Linear Quadratic Control Design - The PhD Course 1994

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Abstract

This is the documentation for the PhD course 1994. Contains course program, session notes, lecture slides, exercises and handin problems.

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Summary

The course 1994 followed the second edition of Anderson and Moore's book "Optimal Control – Linear Quadratic Methods". The course was followed by 10 PhD-students and one MSc-student. The background of the students were a little diverse. Some had for example taken courses in matrix theory and optimization and had taught LQG-theory themselves in the digital control course, others had only the basic course(s) in control theory. However, all students succeeded very fine in covering the material.

The book was found a little boring and old fashioned. There were also problems reconstructing most of the examples in the book. There are obviously a lot of misprints and errors, some more serious than others. The most disturbing error is the statement on p. 48 were it is claimed that detectability is a necessary condition for the existence of a stable closed loop optimal system with finite performance index. That this is wrong has been known since the work of Mårtensson in the 70s (which also is referenced!). Other irritating errors are due to the authors copying examples from old papers without checking the details, such as at pp. 150-156. The choice of designs in the examples also shows a lack of engineering insight which can be disturbing for the ambitious student who checks all details. Such an example is the resonance suppression design example which starts at p. 222 and the lightly damped mode example on p. 233. (Exercise: plot the control signals). Often very central plots are missing.

To ease the reading I prepared explanatory notes to be read before each lecture (2 hours). Each lecture was followed by a problem-solving meeting (2 hours) held by Per Hagander. There were also 7 hand-in problems which were used as the exam on the course.

Material that I used:

- TFRT-7454 A collection of Matlab Routines for Control Analysis and Synthesis, by Kjell Gustafsson, Mats Lilja and Michael Lundh.
- TFRT-7456, Control Design for two Lab-processes, The Flexible Servo – The Fan and the Plate, by Kjell Gustafsson and Bo Bernhardsson.
- TFRT-7475, Discrete Time LQG with Cross-terms in the Loss Function and the Noise Description, by Kjell Gustafsson and Per Hagander.
- Note on LQ-optimal Control Using Lagrange Multipliers ny Per Hagander.
- pp. 188-192 of "Linear Robust Control" by Green and Limebeer.
- Chapter 5 of Multivariable Feedback Design, by Maciejowski.
- "Invariant Subspace Methods for the Numerical Solution of Riccati Equations", from The Riccati Equation, by Bittanti, Laub, Willems, Springer-Verlag 1990.
- "Guaranteed Margins for LQG Regulators", J Doyle, Honeywell note.

Lecture 7 on numerical methods was given by Per Hagander and Lecture 8 on singular problems by Anders Hansson. Their contributions are gratefully acknowledged.

Included in this documentation are

- Course Program
- Session Notes 1-8
- Lecture Slides 1-7 (without the figures)
- Exercise 1-7
- Handins 1-7

Lund, Dec 1994

Bo Bernhardsson

Linear Quadratic Control Theory, 1994

Lecturer

Bo Bernhardsson.

Exercises

Per Hagander.

Literature

- B. D. O. ANDERSON AND J. B. MOORE, *Optimal Control – Linear Quadratic Methods.* Prentice Hall, 1989, ISBN 0-13-638651-2.
- Notes, journal papers, Matlab manual.

Meetings

There will be one lecture and one exercise per week (2+2 hours). Participants are supposed to prepare for the lectures by reading ahead in the book and to take active part of the exercises.

Lecture Plan:

1 Introduction. Summary. Software. Design Challenges. Ch 1.
2 Full Information Problems (LQ). Finite/Infinite Time Horizon. Riccati Equations. Ch 2-3.
3 Properties of LQ regulators. Design Examples. Ch 5-6.
4 State Estimator Design. The Kalman-Bucy Filter. Ch 7.
5 The Separation Principle. Output Feedback Problems (LQG). Ch 8
6 Tracking/Servo Problems. Frequency Shaping. Ch 4,9
7 Numerical Algorithms.
8 Singular Problems.

Examination

7 Hand-in problems

Credits

Nominally 5p.

Session 1

Introduction. Norms. Formulas for the optimal LQG controller. Software. Design challenges.

The course will focus on the state-space approach to optimal linear quadratic design of linear, finite-dimensional systems. The aim is to present both theory and design applications. It is desirable that you familiarize yourself immediately with some LQG-software. I will use the LQGBOX for matlab developed by Kjell Gustafsson at the department. Other alternatives includes the mubox (only continuous time) and the control system toolbox (not recommended). Literature references below correspond to the book by Andersson and Moore if nothing else is mentioned.

It will help if you have taken a course in matrix theory, but you can struggle along without it. Part of appendices A (for instance 18-19 on Jordan forms are not needed but 20 on positive matrices should be known) and B (6,8 11, 12, 14 not necessary) give sufficient prerequisities

Reading Assignment

- pp. 1-6 + appendices A and B.

Get hold off

- TFRT-7454, "A collection of Matlab Routines for Control System Analysis and Synthesis", Gustafsson et.al. Concentrate on the LQG-BOX.
- The Flexible-servo design in TFRT-7456, "Control Design for Two Lab-processes: The Flexible Servo, The Fan and the Plate", Gustafsson and Bernhardsson.
- Internal report TFRT-7475, "Discrete Time LQG with cross-terms in the Loss Functions and Noise Descriptions", Gustafsson-Hagander.

All of the above available via mosaic.

Session 2

Hamilton-Jacobi-Bellman Equation. Full Information Problems. Finite and Infinite Time Horizon.

If you have taken the digital control course what is going to be new for you are 1) the notation 2) the continuous time results 3) the way to obtain the infinite time horizon case from the finite time horizon case.

There are many ways of showing the LQG-results. The way chosen in the text, with the Hamilton-Jacobi-Bellman equation, is not the shortest but it presents some material that is good to know. Also read App. C where Pontryagin Minimum Principle is used instead (without proofs). Remark: It is often preferable to use a $3n \times 3n$ Hermitian formulation instead of (C13) on p. 364

$$\begin{bmatrix} 0 & \frac{d}{dt}I - A & -B \\ \frac{d}{dt}I - A^T & Q_1 & Q_{12} \\ -B^T & Q_{21} & Q_2 \end{bmatrix} \begin{bmatrix} \lambda(t) \\ x(t) \\ u(t) \end{bmatrix} = 0$$

Reading Assignment

- pp. 7-60 + appendices C,D and E

When you start working with LQG-designs in Matlab it will help to read

- TFRT-7454, "A collection of Matlab Routines for Control System Analysis and Synthesis", Gustafsson et.al. Concentrate on the LQG-BOX.
- The Flexible-servo design in TFRT-7456, "Control Design for Two Lab-processes: The Flexible Servo, The Fan and the Plate", Gustafsson and Bernhardsson.

A more complete presentation of the discrete-time results is given in

- Internal report TFRT-7475, "Discrete Time LQG with cross-terms in the Loss Functions and Noise Descriptions", Gustafsson-Hagander.

Session 3

Properties of the LQ-regulator. Weight Selection. Examples.

Note that the results on gain margin and phase margin are based on the return difference formula which is deduced for the "no cross-term case". With cross-terms there is no guarantee forsuch nice margins. Or better said, you have nice gain and phase margins for a transformed system after introducing the new control signal $\bar{u} = u + L_1 x$. You can skip the section on polynomial matrix fractions, you can also skip pp. 122-125 if you find it hard.

It is probably time now to decide on which system to use on the last hand-in. New suggestions are the JAS 39 Gripen model or the wind power plant in Sven Erik Mattssons PhD thesis (details available via BoB and Mosaic).

Reading Assignment

- pp. 101-131 + 139-163.

Session 4

There will probably be nothing new for you on pp. 164-71. If you have not seen Ackerman's formula before do not bother about that we dont prove it (there's a proof in Kailath). It's a bad formula anyway for numerical reasons, see the help-text to **acker** in matlab.

The reduced order estimator (also called luenberger observer) is usually messy reading the first time you see it. It might help to have matlab-code available while reading, see /home/bob/matlab/luenberger.m. This code assumes that C has full row rank. If C does not have full rank it means that some rows are linear combination of other, i.e. that you have redundancy in your sensors. You can then find T and form $\bar{y} = Ty = TCx$ such that $\bar{C} := TC$ has full rank. It is possible to see that Luenberger observers correspond to that you have let the variance on the measurements R_2 tend to zero and have obtained some infinitely fast modes in the observer by having infinite components in K. The observer tends to a lower order system in this way. Of course the observer will be sensitive to noise in y.

Ch. 7.3 is close to the presentation in [Åström,1970] which I however find more readable. If you find it hard to think on matrix-valued Z and M then find the corresponding section in that book instead. He shows that the optimal estimator for Cx, for an arbitrary vector C, is the Kalman filter. Then you can use scalar signals instead.

On p. 196 one can of course change $[F, D]$ stabilizable to $[F, D]$ no uncontrollable $j\omega$-modes and still get a stable $F + K_e H'$ if the proper solution to the Riccati equation (7.3.30) is chosen.

The dual to the return difference equation is (7.3.33)

$$
\begin{aligned}
R_2 + C_2(sI - A)^{-1} R_1 (-sI - A^T)^{-1} C_2^T &= \\
&= [I + C_2(sI - A)^{-1} K] R_2 [I + K^T(-sI - A^T)^{-1} C_2^T]
\end{aligned}
$$

or with cross-terms

$$
\begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_2 \end{pmatrix} = \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix}^T
$$

$$
N(s) = D_{21} + C_2(sI - A)^{-1} B_1
$$

$$
N(s)N^T(-s) = [I + C_2(sI - A)^{-1} K] D_{21} D_{21}^T [I + K^T(-sI - A^T)^{-1} C_2^T]
$$

The discrete time Kalman filter is really easier to prove. There is almost a proof in Ch 11.3 in Computer Controlled Systems. For a complete proof see [Åström,1970]. See also Sec. 4 in TFRT-7475.

Reading Assignment

- pp. 164-206

Session 5

This is a highlight of the book where the output feedback problem is considered. Only the continuous time problem is discussed, there is only little difference in discrete time. Since the mathematics with stochastic integrals become so involved the proof of the separation principle is not made complete. For a rigorous proof see reference [4] on p. 261.

In (8.1-7) notice that $\tilde{x} = x - x_e$ is uncontrollable from u_{ext}. Remember this. In polynomial form this turns up as a cancellation in (notation as in CCS):

$$y = \frac{BT}{AR + BS} r = \frac{B_m A_o}{A_m A_o} r = \frac{B_m}{A_m} r.$$

Of course the observer dynamics shows up in all other aspects of the design, such as attenuation of load disturbances, noise senisitivity, robustness against model errors etc.

For the example on pp. 222-3 I am missing a plot over the power spectral density of the control signal. It looks like a nice reduction of noise in y_f and y_a but a natural question to ask is how much this reduction has costed in increased variance of u? (3 extra points to the first one who hands in a plot of that).

The continuous time separation theorem is often contributed to Wonham.

In our notation the formula for the optimal performance index on p. 227 can be written in two different ways (no cross-terms):

$$\begin{aligned} V^* &= m^T S(t_0) m + \mathrm{tr} S(t_0) R_0 + \int_{t_0}^{t_1} \mathrm{tr}\ S R_1 dt + \int_{t_0}^{t_1} \mathrm{tr}\ L^T Q_2 L P dt \\ &= m^T S(t_0) m + \mathrm{tr} S(t_0) R_0 + \int_{t_0}^{t_1} \mathrm{tr}\ P Q_1 dt + \int_{t_0}^{t_1} \mathrm{tr}\ K R_2 K^T S dt \end{aligned}$$

(compare with formula (8) in TFRT-7475 in discrete time). It is a medium-hard exercise to prove this (also worth 3p). There is a nice interpretation of the different terms in this formula.

(8.3-3) is an example that shows that LQG designs do not automatically have nice gain margins, not even with diagonal weights, cheap control and almost noise free measurements. This was a surprise to many people in the end of the 70's and the breakthrough for a young student called John Doyle.

Section 8.4 presents a remedy to these problems, the loop recovery approach ("the poor man's robust design method"). Here fictious noise is added at the input, "representing" plant variations or other uncertainties. It is shown that for minimum phase systems the LQ loop gain is obtained in the limit. What one does here is really making part of the observer infinitely fast in a certain Butterworth pattern, see the separate handout from PH. Read the discussion in 8.4 but skip the technical details. Skip pp. 242-244, start reading again at "Dual asymptotic ...".

In (8.4-24) the $B_2 u$ is wrong, compare with 8.3-5. I suspect plots 8.4-4 and 8.4-5 are misleading since the measurement noise is not present in these plots. I do NOT think there is less noise on control signal for "robust LQG" than with nominal LQG. Note that the plot 8.4-6 suggests that the controller is unstable. This is a common phenomena when you make the observer very fast.

Section 8.5 treats the so called Q-parametrization which I suggest you skip right now. It is important and sometime in your life you should read pp. 251-255, but I think you can wait until the course in linear systems. There is also a report on the Q-parametrization by Anders, me and Per that gives some more information. Read however the design example on p. 257-8 (and perhaps reproduce the calculations).

We now have all the material to treat realistic design examples.

Reading Assignment

- pp. 207-260, except 242-244, 251-257.

