to the differences we found when reading other researchers using Wenger's theories. Hence, the exploration is divided based on the following three themes: Are communities of practice viewed as pre-existing or are they designed within the study? Are individuals or groups foregrounded in the study? Which parts of the theory are mainly used? These three themes will be presented under each heading followed by a concluding discussion.

SELECTION OF STUDIES

Our selection of studies to explore was limited to those focusing on mathematics teaching or learning and/or mathematics teachers' professional development. We searched 19 databases, using the search words *communities of practice, mathematic** and/or *teach**; the search was limited to peer reviewed journals or books. From this selection, consisting of more than 8000 articles, we limited the search to *communities of practice* and *mathematic** and/or *Wenger*; although that reduced the number of articles, there were still too many in some of the databases. We then removed "or" *teach**. Thereafter we were able to browse through all the titles and keywords to find a selection of research articles using *communities of practice*. This selection is not at all comprehensive, however, the purpose is not to generalise but to illustrate some of the differences we have found. Wenger's theory is also used frequently in studies within economy and management, but such studies are not explored in this paper.

Due to space limitations, this paper cannot present all the articles we have read; instead, we present articles that together illustrate the differences we found based on our three themes. The following ten studies will be discussed in relation to the three themes in the paper: Bohl and Van Zoest (2003); Corbin, McNamara and Williams (2003); Cuddapah and Clayton (2011); Cwikla (2007); Franke and Kazemi (2001); Goos and Bennison (2008); Graven (2004); Hodges and Cady (2013); Pratt and Back (2009) and Siemon (2009).

DESIGNED OR PRE-EXISTING COMMUNITIES OF PRACTICE

Some studies using Wenger's social theory of learning view communities of practice as pre-existing. In some other studies, for example, Bohl and Van Zoest (2003), Cuddapah and Clayton (2011), Goos and Bennison (2008), Hodges and Cady (2013) and Franke and Kazemi (2001), communities of practice are designed by the researcher(s).

In the study by Goos and Bennison (2008), a web-based community of practice is designed within teacher education. After graduation, interaction in the community of practice continues through the web-based tool developing an "online community"

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(p.41). In their article, Goos and Bennison discuss the issue of emergent versus designed communities of practice. Although, in their study Goos and Bennison design the external frames for the community of practice, their interest is in whether or not the web-based community develops into a community of practice. To give the community the best chance to develop into a community of practice on its own, the researchers provide only a minimum of structure concerning how community members are to communicate using the web-based tool. As such, they design a community, but it is its emergence as a community of practice they investigate in their study.

Hodges and Cady (2013) seek to expand on the work of Goos and Bennison (2008) by investigating the development of communities of practice within a professional mathematics teacher's development initiative. In this study a web-based tool is used to "foster the development of communities of practice" (p.302). Hodges and Cady design a virtual space in order to see the emergence of communities of practice. However, unlike Goos and Bennison (2008), Hodges and Cady do not highlight the issue of an emergent or a designed community, even though the emergence of potential communities of practice is in focus.

Cuddapah and Clayton (2011) design a community of practice by arranging physical sessions with a group of novice teachers. They focus on one of several groups of novice teachers that, within a university-sponsored project, meet every second week. The novice teachers meet 15 times during the study. Every session has a theme and the sessions are planned and led by experienced educators. Cuddapah and Clayton write that the group of novice teachers "itself was a community" (p.69) and they use Wenger's theories to analyse the development of the group and its function as a resource for new teacher support. In their analysis they present how the "community was observed throughout and between the data" (p.72). As such, the group of novice teachers being a community of practice was both a precondition and a result of their analysis.

A fourth example of researchers who design communities of practice is Franke and Kazemi (2001). In their study they design communities of practice with mathematics teachers with the purpose of providing teachers with opportunities to learn about mathematics teaching and learning. The teachers in this study do mathematical tasks with their students in their classrooms and then they meet and discuss their experiences. The researchers take part in the discussions and they also visit the teachers at their schools several times. Franke and Kazemi do not describe why or how the group of teachers is a community of practice, but they analyse and describe the interactions in the group connected to teacher professional development.

Examples of studies in which communities of practice are treated as pre-existing, developed before the study began and without the influence of the researchers, are studies by Bohl and Van Zoest (2003), Corbin et al. (2003), Cwikla (2007), Graven

(2004), Pratt and Back (2009) and Siemon (2009). In some studies the communities of practice are identified in the research process based on concepts from Wenger's theory, whereas other studies do not explain how they are identified as communities of practice.

Bohl and Van Zoest (2003), Graven (2004), Corbin et al. (2003), Cwikla (2007) and Pratt and Back (2009) are examples of studies where communities of practice are viewed as pre-existing at the start of the study, where the researchers do not explain how the communities have been identified as such.

Bohl and Van Zoest (2003) analyse how different communities of practice in which novice teachers participate influence their mathematics teaching. They give an empirical example of one novice teacher, in relation to whom they discuss differences in the role of novice teachers in different communities of practice, but they do not present how they identified these as communities of practice, nor do they explain how they identified the novice teacher's membership in these communities.

Graven (2004) investigates teacher learning in a mathematics in-service program. In this study an in-service program is considered to be a community of practice, but it is not explained how this community of practice has been identified as such. This is also the case in the study of Corbin et al. (2003), who investigate numeracy coordinators in an implementation of a national numeracy strategy. They use the notion of communities of practice as a tool to describe the participation of the coordinators in different communities, but they do not explain how they define the communities.

Pratt and Back (2009) investigate participation in interactive discussion boards designed for mathematics students. They simply state that "two idealised communities of practice" (p.119) were adopted as a means to understand the discussion boards. How these communities were created and why they can be seen as such is not explained. They even describe the communities of practice as "hypothetical communities" (p.128). Cwikla (2007) uses the concept of communities of practice in her study of the evolution of a middle school mathematics faculty. The concept of communities of practice is used to identify boundary encounters, but the article does not present any definition of communities of practice, nor does it specify which communities of practice are identified within the study.

Siemon (2009) is an example of a study where communities of practice are viewed as pre-existing at the start of the study, but where the researcher explains how the communities of practice have been identified as such. Siemon (2009) investigates improvements in indigenous students' numeracy skills after they worked on key numeracy issues in their first language. Three pre-existing communities of practice are described and it is explained, using Wenger's concepts, why these are considered to be communities of practice. In the study, the intersection between the acknowledged pre-existing communities of practice is investigated. The members of

these communities are not described in detail, only as, for example "members of the local Indigenous community" (p.225), or "all those that by virtue of their responsibilities are concerned in some way with school mathematics" (p.225). The intersection between the communities of practice is not highlighted, although the author states that the edges of the communities took time to emerge.

FOCUS ON INDIVIDUALS OR GROUPS

Wenger's theory makes it possible to foreground groups (communities of practice) or individuals (learning and/or identity) or both. Since Wenger's theory is very broad and yet detailed, it is not surprising that either groups (communities of practice) or individuals are foregrounded in the studies. Wenger explains that this is not a "change of topic but rather a shift in focus within the same general topic" (p.145). Franke and Kazemi's (2001) study is an exception, however, and an example of "both" since they analyse both the interaction within the community of practice and the identity development of individual participants.

In the studies by Cwikla (2007), Cuddapah and Clayton (2011), Goos and Bennison (2008), Hodges and Cady (2013) and Siemon (2009), groups of teachers are in the foreground and individuals are in the background or are not mentioned as individuals at all. Bohl and Van Zoest (2003), Corbin et al. (2003), Graven (2004) and Pratt and Back (2009), however, foreground the individuals, trying to understand how they are influenced by the different communities of practice in which they participate.

The issue of communities of practice or individuals being foregrounded in the studies as presented in this section is connected to which parts or concepts from Wenger's theory are used in the analyses, which is the focus of the next section.

WHICH PARTS OF THE THEORY ARE MAINLY USED?

Another consequence of Wenger's theory being very broad and yet detailed is that researchers focus on and use smaller parts of the theory, selecting just some of the concepts within it.

Graven (2004) uses the concepts of *practice, meaning, identity,* and *community* to describe and explain teacher learning. These four concepts are, according to Wenger "interconnected and mutually defining" (p.5). Graven also mentions Lave and Wenger's (1991) concepts of *co-participation* and *participation*, but these are not used in her analysis. Even though Graven describes communities of practice in her study, the "three dimensions" (p.72) that according to Wenger are the source of a community of practice, *mutual engagement, joint enterprise* and *shared repertoire*, are not used. However, Graven instead wants to add *confidence* as a supplement to *practice, meaning, identity,* and *community.*

Cuddapah and Clayton, like Graven (2004), initially refer to Lave and Wenger (1991) but to the concept of *legitimate peripheral participation*. They discuss this concept as

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one that can be used when analysing novice teachers as newcomers in teaching. However, as all novice teachers in their study are new members of a new community of practice designed by the researchers, they instead, like Graven (2004), use *practice, meaning, identity,* and *community* when coding their empirical material. They briefly mention the concepts of *mutual engagement, joint enterprise* and *shared repertoire,* but they do not use them in their analysis.

Those three concepts, *mutual engagement, joint enterprise,* and, *shared repertoire*, are used by Goos and Bennison (2008), Hodges and Cady (2013) and Siemon (2009) in their studies. As shown in the last section, these three studies have communities of practice in the foreground. Goos and Bennison (2008) use the three concepts when they analyse the emergence of their designed web-based community of practice. To investigate mutual engagement they count the number of interactions in the web-based tool. By analysing the content in these interactions they also investigate the joint enterprise and the shared repertoire that develops. Siemon (2009) uses the three concepts by making lists of what it is in the different communities of practice identified in the study that indicates joint enterprise, mutual engagement and a shared repertoire. Consequently, in her study communities of practice are pre-existing, but she defines them by mutual engagement, joint enterprise and shared repertoire. Three communities of practice are acknowledged this way. Hodges and Cady (2013) use the three concepts in the same way, but their approach is somewhat different. They use the concept in order to find and/or see development of communities of practice in a designed web-based tool. In their analysis they look for evidence of joint enterprise, mutual engagement and a shared set of ways of interacting in order to see if a community of practice has been developed. As such, the concepts of *mutual engagement, joint enterprise* and *shared repertoire* are used to identify both designed (Goos & Bennison, 2003; Hodges & Cady, 2013) and pre-existing (Siemon, 2009) communities of practice.

In addition to mutual engagement, joint enterprise and a shared repertoire, Siemon (2009) also uses Wenger's concept of *negotiation of shared meaning* when referring to a space where the participants in the different communities of practice can meet. This space is used both as a place to negotiate meaning and as a research tool to "explore the processes involved in building community capital" (p.226). Furthermore, Siemon uses Wenger's concept of *boundary objects* when defining Probe Tasks[3] as a boundary object in the negotiation described above. Cwikla (2007) also uses the concept of *boundary objects.* In her investigation of the evolution of a middle school mathematics faculty, she uses this concept together with the concept of *brokers*, which is also from Wenger. She mentions communities of practice, but she

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[3] A Probe Task is described in the paper as a specifically chosen or designed task to support indigenous teacher assistants as they teach key aspects of number.

does not define them. When using the concept of brokers, she refers to Wenger's definition, stating, "a broker can serve as a conduit for communication and translation between communities of practice" (p.558). Corbin et al. (2003) also use the concept of *brokering* when investigating numeracy coordinators in an implementation of a national numeracy strategy. The concept is used to theorise tensions in the work of the coordinators. Corbin et al. find signs of brokering in their analysis by using three more of Wenger's concepts: the *modes of belonging: engagement*, *alignment* and *imagination*. Pratt and Back (2009) also use the concepts of *engagement, alignment* and *imagination* in their analysis. They also use Lave and Wenger's (1991) concept of *legitimate peripheral participant* as well as *peripheral and central participation* in their analysis. These concepts are used to describe a person's participation, and changes in participation, in two different communities of practice.

