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Multiple scattering by a collection of randomly located obstacles
Part II: Numerical implementation — coherent fields

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Abstract

A numerical implementation of a rigorous theory to analyze scattering by randomly located obstacles is presented. In general, the obstacles can be of quite arbitrary shape, but, in this first implementation, the obstacles are dielectric spheres. The coherent part of the reflected and transmitted intensity at normal incidence is treated. Excellent agreement with numerical results found in the literature of the effective wave number is obtained. Moreover, comparisons with the results of the Bouguer-Beer law (B-B) are made. The present theory also gives a small reflected coherent field, which is not predicted by the Bouguer-Beer law, and these results are discussed in some detail.

1 Introduction

Electromagnetic scattering by randomly located objects are frequently encountered in science. It is an important issue in terrestrial and atmospheric research, biomedical and life sciences, astrophysics, nanotechnology, just to mention a few. The literature is comprehensive, and we refer to the textbook literature and references therein, see e.g., [6, 7, 15, 16, 21-24] for a survey of the field.

The literature contains several methods of computing the effective wave number \( k_{\text{eff}} \) for a half space containing a collection of random spheres, see e.g., [17-19, 25, 26] and [21, Chapter 6], and references therein. The effective wave number is obtained by solving a determinant relation and there are in general many solutions. The new method presented in Part I, [11], does not suffer from these deficiencies and is able to compute the coherent transmitted and reflected fields from an infinite slab containing random scatterers. In this paper results are presented for slabs with different thicknesses and spherical scatterers with the relative permittivity \( \epsilon_r = 1.33^2 \) which corresponds to fresh water at optical frequencies. Both the electrical size of the spheres and the volume fraction have been varied.

2 Theory

The theory of electromagnetic scattering by an ensemble of finite scatterers is reviewed in [6, 7, 13, 15, 16, 21-23].

The underlying theoretical treatment of the problem handled in this paper is presented in detail in Kristensson [11]. The purpose of this section is to review and highlight some of the more important steps in the theory. For a more complete reference we refer to Kristensson [11].

We simplify the theoretical results in [11] to a geometry of a slab \((z \in [0, d])\) and to spherical scatterers of radius \(a\) (dielectric or perfectly conducting). These assumptions simplifies the results considerably simpler, and make the numerical implementation less demanding. The geometry is depicted in Figure 1. Notice that the domain of possible locations of local origins, \([z_0, z_d]\), which defines the domain \(V_s\), is slightly smaller than the extent of the slab, i.e., the interval \([z_0, z_d] = [a, d-a]\). Vectors are denoted in italic boldface, and matrices in roman boldface. A caret over
Figure 1: The geometry of the stratified scattering region. The yellow region denotes the region $V_s$, which is the domain of possible locations of local origins, i.e., the interval $[z_0, z_d]$. A vector denotes a vector of unit length. In this paper, we adopt the multi-index notation $n = \tau \sigma ml$, where the integer indices $\tau = 1, 2, l = 1, 2, 3, \ldots, m = 0, 1, \ldots, l$, and $\sigma = e,o$ (even and odd in the azimuthal angle).

Assume the incident field on the slab is

$$E_i(z) = E_0 e^{ik_0z}$$

The coherent part of the total electric field on either side of the slab is

$$\langle E \rangle (z) = E_t e^{ik_0z}, \quad z > d \quad \text{and} \quad \langle E \rangle (z) = E_0 e^{ik_0z} + E_r e^{-ik_0z}, \quad z < 0$$

where the reflected and transmitted amplitudes, $E_t$ and $E_r$, respectively, are given as

$$E_t = E_0 + \frac{2\pi n_0}{k_0^2} \sum_{l=1}^{\infty} i^{-l} \sqrt{\frac{2l + 1}{8\pi}} \left( \hat{x} \int_{z_0}^{z_d} e^{-ik_0z'} \langle f_{10l1} \rangle (z') + i \langle f_{20l1} \rangle (z') \right) dz' 
- \hat{y} \int_{z_0}^{z_d} e^{-ik_0z'} \langle f_{11l1} \rangle (z') - i \langle f_{21l1} \rangle (z') \right) dz' \right) \tag{2.1}$$
and

\[ E_t = \frac{2\pi n_0}{k_0^2} \sum_{l=1}^{\infty} i^l \sqrt{\frac{2l + 1}{8\pi}} \left( \hat{\mathbf{x}} \int_{z_0}^{z_d} e^{ik_0z'} (<f_{1\alpha l} > (z') - i <f_{2\alpha l} > (z')) \, dz' - \hat{\mathbf{y}} \int_{z_0}^{z_d} e^{ik_0z'} (<f_{1\alpha l} > (z') + i <f_{2\alpha l} > (z')) \, dz' \right) \quad (2.2) \]

in terms of the number density \( n_0 \) and the transmission matrix \( T_{nn'} \) of the scatterers.