Session 6

This session treats the servo problem, i.e. following a non zero reference signal $r(t)$. This is a difficult problem since it is nontrivial to formulate what to optimize. If might for example be unrealistic to require that the output follows the reference signal $y(t) = r(t)$ exactly. One solution is instead to say that the system should react on reference changes as some nominal model, i.e. $y = B_m/A_m r$. Note also that to achieve $y(t) \neq 0$ usually requires a non zero control signal so it is a bad idea to minimize something like

$$\int (y - B_m/A_m r)^2 + u^T Q_2 u$$

which is done in Anderson and Moore. With this optimization problem there will be steady state error for non zero r, since u will be chosen as a trade off between achieving $y = B_m/A_m r$ and keeping $u = 0$.

One remedy is to instead minimize

$$\int (y - B_m/A_m r)^2 + \dot{u}^T Q_2 \dot{u}.$$

Note however that this gives a high-frequency weighting of the control signal. This might be a good idea but can also be unwanted.

Another approach is to minimize something like

$$\int (y - B_m/A_m r)^2 + (u - u_m)^T Q_2 (u - u_m)$$

where u_m is a control signal that "makes the output close to $B_m/A_m r$".

To be able to find the feedforward part by optimal control methods one must make some assumption on what is going to happen with the reference signal in the future. The most common setups are

1. r is known in advance,
2. r belongs to some signal class, for example generated by some known linear system (e.g. $r(t)$ constant in the future),
3. r is a stochastic signal with known spectral density.

Versions of setup number 1 and 2 are treated in Anderson-Moore. I will give a more general solution to setup 1 during the lecture.

There are several ways of formulating servo problems. I do not have sufficient experience to be able to recommend any method as the best.

There are also many ways of introducing integrators in the controller using LQG methods. I will mention the two most common ways: extending the system with integrators and adding fictitious bias signals. The two methods differs in that the order of the model or observer are increased. The most practical reason for wanting an integrator in the controller is to achieve zero steady state error even with constant bias errors. And the most common reason for bias errors is that the input signal "always" has a bias. This is because when we use linearization around some (u_0, y_0) to get a linear model we change variables to $\tilde{u} = u - u_0$ and u_0 is seldom known exactly.

Reading Assignment

- Ch 4. + CCS Ch. 9.5. Skip Ch 9 in Anderson-Moore (or read it very quickly).

Session 7

This lecture is given by PerH. The article by Laub describes several different ways to solve the Riccati equation. There is also a good chapter in Maciejowskis book on numerical algorithms for control if you are more interested.

Reading Assignment

- Read the article by Laub. Take a look at the code in Matlab.

Session 8

This lecture is given by Anders Hansson. He has an internal report with more information for those interested.

Linear Quadratic Control Design 94

Lecture 1

- Introduction
- Norms
- Formulas for the optimal LQG controller
- Software
- Handin problems, design challenges

Ch. 1 (pp. 1-6) + App A and B
TFRT 7454, 7456, (7475)
Mosaic

Course Program

Modern Control

More complex control problems (unstable aircraft, flexible space structures, chemical processes. . .)

Better sensors and actuators – enhanced performance possible

Computers

50-60s: Use optimization to find "optimal controller"

Newton, Gould, Kaiser (1957):

In place of a relatively simple statement of the allowable error, the analytical design procedure employs a more or less elaborate performance index. The objective of the performance index is to encompass in a single number a quality measure for the performance of the system.

Optimization based approach

"Optimal" controller

Absolute scale of merit

Limits of performance

"Euphoria" in late 60s

Classical article: "Good, Bad, Optimal"

LQG Theory

Wiener–Kolmogorov

Kalman–Bucy

Wonham, Willems, Anderson, Åström, Kucera, and MANY others

Still active area

Why so Popular?

The first "automized" design method

Space Program

Good models

Stabilizing

LQ-control $u = -Lx$ gives

- $[1/2, \infty]$-gain margin
- $60\,\text{deg}$ phase margin

Robustness

LQ-control robust

$\hat{x}$ Kalman filter robust (dual)

Output feedback ($u = -L\hat{x}$) NOT necessarily robust.

Norms

A norm is a measure of "size" satisfying

$$\begin{aligned} \|u\| &\geq 0 \\ \|u\| = 0 &\Rightarrow u = 0 \\ \|\alpha u\| &= |\alpha|\|u\|, \quad \forall \alpha \in R \\ \|u_1 + u_2\| &\leq \|u_1\| + \|u_2\| \end{aligned}$$

Norms of Signals

Example: The L_p-norms $(1 \le p \le \infty)$

$$\|u\|_p = (\int |u|^p dt)^{1/p}$$

With $p = 2$, energy, (RMS-value)

$$\|u\|_2 = (\int_{-\infty}^{\infty} u(t)^T u(t) dt)^{1/2} =$$
$$= (\int_{-\infty}^{\infty} U^*(j\omega)U(j\omega)d\omega/2\pi)^{1/2}$$

With $p = \infty$:

$$\|u\|_\infty = \sup_t |u(t)|$$

Note if u is a vector one must define what is meant by $|u|$. We will use $|u|^2 = u^T u$.

Scalar Products

L_2 is actually an inner product space:

$$< u, v >= \int_{-\infty}^{\infty} \bar{u}^T(t)v(t)\, dt$$

$$\|u\|^2 =< u, u >$$

u is called "orthogonal" to v if

$$< u, v >= 0$$

Norms of Systems

$$Y = G(s)U$$
$$y = g \star u$$
$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

The L_2-norm (LQG-norm):

$$\|G\|_2^2 = \sum_i \sum_j \int_{-\infty}^{\infty} |g_{ij}(t)|^2 dt =$$
$$= \sum_i \sum_j \int_{-\infty}^{\infty} |G_{ij}(j\omega)|^2 d\omega/2\pi =$$
$$= \int_{-\infty}^{\infty} \text{trace } G^*(j\omega)G(j\omega)d\omega/2\pi =$$

H_2: As L_2 but with $G(s)$ stable also.

L_1-norm, H_∞-norm etc.

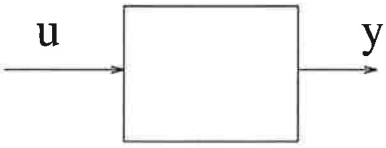
Scalar Product in L_2

$$< G(s), H(s) >_{L_2} = \text{trace} \int_{-\infty}^{\infty} \text{Re } G^*(i\omega)H(i\omega)\, d\omega/2\pi$$

Exercise: Show that if $G(s)$ is stable and $H(s)$ "anti-stable" (i.e. $H(-s)$ is stable) then

$$< G, H >= 0$$

Interpretation of the H_2-norm



u: white noise, mean zero, unit variance

$$E(u(\tau_1)u(\tau_2)^T) = \delta(\tau_1 - \tau_2)I$$

then

$$E(y^T y) = ||G||_2^2$$

Proof

$$E(y^T y) = E(\text{tr } yy^T) =$$
$$= \text{tr} \int_{-\infty}^{\infty} g(t-\tau_1)u(\tau_1)d\tau_1 \int_{-\infty}^{\infty} u(\tau_2)^T g^T(t-\tau_2)d\tau_2$$
$$= \text{tr} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(t-\tau_1)u(\tau_1)u^T(\tau_2)g^T(t-\tau_2)d\tau_1 d\tau_2$$
$$= \text{tr} \int_{-\infty}^{\infty} g(t-\tau_1)g^T(t-\tau_1)d\tau_1$$
$$= ||G||_2^2$$

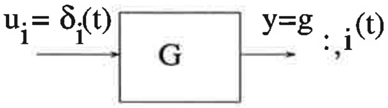
Alternative

$$E(\text{tr } yy^T) = \text{tr} \int S_y(\omega)d\omega/2\pi =$$
$$= \int \text{tr } G^*(j\omega)S_u(\omega)G(j\omega)d\omega/2\pi$$

"Variance of output with white noise in"

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} C^TC & C^TD \\ D^TC & D^TD \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} dt$$

More interprations of the H_2-norm



$$||G||_2^2 = \sum_{i=1}^{m} ||G\delta_i||_2^2$$

"Energy in impulse response"

Yet another interpration in exercises

How to compute the H_2 norm

1) Residue calculus

$$||G||_2^2 = \sum_{i,j} \frac{1}{2\pi i} \oint G_{ij}(-s)^T G_{ij}(s)ds$$

2) Recursive formulas ala Åström-Jury-Schur

3) With $G(s) = C(sI - A)^{-1}B$

$$||G||_2^2 = CPC^T$$

where P is the unique solution to the Lyapunov equation

$$AP + PA^T + BB^T = 0$$

Proof of 3

Alt 1: From CCS (6.43) we now that $P = E(xx^T)$ is given by the Lyapunov equation above. Therefore

$$\|G\|_2^2 = E(yy^T) = CE(xx^T)C^T = CPC^T$$

Proof of 3

Alt 2: Led by the fact that

$$\|G\|_2^2 = \int_0^\infty Ce^{tA}BB^Te^{tA^T}C^T dt$$

we define

$$P := \int_0^\infty e^{tA}BB^Te^{tA^T} dt.$$

By integrating both sides of

$$\frac{d}{dt}e^{tA}BB^Te^{tA^T} = Ae^{tA}BB^Te^{tA^T} + e^{tA}BB^Te^{tA^T}A^T$$

we get

$$-BB^T = AP + PA^T$$

That P is uniquely given by this equation follows from the fact that A stable and matrix theory.

More on Norms of Signals and Systems

Linear Controller Design, Boyd, Barrat

An introduction to Hilbert Space, N. Young

Discrete Time

$G(s) = C(zI - \Phi)^{-1}\Gamma + D$ with A stable

$$\begin{aligned}\|G\|_2^2 &= \frac{1}{2\pi}\text{tr} \oint G^*(e^{j\omega})G(e^{j\omega})d\omega \\ &= \sum_k g^T(k)g(k)\end{aligned}$$

Same interpretations as in continuous time

Exercise to find the corresponding Lyapunov-formula

Remark

Will also discuss finite time horizon problems

Instead of

$$\lim_{T\to\infty} \frac{1}{T}\int_0^T \begin{pmatrix} x \\ u \end{pmatrix}^T Q \begin{pmatrix} x \\ u \end{pmatrix} dt$$

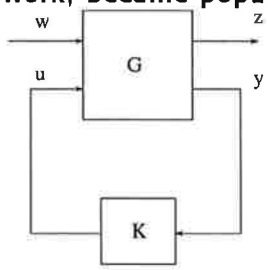
one studies

$$\int_0^T \begin{pmatrix} x \\ u \end{pmatrix}^T Q \begin{pmatrix} x \\ u \end{pmatrix} dt + x^T(T) Q_T x(T)$$

No stability issues.

The Standard Problem

Unified framework, became popular in 80s



$u =$ Control Inputs

$y =$ Measured Outputs

$$w = \text{Exogenous Inputs} = \begin{cases} \text{Fixed commands} \\ \text{Unknown commands} \\ \text{Disturbances} \\ \text{Noise} \\ \vdots \end{cases}$$

$$z = \text{Regulated Outputs} = \begin{cases} \text{Tracking Errors} \\ \text{Control Inputs} \\ \text{Measured Outputs} \\ \text{States} \\ \vdots \end{cases}$$

The H_2 Problem

Closed Loop

$$u = K(s)y$$
$$z = G_{11} + G_{12}K(I - G_{22}K)^{-1}G_{21}w = T_{zw}w$$

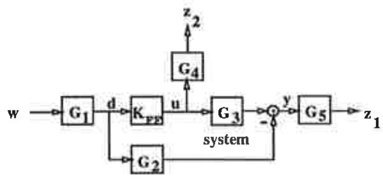
The H_2 problem:

Find $K(s)$ such that the closed loop is stable and

$$\min_{K(s)} \|T_{zw}\|_2$$

is obtained.

Example, Optimal Feedforward



Output

$$y = G_3 u - G_2 d$$

d is a measurable signal $d = G_1 w$

Feedforward regulator

$$u = K_{FF}\, d$$

Minimize a mean square of filtered outputs and filtered control signals:

$$\min \; E(z_1^T z_1 + z_2^T z_2)$$

$$\left(\begin{pmatrix} -G_5G_2G_1 \\ 0 \\ G_1 \end{pmatrix} \quad \begin{pmatrix} G_5G_3 \\ G_4 \\ 0 \end{pmatrix} \right)$$

The Optimal Controller

Let the system be given by

$$\begin{aligned}\dot{x} &= Ax + B_1 w + B_2 u\\ z &= C_1 x + \quad\quad D_{12} u\\ y &= C_2 x + D_{21} w + D_{22} u\end{aligned}$$

under some technical conditions the optimal controller is of order n and is given by

$$\begin{aligned}u &= -L\hat{x}\\ \dot{\hat{x}} &= A\hat{x} + B_2 u + K(y - C\hat{x} - D_{22}u)\\ L &= (D_{12}^T D_{12})^{-1}(D_{12}^T C_1 + B_2^T S)\\ K &= (B_1 D_{21}^T + PC_2^T)(D_{21}D_{21}^T)^{-1}\end{aligned}$$

where $P \geq 0$ and $S \geq 0$ satisfy the Riccati equations

$$\begin{aligned}0 &= SA + A^T S + C_1^T C_1 - L^T D_{12}^T D_{12} L\\ 0 &= AP + PA^T + B_1 B_1^T - K D_{21} D_{21}^T K^T\end{aligned}$$

"Technical Conditions"

1) $[A, B_2]$ stabilizable, i.e. $\exists L : A - B_2 L$ stable

Stable uncontrollable modes allowed.

2) $[C_2, A]$ detectable, i.e. $\exists K$: $A - KC_2$ is stable.

Stable undetectable modes allowed.