Bohl and Van Zoest (2003) mention that communities of practice develop through mutual engagement, joint enterprise and shared repertoire, but in their analysis they use two other concepts of Wenger's: *modes of participation* (their term for what Wenger refers to as *modes of belonging*) and *regimes of accountability*. They use these two concepts to analyse how novice teachers have different roles in different communities of practice and how this influences their mathematics teaching.

As mentioned, Franke and Kazemi (2001) analyse both the interaction in one community of practice and the identity development of individual participants. However, they do this without explicitly using any of Wenger's concepts. The artefacts they mention are not identified explicitly as artefacts used by Wenger but as used in sociocultural theories in general. They also mention *identity* and *negotiation of meaning*, both of which are thoroughly elaborated by Wenger, but they do not refer explicitly to how the concepts are used by Wenger. As such, Franke and Kazemi refer to, and use, Wenger's social theory of learning, but not explicitly or solely; rather, they present it as part of a general sociocultural view of learning.

Overall, several of Wenger's concepts are used in the studies presented in this paper, including *practice, meaning, identity, community, mutual engagement, joint enterprise, shared repertoire, modes of belonging, engagement, alignment, imagination, identity, brokering, negotiation of meaning, boundary objects, regimes of accountability, co-participation* and *participation*. However, seldom are more than three or four concepts used in the same study. Since the theory is broad and yet detailed, it is not surprising that researchers focus on and use only parts of it. Even so, none of the articles referred to in this paper draws attention to the fact that only certain parts of Wenger's theory will be used. Neither do they discuss the eventual consequences of not using the theory in its entirety. Hence, anyone reading only one of the articles may easily believe that the whole of Wenger's theory is used.

DISCUSSION

As seen in the examples in this paper, Wenger's social theory of learning is used in different ways in different studies. Wenger (1998) terms his work a "conceptual framework" (for example, p.5), a "social theory of learning" (for example, p.4) and/or a "perspective" (for example, p.3). According to Eisenhart (1991), there are three kinds of research frameworks: theoretical, practical and conceptual. Eisenhart distinguishes these as theoretical frameworks based on formal logic, practical frameworks based on practitioner knowledge and conceptual frameworks based on justification. Somehow Wenger's social theory of learning comprises all three of these features. According to Niss (2007), theories are stable, coherent and consistent systems of concepts that are organised and linked in hierarchal networks. Those criteria apply to the content of Wenger's book. However, when researchers use only some of Wenger's concepts the criteria are no longer met. Furthermore, Niss (2007) writes that one purpose of theories "is to provide *a structured set of lenses* through which aspects or parts of the world can be approached, observed, studied, analysed or interpreted" (p.100). The diverse uses of Wenger's social theory of learning presented in this paper show that the *structured set of lenses* used in these studies differ substantially.

According to Lester (2005), a framework provides structure in research when it comes to the questions being asked and the concepts, constructs and processes being used. Connected to the overview in this paper, the use of Wenger's social theory of learning appears to coincide with the first (questions), but not the rest. Even though the use of Wenger's social theory of learning differs in the studies presented in this paper, one similarity is the type of questions asked. These questions imply that the theory is considered suitable for studies of mathematics teachers', novice teachers', student teachers' and/or students' learning. Furthermore, in several of the studies (for example, Bohl & Van Zoest, 2003; Siemon, 2009) the social dimension of learning provided by Wenger is emphasised as its main strengths. As such, the use of Wenger's theory in mathematics education research seems to be part of the "turn to social theories in the field of mathematics education" (Lerman, 2000, p.20). According to Lerman (2000), social theories make it possible to foreground individuals (practice in person) or practice (person in practice). However, both elements (person and practice) are always present and part of the analysis, which is in line with Wenger's "shift in focus within the same general topic" (p.145).

As shown in this paper there are differences in the presented studies in terms of communities of practice being viewed as pre-existing or designed as well as communities of practice being identified based on Wenger's concepts or not. In his book Wenger actually writes that since communities of practice are about content and negotiation of meaning – and not form – they are not "designable units" (p.229). That

is, according to Wenger, it is possible to design the outer limits but not the practice that may, or may not, emerge.

As presented above, there is also diversity with respect to whether individuals or (communities of) practice are in the foreground. As also shown, there are differences regarding which of Wenger's concepts is used, even when the same perspective (individuals or communities of practice) is in the foreground. In terms of the concepts used, we were surprised by the rare presence of *reification* and *negotiation of meaning,* as these two concepts recur frequently throughout Wenger's book. Furthermore, there are many other concepts of Wenger's that are not used in any of the studies we read, including *local/global, identification, economies of meaning, ownership of meaning* and *trajectories*.

Finally, what can be learned from this overview of how Wenger's social theory of learning is used in different ways in mathematics education research? Well, often we (think that we) know what researchers imply when they say they have been using a specific theory in their research. However, from the overview presented in this paper, we know that if a researcher says that (s)he has been using Wenger's social theory of learning, we can be quite sure that we do *not* know exactly what that use of Wenger's theory might imply. In this paper we have highlighted some of the diverse uses of Wenger's social theory of learning based on three themes: Are communities of practice viewed as pre-existing or are they designed within the study? Are individuals or groups foregrounded in the study? Which parts of the theory are mainly used? Probably further comparisons based on other themes will reveal other diversities. Further, based on the breadth and wealth of details in Wenger's social theory of learning, the list of themes and diversities may become quite long.

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BEYOND ORCHESTRATION: NORM PERSPECTIVE IN TECHNOLOGY INTEGRATION

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The aim of this study is to bring a socio-cultural dimension to "instrumental orchestration" framework. Our claim is that social and socio-mathematical norms endorsed by teachers are crucial for their pedagogies. A case study was designed to investigate how orchestration types and norms affect each other in technology-enhanced learning environments. Participants are five pre-service mathematics teachers. Data were collected through lesson plans, semi-structured interviews and observations. Analysis of data indicates that there is a two-way interaction between norms and orchestration types. In some cases norms are determinants of orchestration types used by participants. In other cases, orchestration types challenge participants' endorsed norms.

INTRODUCTION

Recently, mathematics teaching in technology-enhanced environments has been widespread and mathematics teachers are faced with a large number of resources (Drijvers, 2012). Various official curriculum documents around the world emphasise the importance of using technology to support learning (NCTM, 2008; DfES, 2013a, 2013b). This requires certain knowledge and pedagogy. For example, International Society for Technology in Education describes technology standards and performance indicators for teachers. Teachers should be able to "plan and design effective learning environments and experiences supported by technology" (ISTE, 2000, p. 9).

Teachers play a key role in effective use of technology in the classroom and the way they integrate technology into teaching affects the way students learn mathematics (Ely, 1996 as cited in Besamusca & Drijvers, 2013). Therefore, mathematics teachers and teacher educators should be guided for the design of learning environments using technological tools and resources (Şay, Kozaklı & Akkoç, 2013).

One of the theoretical categorisation to investigate the use of technological tools in the classroom is "instrumental orchestration" which is based on the framework of instrumental genesis (Trouche, 2004). Considering the literature on orchestration, it can be claimed that this theoretical categorisation focuses on classroom organisation but fall in short to explain psychological and sociological development of teachers. Teachers and pre-service teachers have different pedagogical approaches and go through different professional development phases. Therefore, an investigation of technology integration purely based on physical organisation of technology-enhanced

learning environments and certain teacher behaviours is only one part of the whole picture. Teachers might have different norms and these affect the way they integrate technology into their lessons. There is little research in the literature on how teachers' activities in the classroom are shaped by their norms and very few of them investigated this in the context of technology. The aim of this study is to bring a socio-cultural dimension to instrumental orchestration. Socio-cultural theory aims to investigate human action and its relationship with cultural, institutional and historical situations. Therefore, it focuses on social interactions and the effects of culture on psychological development (Wertsch, del Rio & Alvarez, 1995; Lerman, 2001). Technological tools can turn into effective instruments for learning mathematics via effective classroom interaction. Social and socio-mathematical norms, as one of the aspects of socio-cultural theory, might take a role in shaping student-teacher-tool interaction in the classroom. Furthermore, they are also shaped by this interaction. Therefore, one could elucidate how teachers use technological tools by embracing a norm-perspective within socio-cultural approach. Our claim is that social and socio-mathematical norms (Višňovská, Cortina & Cobb, 2007) endorsed by teachers are crucial for their pedagogies and their choices for different orchestration types. To support this argument, this study investigates the interaction between orchestration types used by pre-service mathematics teachers and social and socio- mathematical norms.

INSTRUMENTAL ORCHESTRATION

An instrumental orchestration is defined as the teacher's intentional and systematic organisation and use of the various artefacts available in a learning environment in a given mathematical task situation, in order to guide students' instrumental genesis (Trouche, 2004).

Drijvers (2012) distinguishes three elements within an instrumental orchestration: a didactic configuration, an exploitation mode and a didactical performance. "A didactical configuration is an arrangement of artefacts in the environment, or, in other words, a configuration of the teaching setting and the artefacts involved in it" (p. 266). An exploitation mode is defined as the teacher's decisions on the way she or he configures a task by providing certain roles for the artefacts to achieve his or her didactical intentions.

A didactical performance involves the ad hoc decisions taken while teaching on how to actually perform in the chosen didactic configuration and exploitation mode: what question to pose now, how to do justice to (or to set aside) any particular student input, how to deal with an unexpected aspect of the mathematical task or the technological tool, or other emerging goals" (Drijvers, p. 266).

Drijvers and his colleagues (2010), Drijvers (2012) and Tabach (2013) distinguish ten types of instrumental orchestrations as seen in Table 1 (The last three orchestration

types are not in the original table and were added from the literature). In this study, pre-service teachers' lessons will be analysed considering the orchestration types in this table.

The orchestration types	Didactical configuration	Exploitation mode
Technical-demo (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The teacher explains the technical details for using the tool.
Explain-the-screen (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The teacher's explanations go beyond techniques and involve mathematical content.
Link-the-screen board (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The teacher connects representations on the screen to representations of the same mathematical objects that appear either in the book or on the board.
Sherpa-at-work (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The technology is in the hands of a student, who brings it up to the whole class for discussion.
Not-use-tech (Tabach, 2011)	Whole-class setting, one central screen	The technology is available but the teacher chooses not to use it.
Discuss-the-screen (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	Whole class discussion guided by the teacher to enhance collective instrumental genesis.
Spot-and-show (Drijvers and his colleagues, 2010)	Whole-class setting, one central screen	The teacher brings up previous student work that he/she had stored and identified as relevant for further discussion.
Work-and-walk-by (Drijvers, 2012)	Students work individually or in pairs with computers	The teacher walks among the working students, monitors their progress and provides guidance as the need arises.
Discuss-the-tech-without-it (Tabach, 2013)	Every students have own laptops or laptops bring classroom with wheeled vehicles	The teacher uses mobile transport system if he/she needs computers in teaching
Monitor-and-guide (Tabach, 2011)	-----	The teacher uses a learning management system by giving guidance to students

Table 1: Orchestration types (Tabach, 2013, p. 3)

SOCIAL AND SOCIO-MATHEMATICAL NORMS

In mathematics education literature, it is widely recognised that social interaction promotes learning opportunities. Norms construct how students learn mathematics and how they become mathematically autonomous (Cobb & Bowers, 1999; Pang, 2000). Norms regulate the way teachers and students participate in learning and teaching activities within a classroom culture (Cobb & Yackel, 1996). While norm emerges from social interaction; belief, value, opinion and attitude are concerned with the individual.