The unknown coefficients \( <f_n > (z) \) are the solution to a system of linear, one-dimensional integral equations in \( z \), viz.,

\[ <f_n > (z) = e^{ik_0z} \sum_{n'} T_{nn'} a_{n'} + k_0 \int_{z_0}^{z_d} \sum_{n'} K_{nn'} (z - z') <f_{n'} > (z') \, dz', \quad z \in [z_0, z_d] \quad (2.3) \]

where the kernel \( K_{nn'} (z) \) can be expressed in terms of spherical waves [11, 12]. The explicit form of the kernel \( K_{nn'} \) is \( (\rho = x \hat{\mathbf{x}} + y \hat{\mathbf{y}}) \)

\[ K_{nn'} (z) = \frac{n_0}{k_0} \sum_{n''} T_{nn''} \int_{\mathbb{R}^2} g(|\rho - z\hat{\mathbf{z}}|) \mathcal{P}_{n''}(k_0|\rho - z\hat{\mathbf{z}}|) \, dx \, dy, \quad |z| < z_d - z_0 \]

where \( g(r) \) is the pair distribution function [3, 14, 23, 28], and \( \mathcal{P}_{nn'}(k_0d) \) is the translation matrix for the outgoing spherical vector waves [2]. The most simple pair distribution function is the hole correction (HC), \( g(r) = H(r - 2a) \), where \( H(x) \) is the Heaviside function and \( a \) is the radius of the spheres. The double integral in the definition of the kernel can be solved analytically for the hole correction in terms of a series of spherical waves [12]. More complex distribution functions, e.g., the hypernetted-chain equation, the Percus-Yevick approximation (P-YA), the self-consistent approximation, and Monte Carlo calculations are not employed in this paper [3, 14, 23, 28].

The spherical scatterers are completely characterized by the transition matrix \( T_{nn'} \), which for a spherical scatterer is diagonal in all its indices. The coefficients \( a_n \) are the expansion coefficients of the incident plane wave in spherical vector waves. If the incident direction is along the positive \( z \)-direction, i.e., \( \hat{k}_i = \hat{\mathbf{z}} \), these are (\( \sigma = e \) is the upper line, and \( \sigma = o \) is the lower line)

\[
\begin{align*}
    a_{1\sigma ml} &= -i^l \delta_{m1} \sqrt{2\pi(2l + 1)} \left( \hat{\mathbf{z}} \times \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \right) \cdot E_0 \\
    a_{2\sigma ml} &= -i^{l+1} \delta_{m1} \sqrt{2\pi(2l + 1)} \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} \cdot E_0 
\end{align*}
\]

where the vector \( E_0 \) denotes the polarization state in the \( x-y \) plane.

Equation (2.4) defines the complex valued transmission and reflection coefficients that maps the incident field to the transmitted and reflected fields, i.e.,

\[ E_t = tE_0, \quad E_r = rE_0 \quad (2.4) \]
respectively.

The transmittivity $T$ and the reflectivity $R$ of the slab is given by

$$T = \frac{|E_t|^2}{|E_0|^2}, \quad R = \frac{|E_r|^2}{|E_0|^2} \quad (2.5)$$

3 Numerical implementation

To compute the reflection and the transmission coefficients of the slab, we need to solve (2.3) for given geometrical and material data. The equation is a linear system of Fredholm integral equations of the second kind [5]. The unknown quantity, $<f_n>$ ($z$), is evaluated at equally spaced points, $z = z_1, z_2, \ldots, z_p$, in the interval $[z_0, z_d]$, and the integral in (2.3) is evaluated by the use of Simpson’s quadrature at the points of discretization. The spatially discretized vector $<f_n>$ is denoted $F$. Remembering that $n$ is a multi-index of $n = \tau\sigma ml$, the entries of the vector are written as