More Technical Conditions

3)

$$\text{rank} \begin{pmatrix} j\omega I - A & -B_2 \\ C_1 & D_{12} \end{pmatrix} = n + m \qquad \forall \omega$$

and D_{12} have full column rank.

4)

$$\text{rank} \begin{pmatrix} j\omega I - A & -B_1 \\ C_2 & D_{21} \end{pmatrix} = n + p \qquad \forall \omega$$

and D_{21} have full row rank.

3) and 4) can be relaxed somewhat. Singular Optimal Control. Special lecture later.

Examples

$$\min |x|^2 + |u|^2, \quad \dot{x} = u_1 + u_2$$

$$\min |x|^2, \quad \dot{x} = u_1$$

Software

Read about the LQGBOX in TFRT-7575

```
[K,P] = lqec(A,C,R1,R2,R12)

[L,S] = lqrc(A,B,Q1,Q2,Q12)

lr = refc(A,B,C,D,L)

[Ac,By,Byr,Cc,Dy,Dyr] = lqgc(A,B,C,D,L,lr,K)
```

lqed, lqrd, refd, lqgd in discrete time

Works reasonably well

$$\begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_{22} \end{pmatrix} = \begin{pmatrix} C_1^T \\ D_{12}^T \end{pmatrix} \begin{pmatrix} C_1 & D_{12} \end{pmatrix}$$

$$\begin{pmatrix} R_1 & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix} = \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} \begin{pmatrix} B_1^T & D_{21}^T \end{pmatrix}$$

Hand In Problems

First Due Oct 31

All info on Mosaic

Design Challenges

Hand-in problem 8, due 941220

LQG-design on (choose one)

- Hot Rolling Mill (LMP) 2 inputs, 3 outputs, control thickness
- Aircraft (p. 152) lateral dynamics, 2 inputs
- Helicopter Control, 2 inputs, 2 outputs, no model available yet.
- IFAC Benchmarks Problems (13 different)
- ACC Benchmark (two cars with spring)

Next Lecture

Prepare by reading chapters 2 and 3.1-3.4 and App C,D,E.

Concentrate on the principle of optimality and how to get the infinite time horizon as a limit of the finite time case.

Next lecture Monday 31/10 13.15

First exercise on Friday 21/10 10.15

Linear Quadratic Control Design 94

Lecture 2

Full information, $y = x$

- Optimal Control
- Hamilton-Jacobi-Bellman
- Pontryagin
- Finite Time Horizon
- Infinite Time Horizon

Ch 2. Ch 3.1-3.4. App C, D and E.

Matrix Theory

$$A > 0 \quad \Leftrightarrow \quad x^* A x > 0,\ \forall x \neq 0$$
$$A > B \quad \Leftrightarrow \quad A - B > 0$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}_{\#} = A - BD^{-1}C \quad \text{"Schur Complement"}$$
$$_{\#}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = D - CA^{-1}B$$

$$\min_u \begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} A & B \\ B^T & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = x'(\ldots)_{\#} x$$

A is Hurwitz if and only if

$$SA + A'S = -Q$$

has unique solution $S > 0$ for any $Q > 0$

If $[A, C]$ is detectable then A is Hurwitz if and only if there exists unique $S \geq 0$

$$SA + A'S + C'C = 0$$

Optimal Control

$$\dot{x} = f(x, u, t) \quad x(t_0) \text{ given}$$

Find u^* that minimizes

$$V(x(t_0), u(\cdot), t_0) = \int_{t_0}^{T} l(x(\tau), u(\tau), \tau)\, d\tau + m(x(T))$$

$$V^*(x, t) = \min_u V(x, u(\cdot), t)$$

Existence ?

Smoothness assumptions important

(easy in LQG)

Principle of Optimality

If V^* exists then

$$-\frac{\partial V^*}{\partial t}(x(t), t) = \min_u \left\{ l(x, u, t) + \frac{\partial V^*}{\partial x} f(x, u, t) \right\}$$
$$V^*(x(T), T) = m(X(T))$$

Hamilton-Jacobi-Bellman

Gives $u^* = u(x(t))$.

LQ

$$\dot{x} = Ax + B_2 u$$
$$\min \int_0^T \begin{pmatrix} x \\ u \end{pmatrix}^T Q \begin{pmatrix} x \\ u \end{pmatrix} + x(T)Q_T x(T)$$

Page 21: If V^* exists then

$$V^*(x(t), t) = x'(t)S(t)x(t)$$

HJB then gives

$$-x'\dot{S}(t)x = \min_u \left(x'S(t) \begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} + \ldots + \right.$$
$$\left. + \begin{pmatrix} x \\ u \end{pmatrix}' Q \begin{pmatrix} x \\ u \end{pmatrix} \right)$$

Riccati equation

Riccati

$$-\dot{\mathcal{S}} = (\mathcal{S}\mathcal{A} + \mathcal{A}'\mathcal{S} + \mathcal{Q})_{\#}$$

where

$$\mathcal{A} = \begin{pmatrix} A & B_2 \\ C_1 & D_{12} \end{pmatrix}$$
$$\mathcal{S} = \begin{pmatrix} S & 0 \\ 0 & 0 \end{pmatrix}$$
$$\mathcal{Q} = \begin{pmatrix} C_1'C_1 & C_1'D_{12} \\ D_{12}'C_1 & D_{12}'D_{12} \end{pmatrix}$$

Existence

$u \equiv 0$ gives an upper bound on $V^*(x, t)$.

Riccati equation hence always has a solution.

Derivation using Pontryagin

Appendix C

$$\dot{x} = Ax + B_2 u$$
$$\min \int_0^T x^T Q_1 x + u^T Q_2 u \, dt + x(T)' Q_T x(T)$$

Hamiltonian

$$H(t, x, u, \lambda) = \frac{1}{2}(x^T Q_1 x + u^T Q_2 u) + \lambda^T (Ax + Bu)$$

$H_u = 0$ gives $u_0(t) = -Q_2^{-1} B \lambda(t)$ and

$$\dot{x} = H_\lambda = Ax + B_2 u$$
$$-\dot{\lambda} = H_x = A^T \lambda + Q_1 x$$

Boundary cond.: $x(t_0) = x_0$ and $\lambda(t_1) = Q_T$.

Pontryagin

$$\begin{pmatrix} \dot{x} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} A & -B_2Q_2^{-1}B^T \\ -Q_1 & -A^T \end{pmatrix} \begin{pmatrix} x \\ \lambda \end{pmatrix}$$
$$= E \begin{pmatrix} x \\ \lambda \end{pmatrix}$$

Trying $\lambda(t) = S(t)x(t)$ gives

$$\dot{\lambda} = \dot{S}x + S\dot{x} = -Q_1 x - A^T \lambda$$

from which

$$-\dot{S} = SA + A'S + Q_1 - SB_2Q_2^{-1}B_2^T S, \quad S(T) = 0$$

follows.

The Euler matrix above is heavily used in numerical algorihtms for solving Riccati equtions

Finite Time Horizon

$$\dot{x} = A(t)x(t) + B_1(t)w(t) + B_2u(t), \qquad x(0) = 0$$
$$z(t) = C_1x(t) + D_{12}(t)u(t)$$
$$y(t) = x(t)$$

$u \in R^m$: control inputs
$w \in R^l$: external disturbances
$z \in R^p$: objectives
$y \in R^q$: measurements

Finite Time Horizon

Wanted : Causal, linear controller

$$u = K(s)y$$

minimize closed loop finite-horizon 2-norm

$$||T_{zw}||_{2,[0,T]} = E\left\{\frac{1}{T}\int_0^T z'z\,dt\right\}^{\frac{1}{2}}$$

Full information: $y = x$.

Simplification

Lets start with simpler subproblem

$$\dot{x} = Ax + B_2u$$
$$z = \begin{pmatrix} Cx \\ Du \end{pmatrix}$$

where $D'D = I$.

No noise

No cross terms

No loss of generality, see later

LQ-optimal control

$$J_t(K, x_t, T, Q_T) = \int_t^T z'z + x'(T)Q_T x(T)$$

If Riccati equation

$$-\dot{S} = SA + A'S + C'C - SB_2B_2'S, \quad S(T) = Q_T$$

has a solution on $[t, T]$. We obtain

$$J_t = x(t)'S(t)x(t) + \int_t^T (u + B_2'Sx)'(u + B_2'Sx)\, d\tau$$

Optimal controller

$$u^*(t) = -B_2S(t)x(t)$$

Optimal cost

$$J_t(K^*, x_t, T, Q_T) = x'(t)S(t)x(t)$$

Non-negativity of S

Since $J_t(K, x_t, T, Q_T) \geq 0 \ \forall \ K, x_t$

$$x_t'S(t)x_t \geq 0 \quad \Rightarrow \quad S(t) \geq 0 \forall t$$

Existence, Yes!

Existence of solution $S(t)$ to Riccati?

$S(t)$ is bounded from above $\forall t \leq T$:

$$x_t'S(t)x_t \leq J_t(0, x_t, T, Q_T) = ||\tilde{z}||^2_{2,[t,T]} + \tilde{x}'(T)Q_T\tilde{x}'(T)$$

where $\dot{\tilde{x}} = A\tilde{x}$ is the open loop trajectory.

Therefore, no finite escape time

Infinite Time Horizon Conjecture

Stability

Necessary to have (A, B_2) stabilizable.

$$u^*(t) = -B_2Sx(t)$$
$$0 = SA + A'S + C'C - SB_2B_2'S$$

Which S?

Closed loop dynamics with noise:

$$\dot{x} = (A - B_2B_2'S)x + B_1w$$

must have $A - B_2B_2'S$ stable

Necessary for existence: (A, C) no unobservable modes on imaginary axis:

$$Ax = j\omega x, \quad Cx = 0 \Rightarrow B_2'Sx \Rightarrow (A - B_2B_2'S)x = j\omega x$$

Infinite Time Horizon

$$J(K,x_0)=\lim_{T\to\infty}\left\{\int_0^T z'z d\tau + x'(T)Q_T x(T)\right\}$$

subject to

$$\dot{x}=Ax+B_2u, \qquad x(0)=x_0$$
$$z=\begin{pmatrix} Cx \\ Du \end{pmatrix}$$

where $D'D = I$ and (A, B_2, C) stabilizable with no unobservable $\jmath\omega$-modes.

Choice of Q_T

Necessary to choose Q_T correctly to have $S(t,T,Q_T) \to S_{opt}$, where $S(t,T,Q_T)$ denotes solution to

$$\dot{S}=SA+A'S+C'C-SB_2B_2'S, \qquad S(T)=Q_T$$

Choose any $Q_T \geq 0$ such that

$$Q_TA+A'Q_T+C'C-Q_TB_2B_2'Q_T \leq 0$$

and

$$\left(A, \begin{bmatrix} C \\ Q_T \end{bmatrix}\right) \quad \text{is detectable}$$

Such a choice is possible: Take any L such that $A - B_2L$ is stable and let Q_T be the unique solution of

$$Q_T(A-B_2L)+(A-B_2L)'Q_T+C'C+L'L=0$$

Monotonicity of $S(t,T,Q_T)$

$$-\ddot{S}=\dot{S}(A-B_2B_2'S)+(A-B_2B_2'S)\dot{S}$$
$$\dot{S}(t)=\Phi(t,T)\dot{S}(T)\Phi'(t,T)\geq 0$$

$S(t,T,Q_T)$ increases with t

Since

$$S(t,T+\tau,Q_T)=S(t-\tau,T,Q_T)$$

it also decreases with T.

$$0 \leq S(t,T,Q_T) \leq Q_T$$

Steady-state solution

$$S=\lim_{T\to\infty} S(t,T,Q_T)$$

(independent of t).

S is solution to Riccati equation:

$$\begin{aligned} S &= \lim_{T\to\infty} S(t,T,Q_T) \\ &= \lim S(t,T_1,S(T_1,T,Q_T)) \\ &= S(t,T_1,\lim_{T\to\infty} S(T_1,T,Q_T)) \quad \text{continuity} \\ &= S(t,T_1,S) \end{aligned}$$

Since $\dot{S}=0$

$$0=SA+A'S+C'C-SB_2B_2'S$$

Stability

Let $L = B_2'S$ where S is obtained as $T \to \infty$ as above. Then $A - B_2L$ is stable.

Proof: Let $L_T = BB^TS_T$ then

$$S_T(A-B_2L)+(A-B_2L)'S_T+L_T'L_T+C'C+\dot{S}_T=0$$

Assume $(A - B_2L_T)x = \lambda x$, show $\text{Re}(\lambda) \geq 0$ impossible (see Green-Limebeer).

Letting $T\infty$ we have $l_T \to L$. Continuity shows that $A - BL$ has all eigenvalues in $\text{Re}\lambda \leq 0$. But $\text{Re}\lambda = 0$ is impossible by detectability on $j\omega$-axis (see Green-Limebeer).

