Propagation inside a bianisotropic waveguide as an evolution problem

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PROPAGATION INSIDE A BIANISOTROPIC WAVEGUIDE
AS AN EVOLUTION PROBLEM

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Abstract

The free source Maxwell system for the bianisotropic medium, in a fixed frequency \( \omega \geq 0 \) and with time convention \( e^{-i\omega t} \), is represented by the equation

\[
\nabla \times \mathbf{J} e = i\omega \mathbf{Me}
\]

where \( \mathbf{e} := (\mathbf{E}, \mathbf{H})^T \) is the electromagnetic (E/M) field; it is defined in a domain \( \Omega \subset \mathbb{R}^3 \), depend on \( \omega \) and take values in \( \mathbb{C}^6 \). We denote

\[
\mathbf{J} := \begin{bmatrix} 0 & -I_3 \\ I_3 & 0 \end{bmatrix}
\]

The matrix

\[
\mathbf{M} := \begin{bmatrix} \varepsilon & \xi \\ \zeta & \mu \end{bmatrix}
\]

characterizes the medium inside \( \Omega \) and its entries are complex functions of the frequency \( \omega \) and the position \( \mathbf{r} \in \Omega \). The Gauss law implies that

\[
\nabla \cdot \mathbf{Me} = 0
\]

Assume that the boundary \( \Gamma := \partial \Omega \) is smooth enough; usually Lipschitz is sufficient for most of the applications. Let \( \hat{n} \) be the exterior normal to \( \Gamma \). For a wide class of boundaries, metallic for example, the perfect electric conductor (PEC) boundary condition for the electric field, \( \hat{n} \times \mathbf{E} = 0 \) on \( \Gamma \), applies.

Let now \( \mathbf{A} = (A_x, A_y, A_z)^T \) be a vector field in \( \Omega \); it can be represented as \( \mathbf{A} = (\mathbf{A}_\perp, A_z)^T \) where \( \mathbf{A}_\perp := A_x \hat{x} + A_y \hat{y} \) is the transverse and \( A_z \) the longitudinal part. It is easily seen that the curl operator reads

\[
\nabla \times \mathbf{A} = \begin{bmatrix} W & 0 \\ 0 & 0 \end{bmatrix} \partial_z \left( \begin{array}{c} \mathbf{A}_\perp \\ A_z \end{array} \right) - \begin{bmatrix} 0 & W \nabla_\perp \\ \nabla_\perp \cdot W & 0 \end{bmatrix} \left( \begin{array}{c} \mathbf{A}_\perp \\ A_z \end{array} \right)
\]

where

\[
W := \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \hat{z} \times I_3
\]

and \( \nabla_\perp := \partial_x \hat{x} + \partial_y \hat{y} \) is the formal transverse gradient.

An infinite waveguide is a cylinder

\[
\Omega = \Omega_\perp \times \mathbb{R}
\]

where \( \Omega_\perp \subset \mathbb{R}^2 \) is a domain with \( \Gamma_\perp \). Observe that the wall of the waveguide is \( \Gamma = \Gamma_\perp \times \mathbb{R} \) and \( \hat{n} \) coincides with its transverse part and is the exterior normal
to $\Gamma_\perp$, whereas $\mathbf{\hat{t}} := W \mathbf{\hat{\nu}}$ is the tangent vector. The PEC boundary condition now reads

$$(0.4) \quad \mathbf{\hat{t}} \cdot \mathbf{E}_\perp = 0, \quad E_z = 0 \text{ on } \Gamma$$

The fact that the longitudinal variable $z$ runs $\mathbb{R}$ allows us to formulate the Maxwell system as an evolution equation with respect to this variable. Indeed, letting

$$C := \begin{bmatrix} 0 & W \nabla \perp \\ \nabla \perp \cdot W & 0 \end{bmatrix}$$

the Maxwell system is written

$$\partial_z \mathbf{v} = (A_0 + i\omega \mathbf{M}) \mathbf{e}$$

where $\mathbf{V} := \mathbf{\hat{z}} \times \mathbf{J}$ and $A_0 := C \mathbf{J}$. Define now a Hilbert space $X$ of functions of the transverse variables and consider the E/M field $\mathbf{e}$ as vector-valued a function

$$\mathbf{e} : \mathbb{R} \ni z \mapsto e(\cdot, \cdot, z) \in X$$

Then $A_0$ can be realized as an unbounded operator in $X$ and the PEC conditions are incorporated in the domain of $A_0$. Actually, if we separate $u \in X$ into “electric” and “magnetic” part

$$u := \begin{pmatrix} u^e \\ u^h \end{pmatrix},$$

then $A_0$ is given explicitly by

$$(0.5) \quad A_0 u = \begin{pmatrix} -W \nabla \perp u^h \\ -\nabla \perp \cdot W u^h \\ W \nabla \perp u^e \\ \nabla \perp \cdot W u^e \end{pmatrix}$$

The first step is to prove that $A_0$ is the generator of a strongly continuous group in $X$. The second is to realize Maxwell system as a perturbed abstract degenerate evolution problem

$$(0.6) \quad \mathbf{v} e'(z) = (A_0 + i\omega \mathbf{M}(\omega))\mathbf{e}(z)$$

and apply relevant perturbation arguments in order to establish well-posedness. The research presented here implements exactly this program.