$$F = \left( <f_{1e01}>(z_1) \cdots <f_{1e01}>(z_p) \cdots <f_{2om_{\max}l_{\max}}>(z_1) \cdots <f_{2om_{\max}l_{\max}}>(z_p) \right)^t$$

where $m_{\max} \leq l_{\max}$. The discretized system has an overall linear dimension of $N = 4(m_{\max} + 1)l_{\max}p$, and the underlying integral equation in (2.3) discretized as

$$F = P + kB \cdot F \Leftrightarrow (I - kB) \cdot F = P \quad (3.1)$$

where $I$ is the identity matrix, and the elements of matrix $B$, $B_{nn'}$, are matrices given by (3.2) which are the Simpson weighted discretized kernel in (2.3) for $n$ and $n'$ respectively. The integration variable is discretized at the same points as the left hand side and ordered the same way as the discrete vector $F$.

$$B_{nn'} = \begin{pmatrix}
  w_1K_{nn'}(0) & w_2K_{nn'}(z_1 - z_2) & \cdots & w_pK_{nn'}(z_1 - z_p) \\
  w_1K_{nn'}(z_2 - z_1) & w_2K_{nn'}(0) & \cdots & w_pK_{nn'}(z_2 - z_p) \\
  \vdots & \vdots & \ddots & \vdots \\
  w_1K_{nn'}(z_p - z_1) & w_2K_{nn'}(z_p - z_2) & \cdots & w_pK_{nn'}(0)
\end{pmatrix} \quad (3.2)$$

where $w_i$, $i = 1, 2, \ldots, p$, are the ordinary Simpson weights for numerical integration. The discretization of the single scattering contribution defines the vector $P$ in the same format as $F$ with vector elements given by

$$P_n = e^{ik_0z_i} \sum_{n'} T_{nn'}a_{n'} \quad i = 1, 2, \ldots, p,$$

We solve for the unknown vector $F$ by the solution of a linear system of equations in MATLAB. The transmitted and reflected fields are then found by using (2.1) and (2.2) respectively.
3.1 Computations of the effective wave number \( k_{\text{eff}} \)

The transmission coefficient \( t_h \) for a normally incident plane wave onto a non-magnetic, \( i.e., \mu = 1 \), homogeneous slab of thickness \( d \) and wave number \( k \) is \([10]\)

\[
t_h(k) = \frac{(1 - \Gamma_h^2)e^{i(k-k_0)d}}{1 - \Gamma_h^2e^{2ikd}} \tag{3.3}
\]

where \( \Gamma_h = (k_0 - k)/(k_0 + k) \) and \( k_0 = \omega/c_0 \) is the wave number in vacuum.

To compute the effective wave number \( k_{\text{eff}} \) of the slab containing randomly distributed non-magnetic scatterers we find the zeros of the function \( G(k) = t - t_h(k) \) where \( t \) is computed by using (2.4) and \( t_h \) is given by (3.3). That is, the effective wave number, \( k_{\text{eff}} \), satisfies \( G(k_{\text{eff}}) = 0 \).

To find the complex roots \( k_i \) of \( G(k) \) in a given domain \( \Omega \) in the complex plane we employ the method described in Theorem A.1 in Appendix A. The area \( \Omega \) is subdivided into \( n \) sufficiently small rectangular domains, \( \Omega_q \), with boundaries \( \gamma_q \). For each \( \gamma_q \), the expression (A.1) is calculated using the midpoint rule. If \( k_i \in \Omega_q \) a test is made with a smaller contour to ensure that \( k_i \) is the single root inside \( \gamma_q \). The process is repeated for every \( \gamma_q \in \Omega \), and then for every frequency. Among the available roots, the root closest to the solution at the previous frequency is chosen. For convenience, \( \Omega \) was restricted to the region \( 0 \leq \text{Im} \Omega \leq \sqrt{0.1k_0}, \sqrt{0.99k_0} \leq \text{Re} \Omega \leq \sqrt{2k_0} \) in our application.

4 Results

The transmittivity using equation (2.5) is compared with the transmittivity computed by using Bouguer-Beer law, which is given below in (4.1).