Cross-Terms

$$z = C_1x + D_{12}u$$

Assumption: D_{12} full rank

Introduce

$$\tilde{u} = (D_{12}D_{12})^{-1/2}(u + D_{12}'C_1x)$$

This gives

$$z'z = x'\tilde{C}_1'\tilde{C}_1x + \tilde{u}'\tilde{u}$$

$$\begin{pmatrix} \tilde{A} & \tilde{B}_2 \\ \tilde{C}_1 & \tilde{D}_{12} \end{pmatrix} = \begin{pmatrix} A & B_2 \\ C_1 & D_{12} \end{pmatrix} \begin{pmatrix} I & 0 \\ -D_{12}'C_1 & (D_{12}'D_{12})^{-1/2} \end{pmatrix}$$

Cross-Terms

$(\tilde{A}, \tilde{C}_1)$ no unobservable modes in $j\omega$:

$$\begin{aligned} 0 &= \begin{pmatrix} \tilde{A} - j\omega I \\ \tilde{C}_1 \end{pmatrix} x \\ &= \begin{pmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix} \begin{pmatrix} I \\ -D_{12}'C_1 \end{pmatrix} x \end{aligned}$$

Detectability condition becomes

$$\text{rank} \begin{pmatrix} A - j\omega I & B_2 \\ C_1 & D_{12} \end{pmatrix} = n + m \quad \forall\omega$$

Control Law:

$$D_{12}'D_{12}u^* = -(D_{12}'C_1 + B_2'S)x$$

With State Noise

We now verify that $u^* = -B_2'Sx$ with $S \geq 0$ satisfying the Riccati equation minimizes also $||T_{zw}||_2$.

Consider any full-information controller

$$\begin{aligned} \dot{\xi} &= F\xi + G_1x + G_2w, \quad \xi(0) = 0 \\ u &= H\xi + J_1x + J_2w \end{aligned}$$

Closed loop system

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{A}\tilde{x} + \tilde{B}w \\ z &= \tilde{C}\tilde{x} + \tilde{D}w \end{aligned}$$

where $\tilde{x} = [x'\xi']'$.

H_2-optimality

Must have $J_2 = 0$ for finite H_2-norm.

$$\tilde{A}'\tilde{Q} + \tilde{Q}\tilde{A} + \tilde{C}'\tilde{C} = 0$$
$$\tilde{P} = \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix}$$

Riccati+Lyapunov gives $\tilde{Q} - \tilde{P} \geq 0$.

Equality if $J_1 = -B_2 S, H = 0$. Then

$$||T_{zw}||_2^2 = \text{trace}(B_1' S B_1)$$

Return Difference Formulas

$$C'C = (-sI - A')S + S(sI - A) + SBQ_2^{-1}B'S$$

Hence

$$Q_2 + G'(-s)G(s) = [I + H'(-s)]Q_2[I + H(s)]$$

where

$$G(s) = C(sI - A)^{-1}B$$
$$H(s) = L(sI - A)^{-1}B$$

SISO case with $Q_2 = \rho$:

Open Loop : $G(s) = B(s)/A(s)$

Closed Loop : $C(sI - A + BL)^{-1}B = B(s)/P(s)$

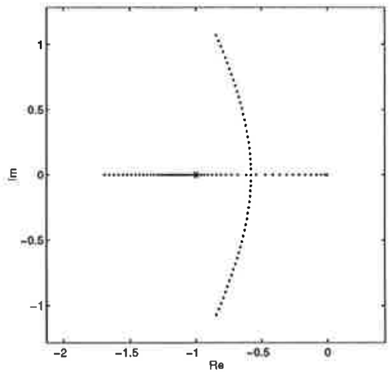
Return Difference : $1 + H(s) = P(s)/A(s)$

$$\rho A^*(s)A(s) + B^*(s)B(s) = \rho P^*(s)P(s)$$

Symmetrisk root locus wrt ρ.

Example

$$G(s) = \frac{1}{(s^2 - 1)s}$$



Linear Quadratic Control Design 94

Lecture 3

Properties of LQ

- Return Difference Formula
- Gain and Phase Margins
- Choise of Weights
- Classical Control Concepts

Ch 5 and 6

Closed Loop

Loop Gain: $L(sI-A)^{-1}B$

Return Difference: $I+L(sI-A)^{-1}B$

A Factorization

$$\begin{pmatrix} 0 & sI-A & -B \\ -sI-A^T & C^TC & C^TD \\ -B^T & D^TC & D^TD \end{pmatrix} = \Phi^T(-s)\Phi_0\Phi(s)$$

where

$$\Phi_0 = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & D^TD \end{pmatrix}; \quad \Phi(s) = \begin{pmatrix} I & S & 0 \\ 0 & sI-A & B \\ 0 & L & I \end{pmatrix}$$

$$\det\Phi(s) = \det(sI-A+BL)$$

Return Difference Formula

Exercise:

$$M^T(-s)M(s) = (I+L(-sI-A^T)^{-1}B)^TD^TD(I+L(sI-A)^{-1}B)$$

where $M(s) = D + C(sI-A)^{-1}B$

If no crossterms:

If $C^TC = Q_1$, $C^TD = 0$ and $D^TD = Q_2$

$$Q_2 + B^T(-sI-A^T)^{-1}Q_1(sI-A)^{-1}B = (I+L(-sI-A^T)^{-1}B)^TQ_2(I+L(sI-A)^{-1}B)$$

$$(I+L(-sI-A^T)^{-1}B)^TQ_2(I+L(sI-A)^{-1}B) \geq Q_2$$

Scalar Case

$$q_2 + |C(sI - A)^T B|^2 = q_2|1 + L(sI - A)^{-1}B|^2$$

therefore

$$|1 + L(sI - A)^{-1}B| \geq 1$$

Gain Margin

$$|1 + L(sI - A)^{-1}B| \leq 1$$

Gain Margin $[1/2, \infty]$

Phase Margin 60 degrees.

Not simultaneously. No cross-terms. All states measurable.

Gain Margin, MIMO

With

$$S = (1 + L(sI - A)^{-1}B)^{-1}$$

$$\bar{\sigma}(Q_2^{1/2} S Q_2^{-1/2}) \leq 1$$

If Q_2 diagonal this gives nice MIMO gain/phase margins see book.

Robustness against nonlinearities

Circle Criterion

Stability with any nonlinear time-varying input gain with slopes in $(1/2, \infty)$.

Note

$$\min \int (u+Lx)^T(u+Lx)$$

gives $u = -Lx$ (if stabilizing).

Cross-terms.

A Matrix Equality

$$(I + L(sI - A)^{-1}B)^{-1} = I - L(sI - A + BL)^{-1}B$$

Proof: Using

$$\begin{aligned} \dot{x} &= Ax + Bu \\ u &= -Lx + r \end{aligned}$$

gives

$$\begin{aligned} r &= u + Lx = [I + L(sI - A)^{-1}B]u \\ u &= r - Lx = [I - L(sI - A + BL)^{-1}B]r \end{aligned}$$

$$(\ldots)^* Q_2(I - L(j\omega I - A + BL)^{-1}B) \leq Q_2$$

Bounded Real

A real rational matrix $S(s)$ is bounded real (BR) if all poles lie in Re$(s) \leq 0$ and

$$S^T(-j\omega)S(j\omega) \leq 1, \qquad \forall\omega$$

Clear that

$$S^T(-s)S(s) \leq 1$$

where

$$S(s) = Q_2^{1/2}(I - L(sI - A + BL)^{-1}B)Q_2^{-1/2}$$

Positive Real

A real rational matrix $Z(s)$ is positive real (PR) if all poles lie in Re$(s) \leq 0$ and

$$Z^T(-j\omega) + Z(j\omega) \geq 0, \qquad \forall\omega$$

Not so clear that

$$Z(s) = Q_2L(sI - A + \frac{1}{2}BL)^{-1}B$$

is PR.

Closed Loop Eigenvalues

Given by eigenvalues of $A - BL$

$$\det \begin{pmatrix} 0 & sI - A & -B \\ -sI - A^T & C^TC & C^TD \\ -B^T & D^TC & D^TD \end{pmatrix} = \det(\Phi^T(-s)\Phi_0\Phi(s))$$

This gives

$$\det(-sI - A^T)(sI - A)\det(M^T(-s)M(s))$$
$$= \det(-sI - A^T + L^TB^T)\det(D^TD)\det(sI - A + BL)$$

Scalar Case, no cross terms

Introduce

$$Q_2 = \rho I$$
$$G(s) = C(sI - A)^{-1}B = B(s)/A(s)$$
$$I + H(s) = I + L(sI - A)^{-1}B = P(s)/A(s)$$

Closed loop characteristic equation $P(s) = 0$

$$Q_2 + G^T(-s)G(s) = (I + H^T(-s))Q_2(I + H(s))$$

$$\rho A(-s)A(s) + B(-s)B(s) = \rho P(-s)P(s)$$

Symmetric Root Locus

`symlocc`, `symlocd` in matlab

Cheap control $\rho \to 0$

Eigenvalues of closed loop tend to stable zeros of $B(-s)B(s)$ and the rest to ∞ as stable roots of

$$s^{2d} = \text{const} \cdot \rho$$

Expensive Control $\rho \to \infty$

Eigenvalues of closed loop tend to stable zeros of $A(-s)A(s)$

Example

$$\min u^2, \quad \dot{x} = x + u$$

$A(s) = s + 1$ unstable.

$u = -2x$ gives

$$\dot{x} = -x$$

$P(s) = A(-s) = -s + 1$

High Frequency Behaviour

$$L(j\omega I - A)^{-1}B = LB/\omega = Q_2^{-1}B^T SB/\omega$$

Controller has "roll-off" 1

Same conclusion for

$$L(j\omega I - A + BL)^{-1}B = LB/\omega = Q_2^{-1}B^T SB/\omega$$

Rules of Thumb

$$Q_1 = \text{diag}(\alpha_1, \ldots, \alpha_n)$$
$$Q_2 = \text{diag}(\beta_1, \ldots, \beta_m)$$

Let $\alpha_i \sim (x_i)^{-2}$ and $\beta_i \sim (u_i)^{-2}$ where x_i and u_i denote allowable sizes on state i and input i

More ideas

Punishing

$$(\dot{x}_i + \alpha x_i)^2$$

"should" give $\dot{x}_i = -\alpha x_i$.

Moving Eigenvalues

Can move one eigenvalue at a time by using

$$Q_1 = qq^T$$

where q is orthogonal to the A-invariant subspace of the rest of the modes

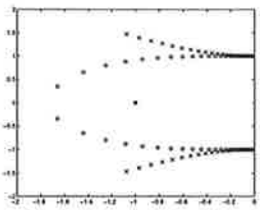
Example

$$G(s) = 1/(s+1)(s^2+1)$$

Increase damping without moving pole in $s = -1$.

```
s =
   0.7071          0.7071          0.4082
        0 + 0.7071i      0 - 0.7071i  -0.4082
        0               0          0.8165
d =
        0 + 1.0000i      0               0
        0               0 - 1.0000i      0
        0               0          -1.0000
```

$$Q_1 = q_i q_i^T, \qquad q_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 2 \\ 0 \\ -1 \end{pmatrix}$$



Classical Control Concepts

$$y = PC(I+PC)^{-1}(r-n) + (I+PC)^{-1}d$$
$$e = r - y - n = (I+PC)^{-1}(r-d) - (I+PC)^{-1}n$$
$$u = C(I+PC)^{-1}(r-n-d)$$

$S = (I+PC)^{-1}$ sensitivity

$T = PC(I+PC)^{-1}$ complementary sensitivity

Trade-offs

Should have S small where good tracking and good disturbance suppression are wanted

Should have T small to have good noise suppression

$$S + T = 1$$

Sensitivity

S sensitivity: Let $H = PC(I+PC)^{-1}$ (scalar)

$$\frac{\partial H}{\partial P} \Big/ \frac{H}{P} = S$$

Another similar interpretation in book

Unstructured Multiplicative Perturbations

If $P(j\omega)$ is changed to $(I + \Delta(j\omega))P(j\omega)$ then the disturbed system is stable if

$$\bar{\sigma}(\Delta) < (\bar{\sigma}(T))^{-1}$$

Should have T small to get robustness against multiplicative perturbations.

Linear Quadratic Control Design 94

Lecture 4

State Estimation

- Preliminaries, Projection Theorem
- Kalman-Bucy Filter Discrete Time
- Duality
- Kalman-Bucy Filter Continuous Time

Ch 7

Prediction

Assume that

$$E(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} m_x \\ m_y \end{pmatrix}$$
$$\text{cov}(\begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} R_x & R_{xy} \\ R_{xy}^T & R_y \end{pmatrix} := R$$

then the vector

$$z = x - m_x - K(y - m_y) \qquad K = R_{xy}R_y^{-1}$$

has zero mean, is uncorrelated with y and has covariance

$$R_z = R_x - R_{xy}R_y^{-1}R_{xy}^T$$

Proof

$E(z) = 0$ obvious

$$E\begin{pmatrix} z \\ y - m_y \end{pmatrix}\begin{pmatrix} z \\ y - m_y \end{pmatrix}^T =$$
$$= E\begin{pmatrix} I & -K \\ 0 & I \end{pmatrix}\begin{pmatrix} x - m_x \\ y - m_y \end{pmatrix}\begin{pmatrix} x - m_x \\ y - m_y \end{pmatrix}^T\begin{pmatrix} I & 0 \\ -K^T & I \end{pmatrix}$$
$$= \begin{pmatrix} I & -K \\ 0 & I \end{pmatrix} R \begin{pmatrix} I & 0 \\ -K^T & I \end{pmatrix}$$
$$= \begin{pmatrix} R_x - R_{xy}R_y^{-1}R_{xy}^T & 0 \\ 0 & R_y \end{pmatrix}$$

This proves it

Note the Schur complement again !

Prediction of Gaussians

Theorem If x, y are Gaussian with mean and variance as above then

$$\hat{x} = E[x|y] = m_x + K(y - m_y) \quad (1)$$
$$E\left\{(x - \hat{x})(x - \hat{x})^T|y\right\} = R_x - KR_yK^T =: R_\Delta \quad (2)$$
$$R_{xy} = KR_y$$

x and $x - \hat{x}$ are independent.