Cobb & Yackel (1996) propose social and socio-mathematical norms to investigate how students' mathematical values and beliefs develop within the classroom culture from the psychological and socio-cultural perspectives. Cobb and his colleagues (2007) also investigated teachers' professional development through social and socio-mathematical norms (Visnoska, Cortina & Cobb, 2007). Social norms apply to any subject matter area. Students' cooperation when solving problems or privileging a logical explanation over other correct answers are examples of social norms (Hershkowitz & Schwarz, 1999). Another example is the way teachers promote students' thinking and value different ideas. On the other hand, socio-mathematical norms are specific to mathematics and are concerned with the way mathematical beliefs and values develop in the classroom. For example, acceptability of a mathematical explanation or a justification is a socio-mathematical norm (Yackel & Cobb, 1996; McClain & Cobb, 2001).

METHODOLOGY

This study embraces the interpretive paradigm to investigate how orchestrations types and norms affect each other in technology-enhanced learning environments. A case study was designed to answer the research question. Participants are five pre-service mathematics teachers who were enrolled in a teacher preparation program in a state university in Istanbul, Turkey. It was a four-month program which will award its participants a certificate for teaching mathematics in high schools for students aged between 15 and 18. The program accepts graduates who have a BSc degree in mathematics. There were two kinds of courses in the program: education and mathematics education courses. The study was conducted during "Instructional Technologies and Material Development" and "Teaching Practice" courses. The former course focused on six software, namely Geogebra, Graphic Calculus, Derive, Geometry Sketchpad, Excel and Probability Explorer. Participants were involved in hands-on activities in front of a computer and prepared teaching materials. Participants also taught lessons in partnership schools during "Teaching Practice" course.

There were thirty-six participants in the program. They were all interviewed on their approach to the use of technology for teaching mathematics. Five participants were

purposefully selected. Two of them (one male and one female) had positive attitudes and two of them (one male and one female) had negative attitudes towards the use of technology. One participant was selected because she had neutral attitude.

The data collection methods are observation and semi-structured interviews. Each participant taught a total of five lessons in a partnership school. At least two of these lessons were technology-based. Each participant taught at least two same classes of students. They were interviewed after their lessons. During the semi-structured interviews they were asked what kinds of norms they endorsed, how they used technology in their lessons and differences between their lessons with or without technology. Their lessons were video recorded. The first author of this paper observed lessons using an observation form. The aim of the observation form was to reveal social and socio-mathematical norms endorsed by pre-service teachers. Interviews and lesson videos were verbatim transcribed. Data from different sources such as interviews, observations and field notes were triangulated. Common themes emerged from verbal discussions among pre-service teachers and students, patterns in pre-service teachers' behaviours and statements about their endorsed norms during the interviews. For instance, the socio-mathematical norm "Answers which are logical are acceptable'' was determined considering pre-service teachers' discussion with students and how they defined "an acceptable answer" during the interview.

FINDINGS

This section presents orchestration types and social and socio-mathematical norms used by participants. First we demonstrated how participants used orchestration types and then how norms and orchestration types affected each other.

Orchestration types	Pre-service teachers
Technical-demo	Nil, Orkun, Melek
Explain-the-screen	Mahir, Orkun, Melek
Discuss-the-screen	Melek
Sherpa-at-work	Orkun, Melek, Nil
Not-use-tech	Oya, Mahir, Nil, Orkun, Melek

Table 2: The orchestration types used by pre-service teachers

The analysis of the data indicated that participants mostly used *technical-demo, explain-the-screen, link-the-screen-board, discuss-the-screen, Sherpa-at-work and not-use-tech* orchestration types as seen in Table 2.

Explain-the-screen promoted the social norm "the authority is the teacher". For example, Mahir taught a lesson on parabolas using Geogebra software. He started his lesson by explaining how to draw a parabola and adding a slide which is defined as

the determinant. He then moved the slide and explained what happened to the graph of parabola. At this stage he did not questioned the mathematical meanings behind what the software performed, but just explained how to use the software.

When participants used *link-the-screen*, they explained a concept or a mathematical idea on the board followed by an elaboration using the software. For example, Orkun taught a lesson on how to draw the graph of *y=sinx* using Geometry Sketchpad. He first plotted a few points and then drew the graph on the board. However students claimed that points should be joint using straight lines. He then moved to the software and constructed a unit circle. He defined a point A on the unit circle and a point B which defines the sine function. Using "trace" feature, he obtained the graph of *y=sinx*. Up until now, the authority was the teacher. Therefore, it can be claimed that *link-the-screen* orchestration type promoted this social norm. Afterwards he asked students how to draw *y=cosx* and *y=tanx* themselves. His question is an indication of a social norm "Students are challenged with the questions of why and how". This social norm required using *discuss-the-screen* orchestration type.

Another participant who used *discuss-the-screen* discussed with their students the actions they performed using the software. For example, Melek used Geometry Sketchpad to explain how to draw trigonometric functions. She first drew the graph of *y=sinx* and then asked one of her students to draw *y=cosx*. Later she discussed with her students how to draw *y=tanx* using the software and tried to reach a common ground (tanx=sinx / cosx):

Melek: Is there anyone who wants to draw the tangent line?

Student A: This time, we will construct a point with *x* and *y* (on the unit circle)...

Student B: Slope

Melek: What else? What is slope? One of the definitions is opposite over adjacent. It's the ratio of opposite side over adjacent side or what is *tan x*?

Student C: *sinx* over *cosx*

Melek: Isn't it *sinx* over *cosx*. That's the expression that everybody knows. Therefore, when we want to find the ratio of *sinx* over *cosx*, that is when we think graphically (showing the point on the unit circle) if we vertically projected this point onto *x*-axis, we say opposite over adjacent to find the tanjant

When pre-service teachers were using *discuss-the-screen* they endorsed the socio-mathematical norm "Answers which are logical are acceptable". For example, Melek who used *discuss-the-screen* aimed at having her students discuss mathematical meanings behind what the software perform. When doing this, she considered different student answers and did not impose her answers or solutions. She encouraged her students discover their own solutions which were meaningful for them. As can be seen in this case, the orchestration type used by this pre-service

teacher affected her endorsed norm. In other words, a norm has emerged which support *discuss-the-screen* orchestration type.

Pre-service teachers chose not to use technology (not-use-tech orchestration type) at least once out of five lessons they each taught. Before participants had teaching experiences with using technology in the classroom, they had the socio-mathematical norm which gives the teacher the mathematical authority and believed that technological tools were not necessary for teaching mathematics:

> Oya: I'm quite conservative. I believe that mathematics should be taught using the blackboard. I think that maths would be better understood this way

Oya was a unique case whose social or socio-mathematical norms did not change after she started using technological tools in her lessons. On the other hand, Orkun who has negative attitudes towards using technology in a mathematics lesson changed his endorsed norms and this situation is illustrated with the following excerpt:

> Orkun: In my first lesson (which he did not use any technological tool) I wished that student would not ask me any questions. Because I was teaching inverse trigonometric functions and the questions I prepared were very difficult ones…students in this school were very clever and I was worrying about receiving different questions. And there was no help from technology. I had to teach on the blackboard. But on the next lessons when I used technology, I wasn't afraid of their questions. When I'm stuck on the board I knew that I could switch to technology

As can be seen from the excerpt above, he sees technological resources as a helpful tool which gives him confidence. This confidence changed his norms and pedagogy.

Another orchestration type observed in this study is *Sherpa-at-work*. Participants in this study used this orchestration type in a different way when compared to the related literature. In the literature, when using Sherpa-at-work students work in front of a computer individually or in pairs and "the technology is in the hands of a student, who brings it up to the whole class for discussion" (Tabach, 2013, p. 3). However, there was a lack of technological resources in partnership schools and students did not get the chance to use their own computers during the lessons. Participants had their own computers which were projected on to a screen. This situation prevented active participation of students. Orkun, Melek and Nil tried to resolve this problem by having students use the teacher's computer. This corresponds to *Sherpa-at-work* orchestration type which emerges as a result of "students who answers correctly go to the blackboard" social norm.

DISCUSSION

This study investigated how orchestration types and norms affect each other in technology-enhanced learning environments. The findings indicated that pre-service mathematics teachers used some of the orchestration types frequently such as *link-*

the-screen-board and *not-use-tech*. On the other hand, some of the orchestration types such as *spot-and-show*, *work-and-walk-by*, *discuss-the-tech-without-it* and *monitor-and-guide* were not used because of lack of technological resources in the partnership school. All classrooms in this school have smart boards but students did not have their own computers or tablets. Therefore, some of the orchestration types were not observed.

Drijvers and his colleagues (2010) claimed that teachers make pedagogical choices based on their views about how to teach mathematics. This study has similar findings by illustrating how orchestration types and norms support each other. Social and socio-mathematical norms endorsed by participants affected their choices of technological tools for teaching mathematics and as a result orchestration types they used.

Our claim was that there was a two-way interaction between orchestration types and social and socio-mathematical norms. This study attempted to justify this claim. As a matter of fact, Drijvers (2012) described *technical-demo*, *explain-the-screen* and *link-the-screen-board* orchestration types as teacher-centred and *Sherpa-at-work*, *spot-and-show* and *discuss-the-screen* orchestration types as student-centred. Therefore, participants who used teacher-centred orchestration types endorsed socio-mathematical norms accept the teacher as the mathematical authority. On the other hand, participants who used student-centred orchestration types endorsed social norms which puts students into the centre.

Findings of this study also revealed that instrumental orchestration categorisation fall in short in explaining the socio-cultural aspect of technology integration. When it comes to teacher-student-tool interaction, technological tools provide a language which supports communication between students and teachers (Noss & Hoyles, 1996). Examining the micro-culture of the classroom provided by this kind of language and social and socio-mathematical norms required by that culture expanded our understanding of orchestration framework. Integrating instrumental orchestration framework into norm perspective provided an insight on the question of why and how particular orchestration types are used.

This study suggests some implications for researchers and teacher educators. First of all, as mentioned above, there is not satisfying research which explicitly investigates norms in the context of technology integration. In this study, this was investigated in the context of a short-term teacher preparation program in Turkey. There is a need for further studies. Second, teacher education programs which aim successful technology integration should develop an awareness of social and socio-mathematical norms and monitor pre-service teachers' development with regard to their endorsed norms.

Acknowledgement

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TOWARDS A PARADIGMATIC ANALOGY EPISTEMOLOGY: SOME EXPLORATORY REMARKS

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In this paper, I present some exploratory remarks on the necessity to build a new kind of epistemology in mathematics education, and, more broadly, in educational sciences as sciences of culture. This epistemological turn asks for theoretical constructs grounded on a system of exemplars, in Kuhn's sense, theoretically and practically related in an analogical way. I give some theoretical and epistemological hints about this kind of process, and I try to sketch a first empirical illustration of the concrete manner to design such paradigmatic analogies.

INTRODUCTION

This paper is devoted to some epistemological considerations. Such a focus lies in the fact that one may conceive of a theory as a body of knowledge relying on background assumptions that are indeed epistemological assumptions, i.e. assumptions referring to the conditions of possibilities of theoretical assertions. For example, claiming that a research result achieved in an experimental setting may be useful in an everyday life situation rests on the (generally) tacit assumption of the sufficient similarity of the two contexts.

In the following, I shall address the general question of the kind of epistemology that could be developed in order to better fit the nature of educational research, and specifically mathematics education research, as a science of culture.

In the first part of the paper, I shall outline some particular features of the Joint Action Theory in Didactics (Sensevy, 2012; Ligozat, 2011; Tiberghien & Malkoun, 2009; Venturini & Amade-Escot, 2013), that will enable me to ground the epistemological analysis that follows in the second part. In the third part of the paper, I shall present a first attempt of an empirical characterization of my reasoning.