The radiative transfer equation (RTE) is frequently used to infer the coherent and diffuse intensities of scattering by random scatterers in a slab geometry \([6]\). The coherent contribution in RTE is Bouguer-Beer law, which specifies the drop in the coherent intensity \( I_c(z) \), due to scattering and absorption in the material. The explicit form of the law is \([6]\)

\[
I_c(z) = I_c(z_0)e^{-n_0\sigma_{\text{ext}}(z-z_0)} \tag{4.1}
\]

where \( \sigma_{\text{ext}} \) is the extinction cross section of the spheres \([8]\).

Slabs with different thicknesses and different electrical size of the scatterers, which are assumed to be spheres with \( \epsilon_r = 1.33^2 \), corresponding to rain drops at optical frequencies, are studied. We also investigate how the volume fraction of scatters, \( f \), affects the results. The volume fraction is kept small so that the hole correction is assumed to be an accurate model.

We also compare the method with the results of \([21, \text{Chapter 6}]\). This approach uses the same underlying theory — translations of spherical vector waves — to obtain a relation for the expansion coefficients of the internal fields of the scatterer. A half-space geometry is employed and the expansion coefficients are assumed to have the form \( a_n e^{ik_{\text{eff}}z} \). This leads to an infinite set of equations, and the effective
wave number, $k_{\text{eff}}$, is found by a determinant relation. To generate results from [21, Chapter 6] a MATLAB code was downloaded from [20]. For comparison reasons, the Percus-Yevick pair distribution function was modified to the hole distribution function, by changing Eq. (6.1.60) in [21, Chapter 6]. Even if the method in [21] and the one presented in this paper are based on the same underlying principles, the analysis diverges, and the agreement therefore is an accurate verification of the method in [21], which uses the assumption mentioned above. In addition, our method predicts the reflection properties of the slab as shown in Figures 9 and 10.

4.1 Computation parameters

The maximum number of terms included in the expansion is determined by the index $l$ which is denoted $l_{\text{max}}$. In all computations $l_{\text{max}} = 12$. This parameter may be set lower for smaller $k_0a$. The Wiscombe criterion states that $l_{\text{max}} = 20$ for $k_0a = 10$ [29]. However, for our application a convergence study shows that $l_{\text{max}} = 9$ is sufficient. The spatial discretization is varied depending on the slab thickness and on $k_0a$. The number of spatial discretization points, $z_p$, is kept such that, $\Delta z = z_{i+1} - z_i$, is smaller than $\lambda/3$, where $\lambda$ is the vacuum wavelength. The index $m$ is fixed and takes the value $m = 1$, due to the excitation and the properties of the transition matrix of a spherical object. This means that in practice the system (2.3) has $4l_{\text{max}}p$ number of unknowns to be solved for.

4.2 Transmittivity as function of $k_0a$ and volume fraction $f$

We compare the transmittivity defined in (2.5) with the transmittivity computed with Bouguer-Beer law (B-B) (4.1) for a slab with thickness $d/a = 100$ consisting of non-magnetic dielectric spheres of radii $a$ and $\epsilon_r = 1.33^2$ as a function of $k_0a$. In B-B, the transmittivity is $T = I_c(d)/I_c(0)$. Two different volume fractions are used, $f = 0.01$ and $f = 0.1$, see Figures 2 and 4, respectively.

In Figure 2 we notice that a very good agreement is achieved between data obtained using the Bouguer-Beer law and the presented method for $f = 0.01$. We also get a very good agreement for $f = 0.1$ when $k_0a < 1$ (see Figure 4). For larger $k_0a$ the agreement in a relative measure is less good, but in absolute values the difference is negligible since both methods predict extremely low transmittivity. In both cases it is noted that the transmittivity has a global minimum, in the studied frequency interval, for $k_0a \approx 6$. The increase in transmittivity at larger $k_0a$ is due to the fact that the extinction cross section $\sigma_{\text{ext}}$ decreases, see Figure 3. This means that the spheres scatter less, and, hence, the coherent transmittivity increases.