Proof: Introduce $\bar{x} := x - m_x$, $\bar{y} := y - m_y$

$$f(x, y) \sim \exp(-\frac{1}{2}\mathcal{D})$$
$$\mathcal{D} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}^T \begin{pmatrix} R_x & R_{xy} \\ R_{xy}^T & R_y \end{pmatrix}^{-1} \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$
$$= \begin{pmatrix} \bar{x} - K\bar{y} \\ \bar{y} \end{pmatrix}^T \begin{pmatrix} R_\Delta & 0 \\ 0 & R_y \end{pmatrix}^{-1} \begin{pmatrix} \bar{x} - K\bar{y} \\ \bar{y} \end{pmatrix}$$

Proof cont'd

$$f(x|y) = \frac{f(x,y)}{f(y)} \sim \exp\left\{-\frac{1}{2} z^T R_\Delta^{-1} z\right\}$$
$$z = x - m_x - K(y - m_y)$$

So $x|y$ is Gaussian with mean (1) and variance (2)

z and y uncorrelated hence independent.

$E(x \mid y) = \text{MLE}$

Corollary: $\hat{x} = E(x \mid y)$ is obtained by

$$\hat{x} = \underset{x}{\text{argmin}} \begin{pmatrix} x - m_x \\ y - m_y \end{pmatrix}^T \begin{pmatrix} R_x & R_{xy} \\ R_{xy}^T & R_y \end{pmatrix}^{-1} \begin{pmatrix} x - m_x \\ y - m_y \end{pmatrix}$$

Therefore $E(x \mid y) = \text{MLE}$.

Geometric Interpretation

Another Theorem

Let x, u and v be jointly Gaussian random vectors and let u and v be independent, then

$$E[x - m_x \mid u, v] = E[x - m_x \mid u] + E[x - m_x \mid v]$$

Proof:

$$E[x - m_x \mid u, v] = \begin{pmatrix} R_{xu} R_u^{-1} & R_{xv} R_v^{-1} \end{pmatrix} \begin{pmatrix} u - m_u \\ v - m_v \end{pmatrix}$$
$$= R_{xu} R_u^{-1}(u - m_u) + R_{xv} R_v^{-1}(v - m_v)$$
$$= E[x - m_x \mid u] + E[x - m_x \mid v]$$

Prediction

$$x(t+1) = Ax(t) + B_1 e(t)$$
$$y(t) = C_2 x(t) + D_{21} e(t)$$

where $E(e) = 0$ and $E(ee^T) = I$.

Add $u(t)$ later

Prediction

$$\hat{x}(t+1 \mid t) = E(x(t+1) \mid \mathcal{Y}_{t-1}, y(t))$$

Introduce

$$\tilde{y}(t) = y(t) - E(y(t) \mid \mathcal{Y}_{t-1})$$
$$= y(t) - C_2 \hat{x}(t \mid t-1)$$

$\mathcal{Y}_{t-1}$ and $\tilde{y}(t)$ independent.

$$E[x(t) - m_x | \mathcal{Y}_{t-1}, \tilde{y}(t)] =$$
$$E[x(t) - m_x | \mathcal{Y}_{t-1}] + E[z(t) - m_z | \tilde{y}(t)]$$

$\tilde{y}(t)$ "innovations"

Gram-Schmidt

Discrete Time Kalman Filter

$$x(t+1) = Ax(t) + B_1 e(t)$$
$$y(t) = C_2 x(t) + D_{21} e(t)$$

Let

$$P(t) := E(\tilde{x}(t|t-1)\tilde{x}^T(t|t-1))$$

be the covariance matrix for

$$\tilde{x}(t|t-1) = x(t) - \hat{x}(t|t-1)$$

Then

$$P(0) = P_0$$
$$\hat{x}(0|-1) = m_0$$

Kalman Filter for $x(t+1 \mid t)$

$$E(x(t+1) - A\hat{x}(t \mid t-1) \mid \mathcal{Y}_{t-1}) = 0$$

$$\begin{pmatrix} x(t+1) - A\hat{x}(t \mid t-1) \\ \tilde{y}(t) \end{pmatrix} = \begin{pmatrix} A & B_1 \\ C_2 & D_{21} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ e \end{pmatrix}$$

Hence $E(x(t+1) - A\hat{x}(t \mid t-1) \mid \mathcal{Y}_t) = K\tilde{y}(t)$

$$\hat{x}(t+1 \mid t) = A\hat{x}(t \mid t-1) + K(y - C_2\hat{x}(t \mid t-1))$$
$$K = (APC_2^T + B_1 D_{21}^T)(C_2 P C_2^T + D_{21} D_{21}^T)^{-1}$$
$$P(t+1) = APA^T + B_1 B_1^T - K[C_2 P C_2^T + D_{21} D_{21}^T]K^T$$

where $P = P(t)$.

Kalman Filter for $x(t \mid t)$

$E(x(t) - \hat{x}(t \mid t-1) \mid \mathcal{Y}_{t-1}) = 0$

$$\begin{pmatrix} x(t) - \hat{x}(t \mid t-1) \\ \tilde{y}(t) \end{pmatrix} = \begin{pmatrix} I & 0 \\ C_2 & D_{21} \end{pmatrix} \begin{pmatrix} \tilde{x} \\ e \end{pmatrix}$$

Hence $E(x(t) - \hat{x}(t \mid t-1) \mid \mathcal{Y}_t) = K_f \tilde{y}(t)$

$$\begin{aligned} \hat{x}(t \mid t) &= \hat{x}(t \mid t-1) + K_f(y - C_2\hat{x}(t \mid t-1)) \\ K_f &= PC_2^T(C_2PC_2^T + D_{21}D_{21}^T)^{-1} \\ P_f(t) &= P - PC_2^T(C_2PC_2^T + R_2)^{-1}CP \end{aligned}$$

where $P_f(t)$ is covariance for $x(t) - \hat{x}(t \mid t)$ and P is covariance for $x(t) - \hat{x}(t \mid t-1)$.

Innovations

Corollary

$$E(\tilde{y}(s)\tilde{y}(t)^T) = \begin{cases} 0 & s \neq t \\ C_2PC_2^T + D_{21}D_{21}^T & s = t \end{cases}$$

Proof $\tilde{y}(t)$ and $\mathcal{Y}_{t-1}$ are independent. If $s < t$ then $\tilde{y}(s)$ is a linear function of $\mathcal{Y}_{t-1}$.

Representation of y

$$\begin{aligned} \hat{x}(t+1) &= A\hat{x}(t) + K\tilde{y} \\ y(t) &= C_2\hat{x}(t) + \tilde{y}(t) \end{aligned}$$

Gives

$$\begin{aligned} &[C_2(sI - A)^{-1}K + I][C_2PC_2^T + D_{21}D_{21}^T][\ldots]^T \\ &= [C_2(sI - A)^{-1}B_1 + D_{21}][C_2(-sI - A)^{-1}B_1 + D_{21}]^T \end{aligned}$$

Duality

For convenience $R_{12} = 0$.

$$\begin{aligned} x(t+1) &= Ax(t) + v(t) \\ y(t) &= C_2x(t) + e(t) \end{aligned}$$

Estimate l^Tx linearly in $\mathcal{Y}_{t-1}$ so

$$\min E(l^T x(t_1) - l^T \hat{x}(t_1))^2$$

As the estimate is linear we have

$$l^T \hat{x}(t_1) = -\sum_{t=t_0}^{t_1-1} u^T(t)y(t) + b^T m$$

for some $u(t)$.

How choose $u(t), b$?

Introduce z as

$$z(t) = A^T z(t+1) + C_2 u(t+1)$$
$$z(t_1) = l$$

Can show

$$l^T(x(t_1) - \hat{x}(t_1)) = z^T(t_0 - 1)x(t_0) - b^T m + \sum_{t=t_0}^{t_1-1} [z^T(t)v(t) + u^T(t)e(t)]$$

Squaring and taking expectations gives

$$E[l^T x(t_1) - l^T \hat{x}(t_1)]^2 = [(z(t_0 - 1) - b)^T m]^2 + z^T(t_0 - 1)R_0 z(t_0 - 1) + \sum_{t=t_0}^{t_1-1} z^T(t)R_1 z(t) + u^T(t)R_2 u(t)$$

Optimal estimate independent of l.

Duality

Optimal Control Problem where

optimal control	state estimation
t	$-t$
t_0	t_1
t_1	t_0
A	A^T
B_2	C_2^T
C_1	B_1^T
D_{12}	D_{21}^T
S	P
L	K

Continuous Time

Stochastic Integrals, hard

In Anderson-Moore the duality calculation is done in continuous time

Hard (?) to prove that linear estimates are optimal

Hard (?) to prove separation principle rigorously

Anderson-Moore/Åström

pp. 182-195 in Anderson-Moore
pp. 241-248 in Åström 1970

$$\dot{x} = Ax(t) + B_1 e(t)$$
$$y(t) = C_2 x(t) + D_{21} e(t)$$

$E(e(t)) = 0$ and $E(e(s)e^T(t) = I\delta(t-s)$.

$E(x(t_0)) = m$ and $E(x(t_0) - m)(x(t_0) - m)^T = P_0$

Everything Gaussian

Assume Linear Estimates

Admissible estimates

$$l^T \hat{x}(t_1) = -\int_{t_0}^{t_1} u^T(t) y(t) dt + b^T m$$

Criterion

$$E[l^T x(t_1) - l^T \hat{x}(t_1)]^2$$

Duality

Introduce z through

$$\dot{z} = -A^T z - C_2^T u$$
$$z(t_1) = l$$

Then

$$E[l^T x(t_1) - l^T \hat{x}(t_1)]^2 = [(z(t_0) - b)^T m]^2 + z^T(t_0) P_0 z(t_0)$$
$$+ \int_{t_0}^{t_1} \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}^T \begin{pmatrix} B_1 \\ D_{21} \end{pmatrix} \begin{pmatrix} B_1^T & D_{21}^T \end{pmatrix} \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}$$

Optimal Control Problem $\Rightarrow u(t)$

Optimal Estimator

Rewriting the estimate

$$l^T \hat{x}(t_1) = -\int_{t_0}^{t_1} u^T(t) y(t) dt + b^T m$$

in recursive form gives

$$\dot{\hat{x}} = A\hat{x} + K(y - C_2 \hat{x})$$

where

$$\dot{P} = AP + PA^T + B_1 B_1^T - K D_{21} D_{21} K^T$$
$$K = P C_2^T + B_1 D_{21}^T (D_{21} D_{21})^{-1}$$

Introducing u

If the state equation is changed to

$$\dot{x} = Ax + B_1 e + B_2 u$$

the estimator changes to

$$\dot{\hat{x}} = A\hat{x} + B_2 u + K(y - C_2 \hat{x})$$

Proof Add a deterministic signal $B_2 u$ to all calculations

Infinite Time

Assumptions 1:

$$[A, C_2] \text{ detectable}$$

Assumptions 2:

$$\text{rank} \begin{pmatrix} j\omega I - A & B_1 \\ C_2 & D_{21} \end{pmatrix} = n + p \quad \forall \omega$$

and rank$D_{21} = p$ ("noise on all measurements")

Then there exists solutions P, K to

$$\begin{aligned} 0 &= AP + PA^T + B_1 B_1^T - K D_{21} D_{21} K^T \\ K &= (PC_2^T + B_1 D_{21}^T)(D_{21} D_{21})^{-1} \end{aligned}$$

so that

$$A - KC_2$$

is asymptotically stable

Proof: Duality

Representation of y

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + K\tilde{y} \\ y &= C_2\hat{x} + \tilde{y} \end{aligned}$$

Gives

$$\begin{aligned} &[C_2(sI - A)^{-1}K + I]D_{21}D_{21}^T[\ldots]^T \\ &= [C_2(sI - A)^{-1}B_1 + D_{21}][C_2(-sI - A)^{-1}B_1 + D_{21}]^T \end{aligned}$$

$\tilde{y}(t)$ orthogonal "innovations"

The H_2 approach

Output Estimation Problem Find $\hat{z}$ which minimize

$$\min E\|\hat{z} - Lx\|_2^2$$

This means minimize H_2 norm from input w to output $\hat{z} - Lx$ where system is

$$\begin{pmatrix} \dot{x} \\ \hat{z} - Lx \\ y \end{pmatrix} = \begin{pmatrix} A & B_1 & 0 \\ -L & 0 & I \\ C_2 & D_{21} & 0 \end{pmatrix} \begin{pmatrix} x \\ w \\ \hat{z} \end{pmatrix}$$

H_2 approach

Idea: Let any estimator be given by

$$\begin{aligned} \dot{\xi} &= F\xi + Gy \\ \hat{x} &= H\xi + Jy \end{aligned}$$

Calculate H_2-norm for extended system with state

$$\begin{pmatrix} x \\ \xi \end{pmatrix}$$

Show that $\|\cdot\|_2^2 \geq \text{trace}(LPL^T)$

Equality obtained for Kalman Filter

Details are left as an exercise

Linear Quadratic Control Design 94

Lecture 5

Output Feedback

Ch 8

- Separation Principle
- Example 1 p. 222
- Example 2 p. 232
- LQG/LTR
- Example 3 p. 233
- Q-Parametrization
- An H_2 proof of continuous time

The Optimal Controller

Let the system be given by

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + \qquad D_{12} u \\ y &= C_2 x + D_{21} w + D_{22} u \end{aligned}$$

Optimal controller

$$\begin{aligned} u &= -L\hat{x} \\ \dot{\hat{x}} &= A\hat{x} + B_2 u + K(y - C\hat{x} - D_{22} u) \\ L &= (D_{12}^T D_{12})^{-1}(D_{12}^T C_1 + B_2^T S) \\ K &= (B_1 D_{21}^T + P C_2^T)(D_{21} D_{21}^T)^{-1} \end{aligned}$$

where $P \geq 0$ and $S \geq 0$ satisfy

$$\begin{aligned} 0 &= SA + A^T S + C_1^T C_1 - L^T D_{12}^T D_{12} L \\ 0 &= AP + PA^T + B_1 B_1^T - K D_{21} D_{21}^T K^T \end{aligned}$$

such that $A - B_2 L$ and $A - KC_2$ stable

"Technical Conditions"

1) $[A, B_2]$ stabilizable

2) $[C_2, A]$ detectable

3) "No zeros on imaginary axis" $u \rightarrow z$

$$\text{rank} \begin{pmatrix} j\omega I - A & -B_2 \\ C_1 & D_{12} \end{pmatrix} = n + m \qquad \forall \omega$$

and D_{12} have full column rank.