UNDERSTANDING DIDACTIC ACTION: THE GRAMMATICAL INQUIRY ON JOINT ACTION

Within the Joint Action Theory in Didactics (JATD), the empirical studies are grounded in the material reality of practice, that one can define as the familiar and concrete structure of teachers' everyday practice. But these studies are carried out in a particular approach that I would describe as a *grammatical* one, following Vincent Descombes:

What makes us able to follow or to draw a straight line? It is nor a psychological question – at what age and by using what resources – neither a question of transcendental philosophy – what the world has to be so that I can go along a straight

line – but a *grammatical*[4] question: in what context saying someone is going along a straight line *means*. (Descombes, 2004, p.444)

In such a perspective, the didactic action (in this expression, "didactic" means "which is related to the teaching and the learning of a particular piece of knowledge") can be studied only if the grammatical necessities of practice have been acknowledged. For example, one can draw interesting consequences from the fact that in order to learn, the student has to reasonably act *on her own*, but cannot learn *without* the teacher. Indeed, the teacher has to be *reticent* (Sensevy & Quilio, 2002; Sensevy, 2011). She has to be tacit about some things she knows, in order to enact student's first-hand relationship to the piece of knowledge. But the teacher has also to talk. She has to be *expressive* in order to accurately orient the students' work. To understand the grammar of teacher's action thus means understand what the teacher chooses to be tacit about and what she chooses to say, what she chooses to hide and what she chooses to show, and how she manages that. With this respect, the JATD draws the hypothesis that every teaching move enacts a *dialectics of reticence and expression*. This dialectics is not confined to the "mere" interaction between the teacher and the student. It is shaped by the system of constraints that delineate the interaction, for example the social and mathematical norms that enact the didactic contract. A further step is accomplished by acknowledging that such a dialectics cannot be understood solely as an *a priori* system of intentions and decisions. This system is reshaped by the *in situ* didactic action, given that the teacher chooses to say (show) something or to be tacit about (hide) it by deciphering, in a specific semiosis process, the student's behavior. Reciprocally, the student is engaged in a particular semiosis process (in a complex dialectics of the Ancient, what she already knows, and the New, what she has to learn) that enables her to make sense or not of the teacher's behavior (Sensevy & Forest, 2011). It is to say that the student directly influences the teacher's behavior. More strongly, it is to say that the didactic action (i.e. the teaching action and/or the learning action) is a joint action, as the grammatical inquiry tells us. This action is a joint action because the essential dialectics of the reticence and the expression cannot unfold concretely without the reciprocal effects of the teacher on the student, of the student on the teacher. This action is also a joint action because it is grounded on a sharing of meanings, which constitutes both the condition and the effect of the didactic action.

TOWARDS A PARADIGMATIC ANALOGY EPISTEMOLOGY

The grammatical inquiry that I described above is a first logical moment essential to the research activity. But this logical research (i.e. which tries to understand the logic of practice) needs a strong relationship to practice at the same time, which asks for both an epistemological and a methodological work.

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[4] The emphasis is mine.

This work rests on the following body of hypotheses.

1. Sciences of culture are sciences of contexts (Passeron, 2013). **2.** The assertions produced within the sciences of culture need to be systematically referred to the contexts they denote. **3.** A good manner to build such a reference consists of instituting some contexts as exemplars (Kuhn, 1979), these exemplars then function as paradigms (sense 2 for Kuhn[5]). We may hypothesize that a given example of practice has to be considered first as an "emblematic example" within a peculiar research endeavor, which further needs to pertain to the common knowledge of a research community to become an exemplar in this research community. **4.** These exemplars/paradigms can be described from an abstract formula (Deleuze, 1988), which enables one to understand them in their main structural features. In order to be scientifically relevant, one has to take continually into account their singularity (Agamben, 2009). One of the fundamental components of the research work consists of revealing and exploring the singularity of exemplars, which provide its substance to the inquiry. **5.** The scientific inquiry thus spreads out in the construction of a paradigmatic system, a constellation of exemplars, within *an epistemology of paradigmatic analogy*. The inquiry requires building a paradigmatic system, then to browse this system in order to compare and to relate the different exemplars it encompasses. **6.** The browsing of the paradigmatic system, that is to say the work of each of the exemplars it contains and their relationships, asks for an "organized plurality of systems of descriptions" (Descombes, 1998). In particular, such a work needs to hold together thin and thick descriptions, in Gilbert Ryle's sense (2009), the "plurality of descriptions" being given by the different densities of each of them (Descombes, 1998), each of them enabling the researcher to document aspects of the exemplar that the other ones cannot account for. **7.** Such an inquiry radically turns upside down the usual relations between the concrete and the abstract, in which the abstract is conceived of as the common part shared by some concrete elements. Its rests on a Marxian dialectical vision of these relations, in the sense that scientific activity allows the "ascent" from the abstract to the concrete (Marx, 2012; Kosic, 1976; Ilyenkov, 1982; Engeström et al, 2012). In the process I have previously described, such an ascent takes place in the comparison of different exemplars of the paradigmatic system, based on the abstract formula, which allows a first apprehension of each of them.

The previous system of hypotheses has to be taken as a kind of compass for the methodological work. They are put into the test through the design of an instrumentation for sciences of culture, grounded on the *study film*. This expression (Sensevy, 2011; Tiberghien & Sensevy, 2012) refers to films of didactic practices,

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[5] Kuhn (1979) acknowledges himself two fundamental meanings to the notion of paradigm. The meaning that I term "sense 1" refers to the notion of "disciplinary matrix". The sense 2, the most important for Kuhn, refers to the exemplar.

which may encompass a long period of time relating to the same teaching. They provide the fundamental bricks for an instrumentation based on hybrid text-pictures systems[6]. The challenge consists in presenting a given activity through pictures (photograms) taken from study films, and texts (of various origins and types) relating to these pictures. Pictures and texts are in mutual annotation, and thus concretely provide the *organized plurality of systems of description* that I mentioned above. Within these hybrid texts-pictures systems, the instrumentation may rely on various pictures and discourses analysis software, and profit of statistical analysis (in which "individuals" are utterances or pictures structures). The hybrid texts-pictures systems thus may provide the conceptual matter of exemplars put in relation within a same paradigmatic system.

It is important to note that the determination of exemplars, notably at the stage of the production of the abstract formulas that allow instituting them, needs models, as seeing-as (Wittgenstein, 2009), which enable the researcher to identify them. In JATD, the exemplars and the paradigmatic systems that they make possible refer to various features of the didactic game as those that I have delineated in the previous section.

TOWARDS PARADIGMATIC SYSTEMS: SKETCHING A FIRST EMPIRICAL CHARACTERIZATION

In this section, I would like to envision how could be developed a first paradigmatic system relating to the reticence-expression dialectics, by focusing on a case study at Kindergarten, the treasures game situation. I will elaborate on a previous study (Sensevy & Forest, 2011) that I try to reconceptualize within a paradigmatic analogy epistemology.

A general description of the situation

Brousseau and his team designed the *treasures game* for kindergarten students at the beginning of the nineteen eighty. This learning sequence takes place over several months and it aims to have students build a system of graphical representations. Brousseau (2004) has presented a strong theorization of this research design, which he considered as a fundamental situation for the notion of a *representation*. The situation was recently re-implemented in some classes in Switzerland (Leutenegger & Ligozat, 2009) and in France. It consists of producing a list of objects to be remembered and communicated. The game is organized in four stages, the rules changing as the game progresses.

The stage 3 of the game aimed at communicating with lists. It takes place in small groups of 5 pupils. 4 object representations are drawn by a pupil, who is "the

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[6] An interdisciplinary collective of educational researchers is participating in this instrumentation endeavor, within the Federative Research Structure (SFR) ViSA: http://visa.ens-lyon.fr/visa

designer", and these objects are hidden in the box. The other 4 pupils have to "read" the graphic representations of the designer, and name each object to get it out of the box. This third stage gives pupils opportunities to debate, firstly in their small group, and secondly in the whole classroom during the 4th stage of the game. We present below some objects aimed to be designed by the students.

Figure 1: some examples from the reference collection of 40 objects.

Solving the problem of pan's representation

The small episode we study takes place when pupils have to read the representation of the designer.

Figure 2: the representation of the pan by the "designer" student

In this episode, the teacher has given the designer four quite different objects; one of them is "the pan" (cf. fig.1, above). When students have to "read" the designer's production (cf. fig.2, shown against), they fail to recognize "the pan" (the designed object, fig.1, above), confusing it with "the lens", another object of the collection of reference, which has a similar form (cf. fig.1, above).

In the short moment of joint didactic action we present, *the teacher has first shared with the students the failure of the designing and recognizing the pan.* She then asks students to find some ideas to better draw and recognize the pan.

Table 1: giving the problem

	Iman: *we have to make a small line, we have to do this...*
	The teacher shows the object in her left hand, and its representation (cf. fig.2 above) by pointing it with her finger, in her right hand, and asks the students for a comparison. A student, Iman, points a part of the object, which is characteristic of it (a hole on the handle). In that way, the student

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	links a sensory feature to a specific characteristic of the object, which is potentially relevant to solve the problem.

Table 2: another strategy is proposed

7'49	**Iman:** *a big ring, and for the lens a small ring*
	The teacher looks herself at the representation drawn by the designer (cf. fig.3 above). She points it by her left hand finger. Iman "writes" with her finger on the table by evoking a possible representation of the difference between the pan and the lens (big ring-small ring).

By showing the representation to the students, and looking herself at it, the teacher drives student's attention onto the design. Iman tries to draw on the table, and a generic sign emerges (a ring). But the representation "big-small" is not a distinctive sign (which may allow to recognize the object in itself) and it will not really work if only one object (the lens *or* the pan) is to be recognized among others: if only the pan (or the lens) is represented, the comparisons of representations (between the lens and the pan) cannot work. Interpreting in a general way the teacher's expectations about the drawings, the student produces a kind of strategy which can be useful, but which is not a winning one in this specific learning game.

Table 3: the less relevant strategy starts spreading

7'50	**Another student**: *...a small ring, and a stick*
	As in table 3, the teacher shows the object and its representation. Again, she asks the students for a comparison.
	Another student writes with his finger on the table, as Iman did. He repeats Iman's proposition (a ring), *adding the possibility of drawing a stick* (for the handle).
	It's worth noticing that the attention of the students focuses on Iman's action. The teacher' attention is oriented to the last student who had made a proposal.

The teacher goes on showing the object and its representation in order to have the students "comparing" them, but Iman's proposal is taken up and meliorated by others.

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The new proposal (a small ring and a stick) possibly solves the problem of a generic representation for the pan and the lens. From this sole point of view, it is relevant. But, as previously, it doesn't work as a distinctive representation between the two objects. So it is less relevant in this representation game.

Table 4: the teacher focuses students' attention on the problem

Showing the pan and its representation, the teacher reminds the students the current difficulty *(we had confused with the lens)*.	In the same time, she reformulates the problem, relating to the lens (*so that one guesses every time that it's the lens…)*	Moving on her right side, near the designer, she formulates the problem for the pan. When talking, the teacher directs her eyes toward different children, finishing toward the designer.
The teacher: *It's true that we had confused with the lens, because it was like the drawing of the pan.*	*We should try to find a drawing for the lens, so that one guesses every time that it's the lens…*	*…and a design for the pan, so that one guesses every time that it's the pan…*

One can see, at this moment, how the teacher works to take on the issues at stake by expressing herself (*we had confused... we should try to find a design... so that one guesses every time*). However, the teacher remains reticent. To support her reticence, she uses the resources of the proxemics, joining gesture to her talk: first, she shows the object and its representation like in the photogram 1, showing them to all the students; second, she evokes the lens (which is absent from the package) putting the pan and its representation on the left side; third, she moves on the other side to finally show the representation which failed, to the designer. The learning objective is strongly restated at this time (*a drawing for the lens/the pan, so that one guesses every time that it's the lens/the pan…*): the students have to produce by themselves a strategy to design the pan, while taking into account its similarity with the lens.