In Figure 5 the transmittivity $T$ is plotted as a function of the volume fraction of the scatterers $f$ at $k_0a = 10$. We note that very good agreement between the two methods is achieved for small $f$ and then the curves start to deviate. A possible explanation to the discrepancies between the Bouguer-Beer law and the proposed method at higher volume fractions, $f$, is that farfield criterion is assumed between the scatterers in the Bouguer-Beer law. At lower concentrations this assumption is more valid, hence the Bouguer-Beer law show better agreement with our method.
Figure 2: The transmittivity $T$ (coherent part) in log scale as a function of the electrical size $k_0a$ for a slab of thickness $d/a = 100$ and constant volume fraction $f = 0.01$ consisting of dielectric spheres of radii $a$. The material parameters of the spheres are $\epsilon_r = 1.33^2$ and $\mu_r = 1$. The dashed line is the result obtained by the Bouguer-Beer law (B-B).

Figure 3: The extinction cross section $\sigma_{\text{ext}}$ as a function of the electrical size $k_0a$ for a single dielectric sphere of radius $a$. The material parameters of the sphere are $\epsilon_r = 1.33^2$ and $\mu_r = 1$. 
Figure 4: The transmittivity $T$ (coherent part) in log scale as a function of the electrical size $k_0a$ for a slab of thickness $d/a = 100$ and constant volume fraction $f = 0.1$ consisting of dielectric spheres of radii $a$. The material parameters of the spheres are $\epsilon_r = 1.33^2$ and $\mu_r = 1$. The dashed line is the result obtained by the Bouguer-Beer law (B-B).

Figure 5: The transmittivity $T$ (coherent part) in log scale as a function of the volume fraction $f$ for a slab of thickness $d/a = 100$ and constant electrical size $k_0a = 10$ consisting of dielectric spheres of radii $a$. The material parameters of the spheres are $\epsilon_r = 1.33^2$ and $\mu_r = 1$. The dashed line is the result obtained by the Bouguer-Beer law (B-B).
Figure 6: The components of the complex-valued transmission coefficient, $t(k_0a)$, in the complex plane as a function of the electrical size $k_0a$ for a slab of thickness $d/a = 100$ and constant volume fraction $f = 0.01$ (black curve) and $f = 0.1$ (green curve) consisting of dielectric spheres of radii $a$. The material parameters are $\epsilon_r = 1.33^2$ and $\mu_r = 1$.

Moreover, no boundary effects are present in the Bouguer-Beer law. In Figure 6 the real and imaginary part of transmission coefficient is plotted for $f = 0.01$ and $f = 0.1$ in the complex $t$ plane with $k_0a$ as a parameter along the curves.

4.3 Computations of the effective wave number $k_{\text{eff}}$

The effective wave number $k_{\text{eff}}$ is calculated using the transmission coefficient for the coherent field, as described in Section 3.1, for three different slab-thicknesses $d = 100a$, $d = 50a$, and $d = 10a$, respectively. In Figures 7 and 8, we compare our results with the results of [21]. The wave number $k_{\text{eff}}$ has been normalized with the wave number of vacuum, $k_0$. We notice that $k_{\text{eff}}$ for $d = 100a$, $d = 50a$, and $d = 10a$, computed with the present method, and the results given in [21], agree well. We also note that the imaginary part of $k_{\text{eff}}$ is small for all $k_0a$, and that the real part of $k_{\text{eff}} < 1$ for $k_0a > 6.6$, and that this holds for the coherent field only. The propagation properties for the incoherent field is not considered in this study.
Figure 7: The real component of the scaled effective wave number, $k_{\text{eff}}/k_0$, as a function of the electrical size $k_0a$ for a slab of thickness $d/a = 100, 50, 10$ (red, blue and green circles and curves, respectively) and constant volume fraction $f = 0.01$ consisting of dielectric spheres of radii $a$. The material parameters are $\epsilon_r = 1.33^2$ and $\mu_r = 1$. The crosses and dashed curve are the result by [21].

Figure 8: The imaginary component of the scaled effective wave number, $k_{\text{eff}}/k_0$, as a function of the electrical size $k_0a$ for a slab of thickness $d/a = 100, 50, 10$ (red, blue and green circles and curves, respectively) and constant volume fraction $f = 0.01$ consisting of dielectric spheres of radii $a$. The material parameters are $\epsilon_r = 1.33^2$ and $\mu_r = 1$. The crosses and dashed curve are the result by [21].
Figure 9: Reflectivity $R$ (coherent part) v.s. reflectivity of a homogeneous slab (dashed) with $R$ in log scale as a function of the frequency for a slab of thickness $d/a = 10$ and constant volume fraction $f = 0.01$ consisting of dielectric spheres of radii $a$. The material parameters of the spheres are $\epsilon_r = 1.33^2$ and $\mu_r = 1$. The dashed line is the result obtained by reflection by a homogeneous slab.