4) "No zeros on imaginary axis" $w \rightarrow y$

$$\text{rank} \begin{pmatrix} j\omega I - A & -B_1 \\ C_2 & D_{21} \end{pmatrix} = n + p \qquad \forall \omega$$

and D_{21} have full row rank.

Separation Principle, basic idea

$$u + Lx = u + L\hat{x} + L\tilde{x}$$

$$\begin{aligned} E[(u+Lx)^T(u+Lx)] = E\ \text{tr}[(u+Lx)(u+Lx)^T] = \\ E(u+L\hat{x})^T(u+L\hat{x}) + \\ E\ \text{tr}(u+L\hat{x})\tilde{x}^T L^T + \\ E\ \text{tr}(L\tilde{x}\tilde{x}^T L^T) \end{aligned}$$

Orthogonality theorem:

$$E(u\tilde{x}^T) = 0 \qquad E(\hat{x}\tilde{x}^T) = 0$$

$\Rightarrow$ minimum for $u = -L\hat{x}$ and

$$\min_u E[(u+Lx)^T(u+Lx)] = \text{tr}\ (LPL^T)$$

where $P = E\ \tilde{x}\tilde{x}^T$.

Bellman Equation

$$G(\mathcal{Y}_t) = \min_{u(\cdot)} E(\sum_{t_0}^{t_1} \cdots \mid \mathcal{Y}_t)$$

Then if $u(t) = f(\mathcal{Y}_{t-1})$:

$$G(\mathcal{Y}_{t-1}) = \min_{u_t} E[G(\mathcal{Y}_t) \mid \mathcal{Y}_{t-1}, u_t]$$

or if $u(t) = f(\mathcal{Y}_t)$:

$$G(\mathcal{Y}_t) = \min_{u_t} E[G(\mathcal{Y}_{t+1}) \mid \mathcal{Y}_t, u_t]$$

- Markov Dynamics
- Decomposable cost function

For a proof see [Åström, 1970]

Discrete time $u = -L\hat{x}(t|t-1)$

Using previous slides on

$$V = x^T(0)S(0)x(0) + \sum_{k=0}^{N-1}[u+Lx]^T[B_2^TSB_2+Q_2][u+Lx]$$
$$+\sum_{k=0}^{N-1} w^TB_1^TSB_1w + \sum_{k=0}^{N-1} w^TB_1^TS[Ax+B_2u] +$$
$$+\sum_{k=0}^{N-1} S[Ax+B_2u]^TSB_1w$$

gives since $E(u(k)w^T(k)) = 0$

$$J(Y_{t-1}) = \min_u EV = \underbrace{m^TS(0)m + \text{ tr } R_0S(0)}_{\text{cost for initial condition}} +$$
$$\underbrace{\sum_{k=0}^{N-1} \text{ tr } B_1^TSB_1}_{\text{cost of state feedback}} + \underbrace{\sum_{k=0}^{N-1} \text{ tr } L^T[B_2^TSB_2+Q_2]LP}_{\text{cost of state estimation}}$$

Discrete time $u = -L\hat{x}(t|t) - L_v\hat{v}(t|t)$

As before but with

$$V = x^T(0)S(0)x(0) +$$
$$+\sum_{k=0}^{N-1}[u+Lx+L_vv]^T[B_2^TSB_2+Q_2][u+Lx+L_vv] + \ldots$$

gives

$$J(Y_t) = J(Y_{t-1}) - \underbrace{\sum_{k=0}^{N-1} \text{tr } M[CPC^T+R_2]M^T[B_2^TSB_2+Q_2]}_{\text{decreased cost when } y(t) \text{ is measured}}$$

Details in TFRT-7475

Continuous Time $u(t) = -L\hat{x}(t)$

Harder mathematics

Derivate $E(x^TSx)$ when $\dot{x} = v$. (answer: SR_1)

Ito-calculus, stochastic integrals

"Such derivations are beyond the scope of this text"

Closed Loop, continuous time

$$\begin{pmatrix} \dot{x} \\ \dot{\tilde{x}} \end{pmatrix} = \begin{pmatrix} A - B_2L & B_2L \\ 0 & A - KC_2 \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix} + \begin{pmatrix} B_1 \\ B_1 - KD_{21} \end{pmatrix} w + \begin{pmatrix} I \\ 0 \end{pmatrix} u_r$$

$$z = \begin{pmatrix} C_1 - D_{12}L & D_{12}L \end{pmatrix} \begin{pmatrix} x \\ \tilde{x} \end{pmatrix}$$

$\tilde{x}$ uncontrollable from u_r

$A - KC_2$ dynamics cancelled from u_r to y.

Controller, continuous time

$$\dot{\hat{x}} = A\hat{x} + B_2u + K(y - C_2\hat{x} - D_{22}u)$$
$$u = -L\hat{x} + u_r$$

$$u = G_{ff}u_r - G_{fb}y$$
$$G_{fb} = L(sI - A + B_2L + KC_2 - KD_{22}L)^{-1}K$$
$$G_{ff} = -L(\ldots)^{-1}(B_2 - KD_{22}) + I$$

Loop gain

$$W_{OL} = G_{fb}(s)P(s)$$

where $P(s) = C_2(sI - A)^{-1}B_2 + D_{22}$

No longer $W_{OL} = L(sI - A)^{-1}B_2$ as in LQ

Loss of guaranteed margins

Example 1, p. 222

LQG1

$$\min E[y_f^2 + y_a^2 + 0.2u^2]$$

LQG2

$$\min E[y_f^2 + y_a^2 + 4x_3^2 + 4x_4^2 + u^2]$$

Plot control signal also !

Results, Example1

Would not recommend the LQG2-design

`/home/fulqg/lqg94/matlab/fig822.m`
(Very ugly code)

Example 2, p.232

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 1 \\ 1 \end{pmatrix} v$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \sigma w$$

$$\min E[x_1^2 + x_2^2 + \rho u^2]$$

What happens as $\rho \to 0$ and $\sigma \to 0$?

Result, Example 2

Terrible gain and phase margins

/home/fulqg/lqg94/matlab/doyle.m

LQG/LTR

Loop Transfer Recovery

Goal: To recover a return difference close to

$$L(sI - A)^{-1}B_2$$

Idea: Add fictious input noise :

$$R_1 := R_1 + qB_2B_2^T$$

For minimum phase systems this gives

$$\lim_{q \to \infty} G_{fb}(s)P(s) = L(sI - A)^{-1}B_2$$

Easy to try this idea, doesn't work always

Dont let $q \to \infty$

Example 3, pp 232-6 and 246-250

See Figure 8.3-4

1 Nominal LQG
2 LQG/LTR
3 (LQG/LTR with colored noise)

Warning: Calculations and plots in book have a lot of errors (also in article [6])

Results, Example3

LQG/LTR with $q = 0.1$ seems good

Hard (?) to find by pole placement (?)

3. Bad idea to do LTR at high frequencies

/home/fulqg/lqg94/matlab/fig834.m

Q-parametrization

$Q(s)$ stable $\Rightarrow$ stable closed loop

Equivalence

All stabilizing controllers can be obtained in this way (for different stable $Q(s)$)!

The closed loop can be written

$$T_{zw}(s) = T_{11}(s) + T_{12}(s)Q(s)T_{21}(s)$$

for some fixed T_{11}, T_{12}, T_{21}.

Idea: Optimize over $Q(s)$.

See linear systems course

An H_2 proof of continuous time

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + \; D_{12} u \\ y &= C_2 x + D_{21} w \end{aligned}$$

Assume for simplicity that

$$D_{12}^T \begin{pmatrix} C_1 & D_{12} \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$$
$$D_{21} \begin{pmatrix} B_1^T & D_{21}^T \end{pmatrix} = \begin{pmatrix} 0 & I \end{pmatrix}$$

H_2 proof

Let L be solution from Riccati, $A - B_2 L$ stable

Then (exercise)

$$T_{zw}(K) = G_c + U G_v(K)$$

where

$$G_c = \left\{ \begin{array}{cc} A - B_2 L & B_1 \\ C_1 - D_{12} L & 0 \end{array} \right\}$$
$$U = \left\{ \begin{array}{cc} A - B_2 L & B_2 \\ C_1 - D_{12} L & D_{12} \end{array} \right\}$$
$$G_v(K) = \left\{ \begin{array}{ccc} A & B_1 & B_2 \\ L & 0 & I \\ C_2 & D_{21} & 0 \end{array} \right\} \star K$$

Note that $G_c(s)$ and $U(s)$ stable

K stabilizes G_v $\Leftrightarrow$ K stabilizes G

H_2 proof

$$||T_{zw}||_2^2 = ||G_c + U G_v(K)||_2^2$$

Stability, so can change H_2 norm to L_2 norm

$$\begin{aligned} ||T_{zw}||_2^2 &= < G_c + U G_v(K), G_c + U G_v(K) >_{L_2} \\ &= < G_c, G_c >_{L_2} \\ &\quad +2 \text{ Re} < G_v(K), U^* G_c >_{L_2} \\ &\quad + < G_v(K), U^* U G_v(K) >_{L_2} \\ &= ||G_c||_2^2 + ||G_v(K)||_2^2 \end{aligned}$$

H_2 proof

Since (exercise)

$$U^T(-j\omega) U(j\omega) = I$$

and

$$U^T(-j\omega) G_c(j\omega) = -B_2^T (j\omega I + A + B_2 L)^{-1} S$$

so

$$< \underbrace{G_v(K)}_{\text{stable}}, \underbrace{U^* G_c}_{\text{anti-stable}} >_{L_2} = 0$$

Analytic function theory version of orthogonality theorem

H_2 proof

$$\min_K ||G_v(K)||_2^2$$

where

$$G_v(K) = \left\{ \begin{array}{ccc} A & B_1 & B_2 \\ L & 0 & I \\ C_2 & D_{21} & 0 \end{array} \right\} \star K$$

Output estimation problem (small modification of previous lecture, add B_2 term).

Solution: Kalman Filter

Minimum:

$$\min_K ||T_{zw}||_2^2 = \text{tr } (B_1^T S B_1) + \text{tr } (L P L^T)$$

(remember: no cross-terms here)

This completes our noise-free proof of continuous time LQG

Linear Quadratic Control Design 94

Lecture 6

Servo Problems, Loop Shaping

Ch 4 + (9)
CCS Ch 9.5

- Servo problems
- Integrators etc

Servo Problems

Which problem to solve?

If optimal control is going to be used we must assume something about future reference signal r

Three approaches (at least)

1. r is known in advance
2. r belongs to some signal class, e.g. is generated by some known linear system
3. r is a stochastic signal with known mean and variance

Servo problems, quick solution

Without any optimal control motivation:

Control signal

$$u = -L\hat{x} + l_r r$$

l_r usually chosen to get unit gain

$$[(C - DL)(-A + BL)^{-1}B + D]l_r = 1$$

Servo Problems ala CCS

Servo Problems ala CCS

Easiest if x and x_m have same physical interpretation

Example

Servo design on pendulum

Collocated sensor

Improve introduction of reference signals

Before

$$u = -L\hat{x} + l_r u_r$$

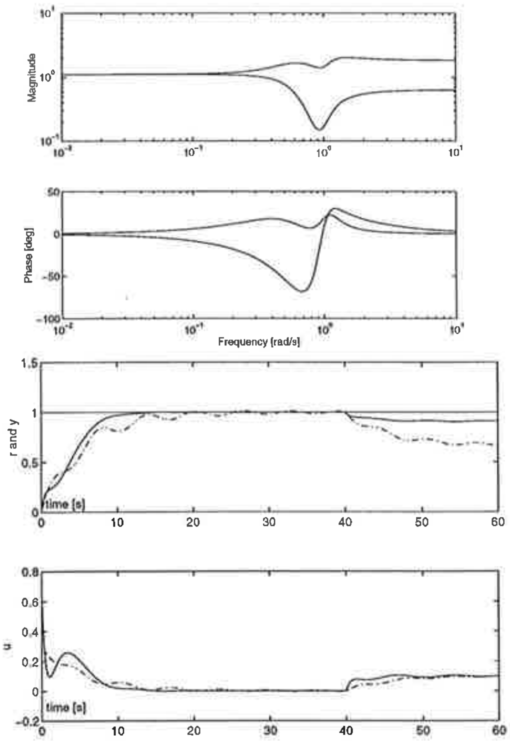
Good idea not to excite the oscillatory modes by the reference signal.

$$\frac{B_m}{A_m} = \frac{a_m}{s + a_m}$$

Example

Servo design on pendulum

Example



Approach 1, future known signals

Criteria

$$\dot{x}(t) = Ax(t) + Bu(t) + \alpha(t)$$

$$J = \int_0^T \left(\right)^T \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} x(\tau) - x_m(\tau) \\ u(\tau) - u_m(\tau) \end{pmatrix} d\tau$$
$$+(x(T) - x_m(T))^T Q_T (x(T) - x_m(T))$$

x_m, u_m, α known in advance.