Table 5: Joint action provides a new relevant move

	Tilio: *Ah yes! so one can make a small hole here.*

	The teacher continues showing the object and its representation. A student (Tilio) points a part of the pan, the same that Iman had pointed on table 3.We can perceive the convergence of student's gaze on Tilio's gesture.

According to the teacher' expectation, Tilio acts by pointing the hole on the object and by naming it (a small hole for the handle). In a more theoretical language, we can say that Tilio deciphered the teacher's expectation about the playing of the "right game". He then plays a relevant move in the representation game, by deciphering a relevant sign on the pan. Indeed, if the representation of the pan includes "a small hole" on the handle, it will allow differentiating the pan from the lens, even though the general form of the two representations is similar (a small ring and a stick). Tilio's proposal will be soon institutionalized as a first way of representing the pan.

A first empirical characterization: some concluding remarks

The example that we present above may be consider as en emblematic example of reticence-expression dialectics. As we have seen, what we term "emblematic example" can be seen as a kind of "potential exemplar", which could be transformed in an exemplar after having been worked out and accepted as such in a research community. This "emblematic example" highlights the following aspects, which concretize the above epistemological reflexion. **1)** In order to produce emblematic examples, one has to use an abstract formula, a theoretical concept that has to be turned into a means of problematization. Here, the reticence-expression dialectics enable to acknowledge the way the teacher expresses herself, for example by reminding students the goal of the game, and the way she is reticent, for example without directly discarding students irrelevant proposals. **2)** In a bottom-up move, the theoretical construct (the reticence-expression dialectics) is reshaped in its concrete unfolding, for example by subsuming under its scope some new phenomena, as we can see in this emblematic example, when the teacher's expression relates to the necessity of re-enacting students' understanding of the problem, or by acknowledging that the reticence could no be maintained without a symbolic and material system which allows the teacher to remain silent. **3)** In a horizontal move, in that by connecting with other theoretical concept, the theoretical construct is accommodated. For example, using another fundamental dialectics, that of contract and milieu, enables the researcher to link the reticence-expression equilibrium to the way in which the contract (as a background of common meanings resulting from the previous joint action between the students and the teacher) and the milieu (as a symbolic and material system which structures the problem to be solved) need to be attuned together. **4)** The "emblematic example" we focus on is built through a text-picture hybrid system of which we try to give a taste in the above section. Such a system is not confined to the sole elements that we have presented above. It may be

built with any kind of texts or picture that is informationally relevant for the inquiry, in itself, and/or in its relations to others texts and/or pictures. **5)** This emblematic example refers to a specific practice that one needs to understand as a connoisseur in order to be able to build such an example. It raises the question of several specificities, notably that of the piece of knowledge at stake (here the representation process), and the educational level (here kindergarten), which are inherent to this example. It is the main reason why this example would have to be integrated in a paradigmatic system, in which one could find a constellation of emblematic examples whose gathering could enable a community of research to jointly build a common reference for the concept of reticence-expression dialectics. A good part of the JATD researchers' endeavour is now devoted to this task, through a common work between teachers and researchers, what we term cooperative engineering (Sensevy, Forest, Quilio, & Morales, 2013).

In this work, I have addressed the general question of the kind of epistemology that could be developed in order to better fit the nature of mathematics education research, as a science of culture. I contend that such an epistemology rests on the reversal of the usual priority of "the abstract" over "the concrete" in classical epistemology. The main issue thus consists in characterizing the "work of the concrete" that researchers need to do to give a contextual reference to the abstract entities. In this paper, I have tried to show how such a work can rely on the description of practice through what I have termed "emblematic examples", which concretize theoretical structures, and that gain their efficiency from being analogically related to other emblematic examples of the same theoretical structures, within a paradigm (exemplar). I argue that the main challenge in developing such an epistemology and the kind of theory it entails consists of building such exemplars on the basis of texts-pictures hybrid systems such the one I have sketched in this paper. These paradigmatic systems could then be proposed not only as ways of representing the meaning of theoretical entities, but also as instruments for Mathematics Education and Teacher's professional Development.

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COMPETENCY LEVEL MODELLING FOR SCHOOL LEAVING EXAMINATION

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A project group was commissioned to develop a content- and action-related competency grid in order to enable quality assessment and comparability of mathematics examination questions in the Austrian Matura at the end of the Secondary School Level II. Based on theoretical grounds, in the competency grid the three dimensions operating, modelling and reasoning are distinguished and described on four levels.

Obtaining information on the development of mathematical competency is a central concern of mathematics education (e.g. Leuders, 2014) and empirical educational research (e.g. Hartig, 2007). In Austria, an approach with the goal of a standardized competency-based written final examination – the so called 'Matura' at the end of Secondary Level II – (cf. AECC, 2009; BIFIE, 2013) in the context of mathematics as a general education subject (cf. Fischer, 2001; Fischer & Malle, 1985; Klafki, 1985; Winter, 1975, 1996) was applied. Examinees are expected to have both mathematical (basic) knowledge and (basic) ability, as well as general mathematical skills such as reasoning skills, problem solving skills, and also the ability to use mathematics in different situations, i.e. modelling skills. However, in PISA 2000, a lack of modelling competency was observed, when students failed to solve (real-life) problems with the help of models in a satisfying way (cf. Klieme et al., 2001). Based on this result, modelling competence was crucial for competency orientation in the curriculum enhancement of mathematics education in the German-speaking region (thus also for the *Matura* in Austria). With reference to Weinert's definition of competencies (2001, p. 27) as "the cognitive skills and abilities which the individual possesses, or which can be learned, to solve certain problems, as well as the associated motivational, volitional and social readiness and skills in order to successfully and responsibly use problem solutions in a range of situations."

In an iterative process we developed a competency level model for the written final exam in mathematics at the end of Secondary School Level II. The process consisted of three elements: the discussion of competency specifications and developments, the discussion of mathematical tasks, task rating in due consideration of the competency model and the discussion of these ratings. Against the background of theoretical and also experience-based ideas about the current development of mathematical skills in school learning, we described the following three domains of mathematical competencies: operating, modelling, and reasoning[7] (O-M-A) on four levels.

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[7] The German word 'Argumentieren' is synonym to reasoning.

In close cooperation with the Federal Institute of Educational Research, Innovation and Development of the Austrian School System ('BIFIE'), we developed a competency level model facilitating the description and comparison of the exam requirements, especially with regard to examination questions in the final examination in mathematics (Siller et al., 2013).

COMPETENCY LEVEL MODEL

In the competency models of the German-speaking countries Austria, Switzerland and Germany (AECC, 2008; HarmoS, 2011; KMK, 2012), content areas (such as geometry or arithmetic), general mathematical competencies (such as reasoning) and skill levels (usually three-stage) are considered. The elements of the model in each country are therefore different when compared to one another. The competency levels are somewhat vague. Therefore, they can only be described on the basis of empirical task difficulty. To put our competency level model in a wider scientific context, we follow Leuders (2014, p. 10): "A model is discussed which (i) a priori postulates levels in acquiring a certain competence, is describing (ii) through stepped task situations and (iii) hierarchically ordered categorical latent ability variable. This allows (iv) determination about which competency pupils possess at each level."

In comparison to earlier statements, the development that has taken place in this area is evident. For example, Helmke and Hosenfeld stated in 2004 (p. 57): "Neither are the currently available versions of the educational standards derived at the time from comprehensive and didactic accepted competency models (...) nor is there already in all relevant areas of content expertise models which meet the abovementioned requirements, particularly theoretically coherent developmental and learning psychology based levels concepts."

Thinking in (competency) levels is common in schools since curricula and teaching materials are based on this view (cf. e.g. Kiper, Meyer, Mischke & Wester, 2004). Competency level models contribute to the diagnosis of the learners' levels of competency by the assessment of their achievements. Moreover, the models aim at describing the development of competencies. Their weaknesses, however, are embodied in the fact that it usually remains undetermined how a change to the next level can be accomplished and what conditions are necessary for this. Furthermore, a fixed sequence is assumed, which implies that neither can any steps be skipped nor regressions occur, but which assumes steady, cumulative learning.

For the present competency level model we have agreed on four stages, which can be identified in a manner analogous to Meyer (2007), who described the following four levels (Meyer, 2007, p. 5):

1. Execution of an action, largely without reflective understanding (level 1)

2. Execution of an action by default (level 2)

3. Execution of an action after insight (level 3)

4. Independent process control (level 4)

The activity theory forms the background for the didactic interpretation of such initially pragmatic levels (cf. e.g. Lompscher, 1985) with the corresponding concept of different cognitive actions and their specific dimensional structure. Nitsch et al. (2014) developed and empirically verified a competency structure model that describes relevant student actions when translating between different forms of representations in the field of functional relationships. For example, they could show that the two basic actions of acquirement *Identification* and *Implementation* (Construction) and the basic cognitive actions *Description* and *Explanation* differ in their cognitive demands, i.e. they are based on different facets of competency. Therefore, we used the theoretical model of hierarchical structure of cognitive actions (Bruder & Brueckner, 1989) for the description of competency levels.

DEVELOPMENT OF A COMPETENCY LEVEL MODEL

Currently existing competency models are primarily based on empirical analyses: Based on the solution probabilities of tasks (items), competency levels are modelled in the context of large-scale studies. An alternative approach is to primarily derive a model from theoretical concepts. This also requires the recognition of central instructional goals such as a sustainable understanding of mathematical relationships, which in turn presupposes a high level of cognitive activation in the teaching processes (cf. Klieme et al., 2006). This can, for example, be achieved by the following measures:

- the preparation of relationships for basic knowledge and skills learned;

- the challenge to describe mathematical relationships or solutions in application contexts;

- the creation of occasions for reasons or reflections.

Such criteria of demanding instruction should also be appropriate to form a competency level model.

THE COMPETENCY LEVEL MODEL O-M-A

Competency level models that are empirically based indicate to what extent tasks differ in their level of difficulty in terms of processing. Evidence of existing difficulties can be obtained by carefully analysing potential and actual solutions. Normative stipulations of difficulty levels imply that it is not possible to successfully process the task on a lower level. The levels of the competency model postulate what skills are needed to solve them. This does not exclude that there are multiple solution strategies, particularly for complex task definitions.

For designing the domains of mathematical competencies, we follow an orientation to Winter's basic experiences (cf. Winter, 1996, p. 37):

1. "To perceive and understand phenomena of the world around us that concern or should concern all of us, from nature, society and culture in a specific way,

2. to learn and comprehend mathematical objects and facts represented in language, symbols, images and formulas as intellectual creations as a deductive-ordered world of its own kind,

3. to acquire task problem-solving skills that go beyond mathematics (heuristic skills)."

While the first basic experience corresponds to mathematical modelling as a fundamental action area in learning mathematics, there are the other two basic experiences "operating" and "reasoning", which serve the second fundamental experience as well as "problem solving" for the third basic experience. In various competency models „communicating" is included to emphasize the linguistic aspects, as well as other domains of mathematical competencies.

"Problem solving" is not separated as an independent domain in the Austrian requirements for the final examination (BIFIE, 2013). "Problem solving" is defined as a more complex aspect of action and therefore includes the domains of the mathematical competencies Operating, Modelling and Reasoning, especially in higher levels of performance. "Communicating" is seen as an important domain of mathematical competencies for teaching mathematics, but cannot be specifically differentiated from Operating, Modelling and Reasoning and is therefore included in the other aspects.