Figure 10: Reflectivity $R$ (coherent part) v.s. reflectivity of a homogeneous (dashed) with $R$ in log scale as a function of the frequency for a slab of thickness $d/a = 50$ and constant volume fraction $f = 0.01$ consisting of dielectric spheres of radii $a$. The material parameters of the spheres are $\epsilon_r = 1.33^2$ and $\mu_r = 1$. The dashed line is the result obtained by reflection by a homogeneous slab.
4.4 Reflectivity as function of $k_0a$

The reflection coefficient for a homogeneous slab of thickness $d$ and wave number $k$ is given by [10]

$$r_h(k) = \frac{\Gamma_h}{1 - \Gamma_h^2 e^{2ikd}}$$  \hspace{1cm} (4.2)

where $\Gamma_h$ is given in (3.3).

Figures 9 and 10 show the reflectivity, $R$, see (2.5), for a volume fraction $f = 0.01$, with slab thickness $d/a = 10$ and $d/a = 50$, respectively. The result is compared with the reflectivity, computed with (4.2), of a homogeneous slab with wave number $k_{\text{eff}}$ obtained in Section 4.3. We obtain good agreement between the two ways of computing reflection data for small $k_0a$. This means that $k_{\text{eff}}$ is a solution to both $G(k) = 0$, see Section 3.1 for definition, and $r - r_h(k) = 0$ when $k_0a$ is small. For these parameter values, when the wavelength of the electromagnetic field is large compared with the size of the scatterers, classical homogenization methods hold (e.g., see [1, 4, 9, 27]). However, at higher frequencies the deviation between the two curves is significant, both in amplitude and periodicity. The deviation is larger for the thick slab, Figure 10, than for the thin slab in Figure 9. This last remark implies that the interference effects due to the thickness of the slab, at least for the thin slab, are the same in the solid slab and the slab of random scatterers. Note that the local extrema are somewhat shifted for the thicker slab (Figure 10) which we conjecture is due to increased damping of the wave in this case.

We believe that the deviations between the curves in Figure 9 and 10 represent a measure of the accuracy of the homogenization procedure, and it provides an upper frequency limit of the validity of the use of homogenization. One possible explanation of the discrepancy between the responses is due to increased incoherent contents in the reflected field at higher frequencies.

5 Discussion and conclusions

We have presented numerical results for the method described in [11] to model the coherent reflected and transmitted field for a slab of finite thickness containing spherical randomly distributed scatterers of equal size and relative permittivity, $\epsilon_r$. The wave number for the transmitted field agrees well with the effective wave number obtained by the method given in [21, Chapter 6].

We have observed that the reflection from this slab is not consistent with the reflection from a homogeneous slab given the above mentioned effective wave number, except for low frequencies when homogenization methods apply.

We have used the hole correction which is applicable for gases. In the future another type of hole correction could be implemented. Extensions to oblique incidence is also planned.
Appendix A  Zeros and poles of an analytic function

The determination of the location of zeros of an analytic function is vital in the computations of the effective wave number in this paper. The following theorem is then useful:

**Theorem A.1.** Let $\Omega$ be an open domain in the complex $z$-plane, and let $f(z)$ be an analytic function in $\Omega$ with a simple zero at $z_0$. Then

$$z_0 = \frac{\oint_{\gamma} \frac{z\, dz}{f(z)}}{\oint_{\gamma} \frac{dz}{f(z)}}$$

where $\gamma$ is any contour that lies inside $\Omega$, and that encircles the zero $z_0$.

We give the proof of this theorem.

**Proof.** Since the zero is simple the function $f(z)$ is

$$f(z) = g(z)(z - z_0), \quad z \in \Omega$$

where $g(z)$ has no zeroes in $\Omega$. The residue theorem then gives

$$\oint_{\gamma} \frac{z\, dz}{f(z)} = 2\pi i \text{Res} \left. \frac{z}{f(z)} \right|_{z=z_0} = 2\pi i \frac{z_0}{g(z_0)}$$

Similarly,

$$\oint_{\gamma} \frac{dz}{f(z)} = 2\pi i \frac{1}{g(z_0)}$$

and the theorem is proved.

\[\square\]

**References**