Solution

$$\begin{pmatrix} 0 & sI - A & -B \\ -sI - A^T & Q_1 & Q_{12} \\ -B^T & Q_{12}^T & Q_2 \end{pmatrix} \begin{pmatrix} \lambda \\ x - x_m \\ u - u_m \end{pmatrix} = \begin{pmatrix} \alpha - \alpha_m \\ 0 \\ 0 \end{pmatrix}$$

where $\alpha_m = (\frac{d}{dt}I - A)x_m - Bu_m$.

Proof: Pontryagin

$$H = J + \lambda(Ax + Bu + \alpha)$$

$$\begin{aligned} \text{First row}: &\quad \dot{x} = \partial H / \partial \lambda \\ \text{Second row}: &\quad \dot{\lambda} = -\partial H / \partial x \\ \text{Third row}: &\quad 0 = \partial H / \partial u \end{aligned}$$

Boundary conditions

$$\begin{aligned} &x(0) = x_0 \\ &\lambda(T) = Q_T(x(T) - x_m(T)) \end{aligned}$$

Future known signals

Using

$$\Phi^T(-s)\Phi_0\Phi(s) \begin{pmatrix} \lambda \\ x - x_m \\ u - u_m \end{pmatrix} = \begin{pmatrix} \alpha - \alpha_m \\ 0 \\ 0 \end{pmatrix}$$

where

$$\Phi_0 = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & D^T \end{pmatrix}; \quad \Phi(s) = \begin{pmatrix} I & S & 0 \\ 0 & sI - A & -B \\ 0 & L & I \end{pmatrix}$$

Inverting $\Phi^T(-s)$ gives

$$D^T D[u - u_m + L(x - x_m)]$$
$$= -B^T(-\frac{d}{dt}I - A^T + L^T B^T)^{-1} S(\alpha - \alpha_m)$$

Recaption

Optimal control signal

$$u = u_m - L(x - x_m) - (D^T D)^{-1} B^T \sigma$$

where

$$-\dot{\sigma} = (A - BL)^T \sigma + S(\alpha - \alpha_m), \quad \sigma(T) = 0$$
$$\alpha_m = (\frac{d}{dt}I - A)x_m - Bu_m$$

Need future x_m, u_m, α.

Remark

Note that if

$$\alpha = 0$$
$$\alpha_m = (\frac{d}{dt}I - A)x_m - Bu_m = 0$$

then

$$u = u_m - L(x - x_m)$$

This structure is recommended in CCS, even for the case that we dont know u_m, x_m in advance. Then put the anticipative term α_m to zero.

Approach 2

Anderson-Moore:

$$\min \int (x - x_m)^T Q_1 (x - x_m) + u^T Q_2 u \, d\tau$$

where $x_m = L_2 y_m$ is the shortest vector x satisfying $y_m = C x_m$.

Approach 2

Solution

$$u = -Lx - L_1 z$$

where L_1 is given by Riccati equations involving both system A, B, Q_1, Q_2 and reference model A_m, C_m

Approach 3, stochastic r

$$r = G_r w_r$$

where w_r is white noise and G_r frequency shaping filter.

Large G_r where good tracking is wanted.

Many possibilities, for example

Integrator 1

CCS 271-273

Extend system with integrators

$$\dot{\bar{x}} = y_m - y$$

$$\min \int x^T Q_1 x + u^T Q_2 u + \bar{x}^T Q_3 \bar{x}$$

gives $\begin{pmatrix} L & \bar{L} \end{pmatrix}$. Kalman filter as before.

$\bar{x}$ noise-free so nonstandard LQG

(D_{21} not full rank).

Integrator 1

Use controller

$$u = -L\hat{x} - \bar{L}\bar{x} + \tilde{u}_c$$

(is this the limit as $\sigma_2 \to 0$?)

Increased order model (A_m in CCS)

Observer order (A_o in CCS) not increased

Integrator, 2

Extend system with fictious bias signals

Non-stabilizable states so nonstandard LQG

Integrator, 2

Use controller (for $D = 0$)

$$\frac{d}{dt}\hat{x} = \begin{pmatrix} A & B_v \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} u + K(y - \begin{pmatrix} C & 0 \end{pmatrix} \hat{x})$$
$$u = -\begin{pmatrix} L & L_{n+1} \end{pmatrix} \hat{x}$$

where L_{n+1} is chosen to cancel bias at outputs

$$C(A - BL)^{-1}(B_v - BL_{n+1}) = 0 \quad \text{(for } D = 0\text{)}$$

Integrator 2

Controller has integrating action

Proof Controller has A matrix (for $D = 0$)

$$\begin{pmatrix} A - BL - K_1C & B_v - BL_{n+1} \\ -K_2C & 0 \end{pmatrix}$$

which is singular. Hence pole at $s = 0$, i.e. integrator in controller.

Increased observer order (A_o)

Not increased model order (A_m)

(what if $D \neq 0$?)

Pre-specified factors in $R(s)$

These approaches can be generalized to other pre-specified modes in the controller

Change $1/s$ to a $1/R_1(s)$.

Prespecified factors in $S(s)$

Want pre-specified transmission zero of LQG-controller

Exercise

Prespecified factors in $T(s)$

Exercise

Numerical Solution of Riccati equations

941212, PH

$$A^TS + SA + Q_1 - SBQ_2^{-1}B^TS = 0$$
$$A^TSA + Q_1 - A^TSB(B^TSB + Q_2)^{-1}B^TSA = S$$

$$S \geq 0, \quad \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} = \begin{pmatrix} C & D \end{pmatrix}^T \begin{pmatrix} C & D \end{pmatrix}$$

Ref. Bittanti-Laub-Willems, 1990.

Solve diff. equation to stationarity

Unique solution? Influence of intial condition? Asymptotic rate of convergence?

Doubling algorithms, square root formulation

Newton refinement – Kleinman

Approximation S_k gives $L_k = Q_2^{-1}B^TS_k$. Solve Lyapunov equation

$$(A - BL_k)^TS + S(A - BL_k) + Q_1 + L_k^TQ_2L_k = 0$$

for new approximation S_{k+1}. Converges for stable $A - BL_0$, quadratically. C.f exercise.

Euler matrix

Lagrange multiplier $p = Sx$, $u = -Lx$
stable invariant subspace

$$\begin{pmatrix} 0 & sI - A & -B \\ -sI - A^T & C^TC & C^TD \\ -B^T & D^TC & D^TD \end{pmatrix} \begin{pmatrix} p \\ x \\ u \end{pmatrix} = 0$$
$$\begin{pmatrix} 0 & zI - A & -B \\ z^{-1}I - A^T & C^TC & C^TD \\ -B^T & D^TC & D^TD \end{pmatrix} \begin{pmatrix} p \\ x \\ u \end{pmatrix} = 0$$

Hamilton Jacobi Bellman – LMI-solution

Completion of squares – Schur complement

$$(Cx + Du)^T(Cx + Du) + (Ax + Bu)^TS(Ax + Bu) =$$
$$(Lx + u)^TG(Lx + u) + x^T(S - \Delta S)x$$
$$G = B^TSB + D^TD, \quad GL = B^TSA + D^TC$$

$$\begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} =$$
$$\begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S - \Delta S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

Similarly for continuous time

$$(Cx + Du)^T(Cx + Du) +$$
$$x^TS(Ax + Bu) + (Ax + Bu)^TSx =$$
$$(Lx + u)^TD^TD(Lx + u) - x^T\dot{S}x$$

$$\begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} A^TS + SA + C^TC & SB + C^TD \\ B^TS + D^TC & D^TD \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} =$$
$$\begin{pmatrix} x \\ u \end{pmatrix}^T \begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} -\dot{S} & 0 \\ 0 & D^TD \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix}$$

Invariant subspace methods

Use stable eigenvectors:

$$E = \begin{pmatrix} A & -BQ_2^{-1}B^T \\ -Q_1 & -A^T \end{pmatrix}, \quad ET = T \begin{pmatrix} -\Lambda & 0 \\ 0 & \Lambda \end{pmatrix}$$

to form $S = T_{21}T_{11}^{-1}$,
or any stable invariant subspace

$$EX = XA_c$$
$$AX_1 - BQ_2^{-1}B^TX_2 = X_1A_c$$
$$-Q_1X_1 - A^TX_2 = X_2A_c$$
$$= X_2X_1^{-1}(A - BQ_2^{-1}B^TX_2X_1^{-1})X_1$$

i.e. the Riccati equation with $S = X_2X_1^{-1}$.
X_1 invertible? $S \geq 0$? A_c stable?

Ordered Schur form, triangular
first step in the QR-iteration
see course in matrix theory or the LAPACK-manual (Mosaic).
Pencils and generalized eigenvalues.
New `qzorder` in Matlab.

Singular Discrete-Time Riccati-Equations

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When solving the discrete-time Riccati equation there are many practical cases where common software does not behave well. We will sort out cases that fail just because of a bad algorithm design from cases that are really difficult. The standard form of the discrete-time Riccati equation is

$$S = A^T SA + Q_1 - A^T SB(B^T SB + Q_2)^{-1} B^T SA$$

where $Q_1 = C_1^T C_1$ and $Q_2 = D_2^T D_2$. Symmetric solutions $S \geq 0$ correspond to

$$\min_{u=-Lx} EJ, \qquad J = x^T Q_1 x + u^T Q_2 u = z^T z$$

$$qx = Ax + Bu + v, \quad z = \begin{pmatrix} C_1 x \\ D_2 u \end{pmatrix}, \quad x \in R^n, \quad u \in R^m$$

where q is the forward shift operator, and where v is discrete time white noise with unit covariance. Also require that $\{A, B\}$ is stabilizable.

Some codes, like the one still in the Matlab Control System Toolbox, require invertible Q_2 and also A. This is just bad choice of solution method. Using such software you are not able to solve a minimum-variance problem or a problem with a pure time-delay. You fail on many examples in standard texts like [1]. Another weakness with many codes is that they don't handle crossterms in the loss J correctly, i.e. when

$$z = Cx + Du$$

with $C^T D \neq 0$. Crossterms appear e.g. for problems on polynomial form, $A(q)y = B(q)u + C(q)v$, and their inclusion should be straightforward for most methods.

The maximal positive semidefinite Riccati-solution S corresponds to closed-loop eigenvalues λ_i of $A - BL$ inside or on the unit circle. Systems with some $|\lambda_i| = 1$ show a first type of singularity. This happens only if λ_i is also a zero of $H(q) = C(qI - A)^{-1}B + D$, the transfer operator from u to z. The solution S is then singular, e.g. [2]. A second type of singularity occurs when $(B^T SB + Q_2)$ is singular giving nonuniqueness in the state-feedback $u = -Lx$

$$(B^T SB + Q_2)L = B^T SA + D^T C$$

Some of the solutions L may give $|\lambda_i| > 1$. That singularity shows up when $H(q)$ lacks left-invertibility, i.e. when there are "redundant" control signals, e.g. [3].

Many codes have difficulties with the first type, and the second type is a major challenge to most algorithms. Still both types are often possible to handle, if great care is taken in the design of the numerical methods. Really singular cases may occur when small changes of $[A, B; C, D]$ make an uncontrollable mode controllable by a control-signal without penalty. Then S could be discontinuous like in $[1/2, \delta; 1, 0]$, where $S = 4/3$ for $\delta = 0$ but $S = 1$ for $\delta \neq 0$.

A general form of the Riccati equation, covering crossterms and singularities, is

$$S + L^T GL = A^T SA + C^T C, \qquad GL = B^T SA + D^T C, \qquad G = B^T SB + D^T D$$

i.e. $$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

Singular Discrete-Time Riccati-Equations

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Discrete-time Riccati equation

$$S = A^T SA + Q_1 - A^T SB(B^T SB + Q_2)^{-1} B^T SA$$

Symmetric solutions $S \geq 0$ correspond to

$$\min_{u=-Lx} EJ, \qquad J = x^T Q_1 x + u^T Q_2 u = z^T z$$

$$qx = Ax + Bu + v, \qquad z = \begin{pmatrix} C_1 x \\ D_2 u \end{pmatrix}, \; x \in R^n, \; u \in R^m$$

where $Q_1 = C_1^T C_1$ and $Q_2 = D_2^T D_2$.

Bad algorithms or difficult problem?

Standard codes often fail for standard ex's.

Singular Q_2	– minimum-variance control
Singular A	– time-delays
Cross-terms, $z = Cx + Du$ with $C^T D \neq 0$	– sampling or polynomial form

Two types of singularity

Type 1:

λ_i of $A - BL$ on unit circle
λ_i also a zero of $H(q) = C(qI - A)^{-1}B + D$

Ex:

$$H(q) = (q^2 - q + 1)/q^3$$

Notice $H(q)$ has no zeros when $Q_2 > 0$

Type 2:

$(B^T SB + Q_2)$ is singular
State-feedback $u = -Lx$ nonunique
Some solutions L may give $|\lambda_i| > 1$
$H(q)$ lacks left-invertibility

Ex:

$$qx = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} x + u$$
$$J = (u_1 + u_2 - x_1)^2 + (x_1 + x_2)^2$$

How hard?