The domain "Reasoning" is related to the suggestions of Bruder and Pinkernell (2011), who also pick up on considerations of Walsch (1972). "Modelling" served as the basis of the

Domains of mathematical competency			
Level	Operating	Modelling	Reasoning
1	Identifying the applicability of a given or familiar method; Implementing/executing a given or familiar rule	Implementing a representational change between context and mathematical representation. Using familiar and directly recognizable standard models for describing a given situation with appropriate decision	Performing basic technical language reasoning. Examining the application of a relationship or method and the fit of a term for a given (intra-mathematical) situation
2	Implementing/executing multi-step methods/rules, possibly with the use of computers and use of control options	Describing the given situation by mathematical standard models and mathematical relationships. Recognizing and setting general conditions for the use of mathematical standard models	Understanding, comprehending, explaining mathematical concepts, principles, methods, representations, reasoning chains and contexts
3	Determining whether a particular method/specific rule is appropriate for a given situation, make and perform the appropriate method/rule	Applying standard models to novel situations, finding a suitable fit between suitable mathematical model and real situation	Examining and completing mathematical reasoning, perform and describe multi-step mathematical standard reasoning
4	Develop/form macros[2] and join together macros already available	Complex modelling of a given situation; reflection of the solution variants or model choice and assessment of the accuracy or adequacy of underlying solution methods	Form independent chains of reasoning, technically correct explanation of mathematical facts, results and decisions

[2] aggregated mathematical rules

Figure 1 O-M-A Grid

fundamental work of Niss (2003) and other ideas, e.g. of Boehm (2013) or Goetz and Siller (2012). There are relatively few preparations for a levelled conception of competencies in the mathematical domain "Operating". For example, Drueke-Noe

(2012) shows that complex algorithms are required already in early grades. But for a high level of expertise, it is not only necessary to use complex algorithms, but also to find the right algorithm to apply in a given situation and to combine different algorithms where appropriate.

The result of our considerations as part of this project is a model with three domains of mathematical competencies (cf. Fig. 1) that substantially captures the key aspects of mathematical work at school. The competency level model is aimed at fulfilling all essential requirements with regard to the conception of mathematical learning outcomes in Austrian mathematics education of the Secondary School Level II (cf. BIFIE, 2013). Complex problem solving situations can be described by the interaction of the three domains of mathematical competencies.

WHAT EMPIRICAL EVIDENCE EXISTS FOR THE DIFFERENTIATION OF THE THREE DOMAINS OF COMPETENCIES OPERATING, MODELLING, REASONING AND ITS GRADATION IN 4 LEVELS?

The question about an empirical verification of the theoretical competency level model with respect to the separation of domains of mathematical competencies and the gradation can be answered only in the context of a sufficient number of processed tasks for each area of expertise.

Data were taken from the so-called "school experiment" in 2014. Before the central final examination throughout Austria will be implemented in the school year 2014/2015, secondary academic schools and maths teachers were invited to voluntarily take part in a pilot study on graduating students' math competences. In this study, the math tasks were processed under the same conditions as they would be processed at the mandatory central final examination. It is important to note that the performance in the tasks contributes to students' final grade. For the school experiment whose data are being reported here, there were 803 students (m = 345, f = 458) from 42 classes from 9 districts in Austria. The examination consisted of two separated parts with so-called type 1 and type 2 tasks (cf. https://www.bifie.at/node/2633).

Type 1 tasks "focus on the basic competencies listed in the concept for written final examination. In these tasks, competence-oriented (basic) knowledge and (basic) skills without going beyond independence are to be demonstrated." (cf. BIFIE, 2013, p. 23). They are coded as solved against non-solved. The various bound response formats such as multiple-choice format and a special gab-fill format enable accurate scoring. For the award of points in tasks with open and semi-open response format, solution expectations and clearly formulated solution keys are specified.

The characterization of type 2 tasks presents serious challenges to the basic principles of modern test theory. The tasks are considered "for the application and integration of the basic competencies in the defined contexts and application areas. This is concerned with

extensive contextual or intra-mathematical task assignments, as part of which different questions need to be processed and operative skills are, where appropriate, accorded greater importance in their solution. An independent application of knowledge and skills is necessary" (cf. BIFIE, 2013, p.23). These tasks are also consistently structured in design and presentation, as well as in terms of scoring (cf. BIFIE, 2013).

A total of 16 (type 1) tasks in the competency domain of operating, 2 tasks (type 1) in the competency area of modelling, and 4 tasks (type 1) in the competency area of Reasoning were tested in the 2014 school experiment. Thus, no level analyses could be conducted.

There is a relatively high variation of the solution frequency within competency domain Operating (cf. fig. 2), which can be explained by the heterogeneity of tasks presented, especially with regard to high profile/over-training. Variation of solution frequency was also observed for the competence domain Modelling (cf. fig. 3) as well as Reasoning (cf. fig. 4). The parameter "difficulty" was not measured, only the percentage of solution as an indicator for the level of difficulty of a task.

Task	Competence domain	level of difficulty (% solved)
3	Operating	0.81
4	Operating	0.84
7	Operating	0.98
8	Operating	0.94
9	Operating	0.84
10	Operating	0.88
12	Operating	0.87
13	Operating	0.59
14	Operating	0.88
15	Operating	0.56
17	Operating	0.84
18	Operating	0.57
21	Operating	0.80
22	Operating	0.49
23	Operating	0.81
24	Operating	0.58

Figure 2 Difficulty level of the tasks for the competence domain Operating (n = 803)

Task	Competence domain	level of difficulty (% solved)
19	Modelling	0.51
20	Modelling	0.91

Figure 3 Difficulty level of the tasks for the domain Modelling (n = 803)

Task	Competence domain	level of difficulty (% solved)
1	Reasoning	0.80
5	Reasoning	0.73
6	Reasoning	0.94
11	Reasoning	0.64

Figure 4 Difficulty level of the tasks for the domain Reasoning (n = 803)

Two of the type 1 tasks are positioned on competency level 2 and could be analysed. A heterogeneous picture emerged for these two tasks: While task 2 could rarely be solved, task 16 was easily mastered by the students.

Can the pre-defined four levels be confirmed empirically in all the three areas of competency? This question can be answered in a first approximation only on the basis of type 2 tasks for levels 1 and 2 due to the fact that higher
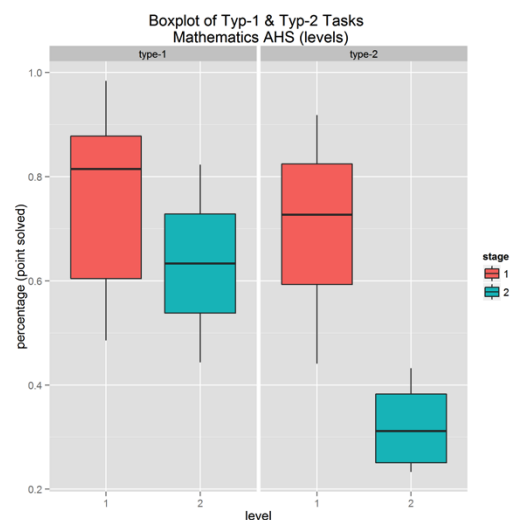

Figure 5 Empirical difficulties of type 2 tasks separated by type and level (n = 803)

graduations did not appear in these exam booklets.

As can be seen in Figure 5 (in general) and Figure 6 (separated among O-M-A), the level 2 tasks seem to be more difficult in general. Thus the competency level of the task gives us a good statement about the level of difficulty.
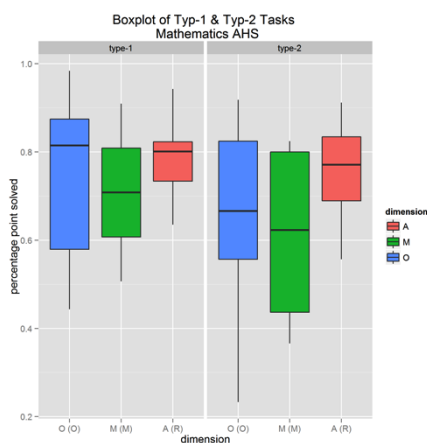
SUMMARY AND OUTLOOK

Figure 6 Empirical difficulties of the type 1 and type 2 tasks among O-M-A (n = 803)

The provided model with the three domains of mathematical competencies Operating, Modelling and Reasoning (O-M-A) distinguishes three basic mathematical operations on four levels. It is based on considerations from educational sciences and learning theories as well as insights and experiences with regard to relevant factors for learning mathematics in school. It is part of a complex effort to gain a sound basis for competency diagnostics and performance assessment in mathematics in the German-speaking countries. It differs from other models by its consistent theoretical foundation and by the focus on potential lines of development for long-term competency building. The model O-M-A provides a normative setting for relevant levels of requirement in the three domains of mathematical competencies. This facilitates a certain comparability of type 1 and type 2 tasks provided for the final examination.

The added value of the developed model lies in several areas:

- It provides guidance both for the assessment of (written) performance and for the learning tasks in the classroom.

- It serves the purpose to reveal potential for development in the classroom.

- It allows for the identification of development potential in the task structure

Limitations of the competency level model O-M-A lie in the coarseness of the approach. Neither can all the differences between the test tasks relevant to their level of difficulty be considered in detail (such as linguistic complexity), nor can individual developmental trajectories be mapped in learning processes. Further restrictions of the model are also indicated by the fact that of all the mathematical content and activities implied in each task only a basic competency referring to the list of basic skills (cf. BIFIE, 2013) can be adopted. The specific situation of each school class or priorities of teachers cannot be reflected. Thus, many tasks can prove to be easier, but also more difficult than in the rating.

The competency level model O-M-A aims at describing levels of competencies by identifying the qualitative differences of each competence. The growing body of research on maths learning served as the theoretical background. The data and results presented so far are preliminary and did not account for not controllable influence factors such as training effects. However, they can be interpreted as a first clue that the O-M-A can be rudimentarily verified empirically. Therefore, further research is necessary to empirically test the levels of the model and to test the model against level 3 and level 4 tasks.

The model O-M-A is indefinite in explaining the attainment of the next higher level. For this reason we define it as a competency *level* model and not a competency *development* model. To answer the question as to whether this model could map potential lines of students' long-term competency development, more theoretical and empirical work is needed. So far, it cannot be applied to the development of a math learning process.

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STRUCTURALISM AND THEORIES IN MATHEMATICS EDUCATION

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We present the structuralist conception of scientific theories as a Deus ex Machina *which allows to resolve the entanglements of theories in Mathematics Education. We illustrate with examples how this conception, which forms a solid and solvent body of knowledge in Philosophy of Science, provides us with tools to perform a careful analysis of a theory, both by itself and in connection with other theories.*

RECONSTRUCTION OF SCIENTIFIC THEORIES

As it is the case in many other disciplines, in Mathematics Education there are several theories living together: Theory of Didactic Situations (Brousseau, 1997), Anthropological Theory of the Didactic or ATD (Chevallard, 1999; Bosch et al. 2011), APOS[8] theory (Dubinsky & McDonald, 2002), Onto-Semiotic Approach (Godino et al., 2006), Theory of Abstraction in Context (Dreyfus et al., 2001), Theory of Knowledge Objectification (Radford, 2003)… Whereas the cohabitation of theories is perfectly normal, efforts aiming to connect some of them, especially from the CERME working team "Theoretical perspectives and approaches in mathematics education research" (CERME 8, 2013), are also very natural and desirable.

We defend in this work that, for a better understanding of the possibility of connection of two theories, we must reconstruct them by using the same language. The reconstruction of a theory can be carried out from different conceptions. When we speak of 'conceptions' we mean ways of giving an account of what a scientific theory is, and not of how a scientific theory (in particular, a scientific law) is constructed. Thus, a priori these conceptions do not pay attention to methodological aspects.

The one favoured here is the so-called *structuralist conception* (Balzer et al., 1987). This is an elaboration of the *semantical conception* (initiated by Suppes and Adams in the 1970s), and it seems to reconcile the most important aspects of the *syntactical conception* (advocated by Reichenbach, Ramsey, Bridgman, Campbell, Carnap in several works from the 1920s to the 1950s) and the *historicist conception* (advocated by Kuhn, Lakatos, Laudan in several works in the 1960s), while avoids their problems (Diez and Moulines, 1997).

Now we will give a brief explanation of the main points of the structuralist conception. For a more extensive treatment see (Balzer et al., 1987).

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[8] This is the short form for "Action, Process, Object and Scheme".

According to the structuralist conception, a scientific theory is a net of many nodes (which will be called *elements of the theory* or *theory-elements*) connected in several via *specialization*, see Definition 2 below. Of course, such a net does not appear out of the blue, but it is developed little by little along the time. This is how the structuralist captures the diachronic character of a theory. The synchronic character of a theory appears in the description of the theory-elements.

Definition 1: To determine a *theory-element* one has to specify:

i) The portions of reality the theory-element conceptualizes, i. e. the portions of reality the theory can speak of, called *potential models*. These potential models are described as portions of reality which can be modelled by using a *structure* (that is to say, a tuple $(D_1, D_2, …, R_1, R_2, …)$ of sets D_i and relations R_j between these sets) and a list of properties applicable to the structures of the former type. We call M_p the set of potential models.

ii) The *laws* with which the theory-element aims to enlighten reality. Each law is a property applicable to the structures of the specified type. The laws distinguish the so-called *actual models* among the potential models. We call M the set of actual models.

iii) The *partial potential models*, which are these portions of reality which can be checked to be potential models without assuming the laws of the theory-element. Notice that to verify that a portion of reality is a potential model we check, in particular, that the relations R_j appearing in the type of structure are satisfied. In this checking we use some method and this method might, or might not, assume the laws of the theory-element. A relation R_j is *theoretical with respect to a theory element* T (or, in short, T-*theoretical*) if every method of determination of R_j assumes the validity of the laws of this theory-element. Thus, a partial potential model of a theory-element is nothing but a potential model in which we omit the theoretical relations. We call M_{pp} the set of partial potential models.

iv) Those partial potential models that are expected to be actual models. These partial potential models are, after all, the *intended applications* of our theory-element. We call I the set of intended applications of our theory-element.

Thus a *theory-element* is an ordered pair T = (K, I) where I is the set of intended applications and K = (M_p, M_{pp}, M) is the *core*, formed by the set of potential models the set of partial potential models, and the set of actual models.

The *empirical claim* of a theory-element is just the statement which asserts that the intended applications are actual models, I $\subseteq$ M, that is to say, that in certain portions of reality, which can be detected without assuming the laws of the theory-element, these laws actually hold.

In the next section we will give several examples of theory-elements but, unfortunately, we will not point out a theoretical relation in any of them. It is an

important open question whether there are theoretical relations in the current theories of Mathematics Education. In Classical Mechanics (CM), the relations of *position* or *time* are not CM-theoretical, since you can determine them without assuming any proper law of Classical Mechanics. However, the relation *mass* is CM-theoretical, since any method of determination of the amount of mass of an object assumes a law proper of the CM. For examples in other disciplines see (Balzer et al. 1987).

NETWORKING THEORIES

In what follows we use the structuralist approach to present different kinds of possible connections between theory-elements.

Definition 2: A theory-element T' is an *specialization* of another theory-element T, and we write T' σ T, if:

(1)

1.1) $M'_p = M_p$, that is to say, both theory-elements conceptualize the world in the same way.

1.2) $M'_{pp} = M_{pp}$, that is to say, both theory-elements consider the same theoretical relations.

1.3) $M' \subseteq M$, that is to say, every law in T is also a law in T'.

(2) $I' \subseteq I$, that is to say, every portion of reality aimed to be explained by T' is also a portion of reality aimed to be explained by T.

In short, to specialize consists of increasing the amount of laws without changing the conceptual architecture.

Definition 3: A *net-theory* is a pair N = ({T_i}, σ) where {T_i} is a non-empty set of theory-elements and σ is a specialization relation on {T_i}.

Next we are defining the notion of *theorization*, but first we need the following:

Definition 4: Given two structures (see Definition 1) x = (D_1, …, D_m, R_1, …, R_n) and y = (D'_1, …, D'_p, R'_1, …, R'_q), we say that y is a *substructure* of x if:

1) $p \leq m, q \leq n$.

2) Every D'_i is a subset of some D_j.

3) Every R'_i is a subset of some R_j.

Definition 5: A theory-element T' is a *theorization* of a theory-element T if:

1) Every intended application of T' admits an actual model of T as substructure.

2) There are potential models of T' which are not substructures of potential models of T (because they contemplate new domains and/or new relations).

The first condition says that every portion of reality T' aims to explain satisfies the laws of T. The second condition says that T' includes new (not necessarily T'-theoretical) concepts not contemplated by T.

Next I will show in examples some tentative structuralist descriptions of some elements of the ATD.

Example of theory-element: Our first example is inspired in the so-called *Herbartian scheme* (Chevallard, 2012), which is probably the most general structure proposed by the ATD to deal with situations of study. In this structure there are things like a task or question which requires some answer, a series of partial answers, and a final answer. Therefore, the structure corresponding to our theory-element T_1 will be the tuple $(\{1, …, n\}, P, s)$ where $\{1, …, n\}$ is the set of the first n natural numbers, P is a non-empty set P, and s is a map from $\{1, …, n\}$ to P. The image of 1 is said to be a *generating question*, the image of n is said to be a *final answer* and the other images are said to be *partial answers.* Since no law is stated, there is no distinction between potential, partial potential and actual models. Notice that every temporal sequence of n events fits in this structure, but, of course, not every such sequence is an intended application of T_1. This is why it is important to explain which are our intended applications, namely, those sequences of events consisting in finding an answer to a question.

Example of theorization: If, moreover, in each of the partial answers of T_1 we distinguish between tasks, techniques and logos elements[9], that is, if we look at the constituent parts of the so-called *praxeologies* (Chevallard, 1999), we would have reached a theorization, T_2, of T_1. The structure corresponding to T_2 will be a tuple $(\{1, …, n\}, S_T, S_t, S_L, s)$ where S_T, S_t, and S_L are non-empty sets whose elements are called *tasks*, *techniques* and *logos-elements*, respectively, and s is a map from $\{1, …, n\}$ to $S_T \times S_t \times S_L$. Since no law is stated there is no distinction between potential, partial potential and actual models. Now not every temporal sequence of n events fits in the structure of T_2. Not even every temporal sequence of n events consisting in finding an answer to a question! In fact, our intended applications are temporal sequences of events consisting in finding an answer to a question such that in each of these events we find three components and such that, moreover,

- all the first components of the events are 'of the same nature' (this is encoded in the fact that they belong to the same set), namely, tasks,

- all the second components of the events are of the same nature, namely, solutions to the task specified in the corresponding first component, and

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[9] For the sake of simplicity, we do not distinguish between *task* and *type of task*, and between *technological* and *theoretical* elements among the logos elements, even if they are important distinctions in the ATD.

- all the third components of the events are of the same nature, namely, explanations of why the corresponding second element solves the corresponding first element.

Example of theorization: If, moreover, we take into account the dynamics of each of these praxeologies, recognizing the so-called *study moments* (Chevallard, 1999), we would have a theorization, T_3, of T_2. The structure corresponding to T_3 will be a tuple $(\{1, \ldots, n\}, S_T, S_t, S_L, \{0, 1\}, \{*\}, s)$ where

- $\{1, \ldots, n\}$, S_T, S_t and S_L are as before.

- s is a map from $\{1, \ldots, n\}$ to $S \times (S \cup \{*\}) \times (S \cup \{*\}) \times \{0, 1\} \times \{0, 1\} \times ([0, 1] \cup \{*\}) \times (\mathbf{N} \cup \{*\}) \times (\mathbf{N} \cup \{*\})$, where S is the union of S_T, S_t and S_L, called the *study sequence map*, and its images are called *events*.

The structure is now more complicated because it has to model more ambitious intended applications. Indeed, in the events of the sequence we still look at tasks, techniques and logos, but we also pay attention to the way they are related:

- The first (respectively, second, third, fourth and fifth) component of an event refers to the first (respectively, second, third, fourth and fifth) study moment (Chevallard, 1999).

- The last three components of an event refers to the sixth study moment, namely, to the evaluation moment. More precisely, the sixth component refers to the scope of the technique (it is a bounded magnitude which reaches the value 1 if the technique covers all the possible cases of the task), the seventh component refers to its economy and the eighth component refers to its reliability (see Sierra Delgado et al., 2013)[10].

For example, an event which is an element of $S_T \times S_t \times S_L \times \ldots$ is regarded as a task followed by an elaboration of a technique followed by an explanation of why this technique works, whereas an element of $S_t \times S_T \times \{*\} \times \ldots$ is regarded as a technique followed by a task which is solved by the technique followed by no explanation of why the technique works. We use * to express absence of activity in the second, third, sixth, seventh and eighth components, and we use 0 (respectively, 1) to express absence (respectively, presence) of activity in the fourth and fifth components. We can add some axioms devoted to prevent us from considering impossible events, for example:

Axiom 1: There are not events starting with a task and continuing with a logos element.

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[10] Actually, the last two components should be interpreted as evaluations of a technique in comparison with another technique. Indeed, we typically speak of a technique as being more or less economic or reliable than another technique. However, for the sake of simplicity, we do not take into account this aspect here.

Axiom 2: If an event starts with a technique, then it cannot continue with a logos element.

Axiom 3: In an event there are not two tasks, two techniques or two logos elements. Thus, for instance, there are not events which are elements of $S_T \times S_T \times \dots$

Axiom 4: If in an event there is no task, then the last three components of the event are (*, *, *).

Axiom 5: If the fourth component of an event is *, then the last three components are (*, *, *).

Examples of specialization: Imagine we create a new theory-element T_4 by adding the following law to the theory-element T_3:

Law: The last three components of every event are (*, *, *).

The new theory-element T_4 is a specialization of T_3. Indeed, there are actual models in T_3 which are not actual models in T_4, namely, those study sequences having at least an event in which the last three components are not (*, *, *). After the axioms, it is clear that the former law holds for those study sequences in which each event $s_i = (s_{i1}, s_{i2}, s_{i3}, s_{i4}, s_{i5}, s_{i6}, s_{i7}, s_{i8})$ satisfies that none of the s_{ij} are a task or that $s_{i4} = *$. Hence, those study sequences would be actual models of our theory-element T_4.

The notion of *didactic contract* (Brousseau, 1996) is a good source of laws for theory-elements dealing with study sequences. Indeed, a didactic contract can be regarded as a special family of *clauses* or *conventions*, and, inspired in Lewis (1969), we could express a convention as a law stating that a certain regularity in the events of a study sequence holds (see for instance the law above).

Remark: In (Chevallard, 1988b) there is a sketch of the possible sets and relations of the structures an anthropological theory of the didactics would deal with. It would be interesting to compare them with the ones used in our examples above.

Remark: Brousseau (1986), inspired among others by (Suppes, 1969, 1976)[11], used finite automata to give a structuralist formulation of the notion of *situation*. Our structuralist formulations of notions of the ATD are more in the spirit of the Stimulus-Sampling Theory (Estes et al., 1959). It is worth noting that, as proved in (Suppes, 1969), given any finite connected automaton there is a stimulus-response model that asymptotically becomes isomorphic to it.

Finally, let us consider the relation of *reduction* between theory-elements.

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[11] It is remarkable the fact that Suppes was the main promoter of the semantic conception, direct precedent of the structuralist conception.

Definition 6: A theory-element T is *reducible* to a theory-element T* if there exists a relation $\rho \subseteq M_p(T) \times M_p(T^*)$ such that:

1) If $(x, x^*) \in \rho$ and $x^* \in M(T^*)$, then $x \in M(T)$.

2) If $y \in I(T) \cap r(M(T))$ then there exists $y^* \in I(T^*) \cap r(M(T^*))$ such that $(y, y^*) \in r(\rho)$.

The underlying idea is to regard the elements (x, x^*) of ρ as pairs of portions of reality so that x^* is the T*-version of x. The first condition says that the laws of T can be derived from those of T*. The second condition says that all the successful applications of T admit T*-versions which are also successful applications of T*. In other words, the successes of T can be explained in virtue of those of T*. Notice that, in contrast to what happened with the *theorization* (Definition 5), reduction does not require an increase in the conceptual map, that is to say, the kind of structures contemplated as potential models. Indeed, the conceptual map of T* might be completely different to the conceptual map T.

Examples of reduction: The *classical mathematics education* (see Gascón, 1998) explains certain phenomena with laws involving cognitive or motivational concepts. Indeed, these would be the kind of concepts used by the classical mathematics education to explain the kind of phenomena presented in (Equipe Elémentaire IREM de Grenoble, 1982). One can use (Chevallard, 1988a) to sketch how part of this classical mathematics education can be reduced to a theory-element including among the laws the clauses of the *didactic contract*. On the other hand, one can also use (Chevallard et al., 1997) to reduce part of this classical mathematics education to a theory-element with laws stating the *incompleteness* of scholar study processes (this incompleteness can be expressed in terms of the study moments, for example, by saying that *the moment of the construction of the technological-theoretical frame* or *the moment of the work of the technique* is lost).

In Bikner-Ahsbahs and Prediger (2010) the following "networking strategies" are presented: to ignore other theories (as an extreme strategy of non-connection), to make your own theory and foreign theories understandable, to compare/contrast, to coordinate/combine, to integrate locally/synthesize and to unify globally (as an extreme strategy of total connection). Next, let us clarify this taxonomy by presenting, in a brief and simplified way, possible translations of these strategies to the structuralist language:

- To ignore other theories: not to consider the possibility of (even partial) specialization, theorization or reduction (see Definitions 2, 5 and 6) as a relation among two theory-elements.

- To make your own theory and foreign theories understandable: to accomplish this, as we said at the beginning of this paper, one need to translate both theories to the

same language. What we suggest here is to use the structuralist language. So, in a sense, in the present work we take seriously this second networking strategy.

- To compare/contrast: to check which are the potential models shared by two theory-elements. Thus, when comparing/contrasting we could be performing a theorization.

- To coordinate/combine two theory-elements T and T': consists in saying that a common intended application is both an actual model of T and an actual model of T'. It is important to notice that, for this to happen, T and T' must share the partial potential models. This last sentence explains the meaning of the following statement of (Bikner-Ahsbahs & Prediger, 2010): "Whereas all theories can of course be compared or contrasted, the combination of (elements of) different theories risks becoming difficult when the theories are not compatible."

- To integrate locally/sythesize two theory-elements T and T': to find a third theory-element T" to which we can reduce the theory-element derived from T when considered just some sub-structures z of the structures x of T, as well as the theory-element derived from T' when considered just some sub-structures z' of the structures x' of T'. Notice that the structures x" of T" should admit both z and z' as sub-structures.

- To unify globally: to find a theory to which any other theory could be reduced.

CONCLUSION

Here we suggest to use the structuralist formalization of scientific theories to the benefit of the questions about the theoretical status of different approaches in Mathematics Education. Needless to say, we do not mean one can not work properly in theory unless this is formalized. For example, it is not reasonable to say that Newton were not doing Mechanics just because he did not have at hand an strict formalization. On the other hand, theories in Mathematics Education are still far from being formalizable, being this (even partial) formalization a long-term project in any case. Concerning this, it is important to point out that the degree of resistance of a theory to be formalized is inversely proportional to its degree of development. For example, if we can not distinguish the actual models among the potential models, then we can not identify any law of the theory (and at this point we should wonder whether this forces us to accept this theory is nonexistent…). Anyway, regardless of the difficulty of a complete formalization, we defend that:

- The framework offered by the structuralist conception of scientific theories is illuminating to the extend that it provides us with high order tools which allow a better understanding of the theoretical scene in Mathematics Education.

- Even if we were not interested in networking theories, the attempt to formalize a theory in the structuralist way forces us to consider extremely interesting questions

about this theory. For instance: which are the underlying structures?, which are the laws?, which are the theoretical relations?,…

Among many other things, it is still an open question which are the links between our structuralist approach, the definition of *theory* by Radford (2008) and the notion of *research praxeology* by Artigue et al. (2011a, 2011b).

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POSTERS

Inferentialism in mathematics education: Describing and analysing students´ moves in sorting geometrical objects

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This poster presents a language for describing and analysing students' language "moves" while reasoning in an open-ended sorting activity. A close analysis of one individual student's language moves in a collaborative activity is supposed to shed light on individual contribution to the collaborative reasoning process. Furthermore, these moves give indications on what pupils decided to be relevant in the simultaneous enterprise of reasoning in collaboration, and even the prior knowledge available in the classroom.

Keywords: inferentialism, deontic scorekeeping, mathematical reasoning, collaboration.

THEORETICAL FRAMEWORK

The purpose of this poster is to present an inferentialist language for describing reasoning in terms of moves in language game. Inferentialism is introduced by Robert Brandom (1994), and it advocates a new order of explanations. Inference is to prioritize over reference or representation, and it is set at the heart of human knowing. Inferentialism identifies the meaning of an expression by its inferenetial relationship to other expressions. Brandom (2000) stated:

> To grasp or understand […] a concept is to have practical mastery over the inferences it is involved in – to know, in the practical sense of being able to distinguish (a kind of know-how), what follows from the applicability of a concept, and what it follows from. (Brandom, 2000, p. 48, his italics)

The introduction of inferentialism to mathematics education research is recent. Nevertheless, ideas based on inferentialism have already been used in different ways in mathematics education research. Schindler and Hußmann (2009) used the status of claims (commitments and entitlements) to investigate 6th grade students' individual learning process and concepts formation in the topic of negative numbers. Bakker and Derry (2011) draw upon inferentialist epistemology, to design tasks in teaching statistics inferences. Based on inferentialism, this poster will present a language to describe and analyze young learners' collaborative mathematical reasoning.

Geometrical objects (2D) of different sizes and shapes were presented to groups of four first grade young learners (6-7 year olds) by the teacher. They were challenged to collaboratively come to an agreement on ways of sorting. The open-ended aspect of this task creates favorable conditions for a fruitful game of giving and asking for reasons.

Brandom (1994; 2000), used the term deontic scorekeeping to name a process embedded in the game of giving and asking for reasons. During this process, different competent interlocutors keep track of their own and others' linguistic performance. It describes the course by which different interlocutors converge toward the same meaning, in search of agreement or/and objectivity. It comprises both collaboration and the reasoning. The analysis of the deontic scorekeeping, especially the "moves" will serve as tool to characterize and analyze young learners' collaborative mathematical reasoning.

The "moves" express how claims are put forward: *Attributing*, *acknowledging* and *undertaking*. Attributing is just a kind of reporting, and it does not indicate an understanding or knowing. Acknowledging is to take something to be true. Undertaking a claim is to be aware of the premises and consequences of it. The moves are interrelated and depend on each other. For instance undertaking a commitment is something that makes it appropriate for others to attribute it (Brandom 2000).

I believe the "moves" could also show signs of participating norms/rules in a classroom if they are investigated with appropriate quest. To illustrate the analytical points of the proposed theoretical framework, the poster presents video recorded data from a Swedish classroom

POSTER DESIGN

The poster will be designed in three columns. The first presents excerpts to illustrate the three moves and their significance on individuals' learning trajectory. The second column will conceptualize how these moves are interrelated in the deontic scorekeeping. The last column will draw on possible implications of the previous columns for teaching and learning in classrooms.

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CROSSROADS OF PHENOMENOLOGY AND ACTIVITY THEORY IN THE STUDY OF THE NUMBER LINE PERCEPTION

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The problem of the development of perception is investigated in mathematics education by Luis Radford from a Marxist perspective (Radford, 2010). Radford supposes that only through social practice the "domestication" of a perceptive organ (an eye) can occur, and that the phenomenological approach towards perception as a system of intentional acts cannot explain acquiring of the new, culturally specific ways to approach objects. Our research shows the productivity of a dual analysis of the same phenomenon from the cultural-historical activity theory (CHAT) and the phenomenological perspectives.

We analysed the eye-movements of participants while they operated with the number line. The SMI RED eye-tracker was used with sample rate 120 Hz. The task was to answer on which point on the number line from 0 to 10 the grasshopper was sitting. There were 6 pairs of adults and pre-schoolers. We consequently collected three sorts of data, concerning adults' perception, the interaction between children and parents when we asked each parent to teach her/his child, and the children's perception. The results derive from the detailed qualitative analysis and they will be presented as several series of pictures, which represent synchronized data of short time interval (0,5-5 seconds) tracks of eye-movements, speech oscillograms, audio transcriptions and pictures of gestures, taken by an external camera. Here is a space-saving summary of interweaving ways to address results from the two perspectives.

(1) Adults' perception and teaching strategies revealed a vivid difference in how the parents detected the point on the number line themselves, and how they taught the children to do it. All adults either immediately grasped the answer by one fixation, or by a couple of fixations while they were counting from the midpoint or the last point. While teaching, most of our adults showed the child a strategy of counting from zero up to the point, making arc movements with their finger from point to point, and rhythmically counting or making pauses for the child to count. From the CHAT perspective that focuses on cultural practices and artefacts (Vygotsky, 1981) we need to interpret the adults' kind of perception as *mediated* by previous knowledge and by the number line itself—which is a visual semiotic means that has sedimented the activity of counting. We can judge the way that adults performed counting as a developed perception that has a form of mental action that keeps only a general form of real action and misses the intermediate parts (Davydov, 1959). Following the Husserlian phenomenology by adopting the first person perspective in our analysis, we see the adults' perception as *immediate,* where a number line is "taken for

granted" (Husserl, 2001), i.e. approached as a natural object of their living world without treating it as a product of previous mathematical work.

(2) In the teaching stage the attention of the children was strongly driven by a system of means used by the adult. Each pointing movement of the finger of the adult corresponded to a prosodic stress and to a fixation of the child's glance on a point on the number line; the adult made a theoretical perception possible through involving the child into *social practice*, "through an intense recourse to pointing gestures, words, and rhythm", as Radford (2010, p. 5) puts it. Now let us look more closely at the children's eye-movements. In one case the child followed the adult's movements precisely in counting from zero up to ten, but at the moment when her finger was passing the point where the grasshopper was sitting, the child found time to look at the written question of the task. In another example a child misperceived an adult's gestures: she was moving her eyes one point ahead of where the adult was pointing. But being in contact with the original task she managed to correct herself at the end of the counting: she was making two fixations on the same point where the grasshopper was, corresponding to two separate arc movements of her father's finger. Thus, being "moved" by the adult activity the children caught up the meaning of it through the goal, which was retained at a grounding level of the children intentionality. So, our data show that it is exactly a complex system of *intentionality* that makes possible the "crucial form of communication in which two consciousness meet in front of the cultural mathematical meaning" (Radford, 2010, p. 6) within the social practice. From a phenomenological perspective this meaning should not be perceived as a "ready-made entity" which a child is expected to follow; instead, only a serious and genuine move back to the intentional origins of this meaning gives us a real understanding of its constitutive potential (Husserl, 1970, 2001).

Conclusions. CHAT analysis focuses on cultural means and social practices, which necessarily mediate the transformation of perception, while phenomenological analysis aims towards understanding of the role of intentionality in acquiring new forms of immediate perception. Hence these two perspectives attempt to grasp two important aspects of the educational/learning complexity, and seem to be neither contradictory nor reducible to each other, but rather essentially complementary.

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