Type 1 difficult for many algorithms
Type 2 major challenge to most algorithms

Both types often possible to handle, provided great care in design of the numerical methods.

For Type 2 we have that S is often discontinuous for some small changes of $[A, B; C, D]$.

Ex: $qx = x/2 + \delta u$, $J = x^2$ giving

$$S = \begin{cases} 4/3 & \text{if } \delta = 0 \\ 1 & \text{otherwise} \end{cases}$$

Here a mode becomes controllable by a control-signal without penalty.

The parameter structure is important.

LQ-optimal control using Lagrange multipliers

First order optimality conditions:

$$\begin{pmatrix} 0 & E & 0 \\ -A^T & 0 & 0 \\ -B^T & 0 & 0 \end{pmatrix} \begin{pmatrix} \lambda(k+1) \\ x(k+1) \\ u(k+1) \end{pmatrix} = \begin{pmatrix} 0 & A & B \\ -E^T & C^TC & C^TD \\ 0 & D^TC & D^TD \end{pmatrix} \begin{pmatrix} \lambda(k) \\ x(k) \\ u(k) \end{pmatrix} \tag{1}$$

$$E^T\lambda(N) = C_N^TC_Nx(N)$$

$$x(0) = x_0$$

$\lambda(k+1) = S(k+1)Ex(k+1)$ gives

$$D^TCx(k) + D^TDu(k) = -B^T\lambda(k+1) = -B^TS(k+1)\left[Ax(k) + Bu(k)\right]$$

$$u(k) = -L(k)x(k)$$

$$\left[D^TD + B^TS(k+1)B\right]L(k) = D^TC + B^TS(k+1)A$$

Now find (using `qzorder`) orthogonal Q and Z such that

$$\begin{aligned} Q^T \begin{pmatrix} 0 & E & 0 \\ -A^T & 0 & 0 \\ -B^T & 0 & 0 \end{pmatrix} Z &= \begin{pmatrix} E_c & * & * \\ 0 & E_u & * \\ 0 & 0 & 0 \end{pmatrix} \\ Q^T \begin{pmatrix} 0 & A & B \\ -E^T & C^TC & C^TD \\ 0 & D^TC & D^TD \end{pmatrix} Z &= \begin{pmatrix} A_c & * & * \\ 0 & A_u & * \\ 0 & 0 & * \end{pmatrix} \end{aligned} \tag{2}$$

where $[zE_c - A_c]$ has all eigenvalues inside (or on) the unit circle. With

$$\begin{pmatrix} \lambda(k) \\ x(k) \\ u(k) \end{pmatrix} = \begin{pmatrix} Z_{11} \\ Z_{21} \\ Z_{31} \end{pmatrix} Z_{21}^{-1}x(k) = \begin{pmatrix} SE \\ I \\ -L \end{pmatrix} x(k)$$

we get that the "stable", "optimal", closed loop

$$Q^T \begin{pmatrix} 0 & E & 0 \\ -A^T & 0 & 0 \\ -B^T & 0 & 0 \end{pmatrix} Z \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} Z_{21}^{-1}x(k+1) =$$

$$Q^T \begin{pmatrix} 0 & A & B \\ -E^T & C^TC & C^TD \\ 0 & D^TC & D^TD \end{pmatrix} Z \begin{pmatrix} I \\ 0 \\ 0 \end{pmatrix} Z_{21}^{-1}x(k)$$

or

$$\begin{pmatrix} E_c \\ 0 \\ 0 \end{pmatrix} Z_{21}^{-1}x(k+1) = \begin{pmatrix} A_c \\ 0 \\ 0 \end{pmatrix} Z_{21}^{-1}x(k) \tag{3}$$

satisfies the first order optimality conditions.

Exercise 1

1. Compute the H_2-norm of the continuous time systems $G(s) =$ a) 1, b) $\frac{1}{s+a}$, c) $\frac{1}{s^2+2\zeta\omega s+\omega^2}$

2. Is the H_2-norm invariant under time-delays, i.e. is

$$\|G(s)\|_2 = \|G(s)e^{-s\tau}\|_2?$$

3. Show that if $G(s)$ is stable and $H(s)$ anti-stable (both strictly proper) then

$$< H, G >_{L_2} = 0$$

"Stable and anti-stable systems are orthogonal.

4. Prove or disprove that

a.

$$\|G_1 + G_2\|_2 \leq \|G_1\|_2 + \|G_2\|_2$$

b.

$$\|G_1 G_2\|_2 \leq \|G_1\|_2 \|G_2\|_2$$

5. Prove that if g is a SISO system then

$$\max_{u \neq 0} \frac{\|gu\|_\infty}{\|u\|_2} = \|g\|_2$$

where $\|z\|_\infty = \sup_t |z(t)|$ and $\|z\|_2^2 = \int_{-\infty}^{\infty} |u(t)|^2 dt$ for a signal $z(t)$.

6. Which of the following qualifies as norms of smooth signals $u(t)$?

a. $\sup |\dot{u}(t)|$

b. $|u(0)| + \sup |\dot{u}(t)|$

7. Prove that

$$\|G\|_2^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_i \sigma_i^2(G(j\omega)) d\omega$$

where σ_i denotes the singular values.

8. Prove or disprove that

$$\sup_{\|u\|_a \leq 1} \|Gu\|_b = \sup \frac{\|Gu\|_b}{\|u\|_a}$$

where $\|\cdot\|_a$ and $\|\cdot\|_b$ are arbitrary signal norms.

9. Use the Riccati equation

$$-\dot{S} = SA + A^T S + Q_1 - SBB^T S, \quad S(t_1) = Q_0$$

to rewrite

$$\int_{t_0}^{t_1} \frac{d}{dt}(x^T Sx)dt$$

in order to prove that

$$\begin{aligned} J &= \int_{t_0}^{t_1} x^T Q_1 x + u^T Q_2 u dt + x^T(t_1) Q_0 x(t_1) \\ &= \int_{t_0}^{t_1} (u + Q_2^{-1} B^T Sx)^T Q_2 (u + Q_2^{-1} B^T Sx) dt + x^T(t_0) S(t_0) x(t_0) \end{aligned}$$

Remark: This trick probably gives the most direct way to solve the LQ-problem, and it actually provides both necessary and sufficient conditions. The discrete time version is given in CCS p. 343

In the following problems L can be any linear mapping between two inner product spaces. The norm is given by $||x||^2 = <x, x>$ and one says that L^* is an adjoint to L if $<L^*y, x> = <y, Lx>$. If you do not understand what this means, think of L as a matrix and L^* as the transposed matrix.

10. Show that

$$\min_u ||u||, \quad \text{such that } <z, u> = a$$

is solved by

$$u_{min} = az/||z||^2$$

11. If LL^* is invertible then show that

$$\min_u ||u||, \quad \text{such that } Lu = x$$

is solved by

$$u_{min} = L^*(LL^*)^{-1}x$$

the minimum is given by

$$||u_{min}||^2 = <(LL^*)^{-1}x, x>$$

12. Let $Lu_{[0,t_1]} = \int_0^{t_1} e^{(t_1-t)A} Bu(\tau)dt$

a. Show that $L^* x_1 = B^T e^{(t_1-t)A^T} x_1$.

b. Show that

$$\exists u : ||u||_2 \leq 1, x = Lu \quad \Leftrightarrow \quad x^T P^{-1} x \leq 1$$

where $P = \int_0^{t_1} e^{At} BB^T e^{A^T t} dt$.
"The Gramian P describes the directions in which it is easy to control".

Exercise 2

1. =2.2.1

2. =2.3.5

3. =2.3.6

4. =3.2.1

5. =3.2.2

6. =3.2.4

7. =3.2.8

8. =3.2.9-12

Exercise 3

1. =5.2.1

2. =5.3.1

3. =5.4.4

4. Show that

$$\begin{pmatrix} 0 & sI - A & -B \\ -sI - A^T & C^TC & C^TD \\ -B^T & D^TC & D^TD \end{pmatrix} = \Phi^T(-s)\Phi_0\Phi(s)$$

where

$$\Phi_0 = \begin{pmatrix} 0 & I & 0 \\ I & 0 & 0 \\ 0 & 0 & D^TD \end{pmatrix}; \quad \Phi(s) = \begin{pmatrix} I & S & 0 \\ 0 & sI - A & B \\ 0 & L & I \end{pmatrix}$$

and that

$$\det\Phi(s) = \det(sI - A + BL)$$

Also deduce the return difference formula from this.

5. In the MIMO case: Investigate the closed loop poles as $\rho \to 0$ for CB invertible. A reference could be Molander, Egardt, Int. J. Control 28, p. 253.

6. An iterative method for refinement of a solution is the Kleinman "Newton-Raphson" method. You first stabilize the system using a state feedback L_0. Then solve S_0 from the Lyapunov equation

$$S_0(A - BL_0) + (A - BL_0)^T S_0 = -Q_1 - L_0^T Q_2 L_0$$

Then choose $L_1 = -Q_2^{-1}B^TS_0$ and use that as the suggested feedback instead. Solve for S_1 etc. Show that $S_{i-1} - S_i \geq 0$ and that S_i converges to the Riccati Solution.

7. How does the formula

$$\rho A^T(-s)A(s) + B^T(-s)B(s) = \rho P(-s)P(s)$$

change when cross-terms are introduced?

8. Choose a system for the last hand-in.

Exercise 4

1. 7.2.7-8 (but with -2 changed to 0, i.e. a double integrator).

2. Show that when R_1 and R_2 are changed to αR_1 and αR_2 then the estimator dont change.

3. 7.3.3

4. 7.3.4

5. 7.3.6

6. 7.3.7

7. 7.3.8

8. Construct the optimal estimator for

$$\dot{x} = 0; \qquad y = x + e$$

where $E(x(0)) = m$, $E(x(0)^2) = \sigma_0^2$, $E(e) = 0$ and $E(e^2) = \sigma^2$. What happens in stationarity? *(This is a common model for an unknown bias.)*

Exercise 5

Remember: Bode diagram, Nyquist, Pole/zeros, Root-loci, sensitivity ....
MATLAB: `lqrc, lgec,lqgc,frbox,frcss,symlocc`

It would be nice if every exercise have been done by at least one of you. You probably dont have time to do all of them. Hence it might be a good idea if you randomly choose which to do first. What "Comment" means below is up to you!

1. Plot symmetric root-loci for

a. $G_1(s) = 1/s^2$

b. $G_2(s) = (s+1)/s^2$

c. $G_3(s) = 50(s+1)/(s+10)(s-5)$

d. $G_4(s) = 20/(s-1)(s^2+10s+25)$

e. $G_5(s) = (s-1)/s^2$

2. Consider the system from AC Aug 79 by Doyle and Stein:

$$\dot{x} = \begin{pmatrix} -4 & -3 \\ 1 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u + \begin{pmatrix} -61 \\ 35 \end{pmatrix} v$$
$$y = \begin{pmatrix} 1 & 2 \end{pmatrix} x + w$$

Check the details in the article. Calculate the loop transfer for LQG and LQG/LTR with $q = 0, 10^2, 10^3, 10^4$. Evaluate the designs in all ways you find interesting. Comment!

3. Read the design example in Franklin et.al "Feedback Control of Dynamical Systems" pp. 506. Comment!

4. Read the design example in the CCS-course (flexible servo), see TFRT-7456. Comment!

5. What happens if you do LQG/LTR on the nonminimumphase system

$$\frac{s-1}{(s+1)^2}$$

Evaluate the designs in all ways you find interesting. Comment!

6. Read the design example of LQG in Maciejowski Ch 5.8 pp. 244-259. (We will talk more about introduction of integrators in the next lecture.) Comment!

7. Read the mosaic-page on LQG-surf. Find an LQG-article.

8. *For the H_2-fan:* Check that

$$T_{zw}(K) = G_c + UG_v(K)$$
$$U^T(-j\omega)U(j\omega) = I$$
$$U^T(-j\omega)G_c(j\omega) = -B_2^T(j\omega I + A + B_2 L)^{-1}S.$$

(notation as in the lecture). Check also the formula for the optimal H_2-norm. *(Hard. Only for the duality master* Sketch a dual proof to the H_2-proof in the lecture. Also write down the discrete time H_2 proofs of LQG.

Exercise 6

1. Find the optimal control signal for $\dot{x} = -x + u, \;\; x(0) = 0$ that minimizes

$$\int_0^{10} (x - x_m)^2 + (u - u_m)^2 \, dt$$

where $x_m = 0$ for $t < 5$ and $x_m = 1$ for $t \geq 5$. Do this both for the case $u_m = 0$ and $u_m = x_m$. Note: u_m and x_m are signals known in advance.

2. Show that extending the system with integrators at the output and using

$$u = -L\hat{x} - L_{n+1}x_{n+1}$$

gives a controller with integrating action.
How about extending the system with integrators at the input and minimizing

$$\int y^2 + \rho(\dot{u})^2$$

using state feedback with Kalman filter. Will this always give integrating action in the controller?

3. Show how to introduce prescribed factors of R, S and T in the LQG-formalism (when the LQG controller is written in the form $R(s)u = -S(s)y + T(s)r$).

4. (Hard) Show how to obtain smoothing formulas (giving $\hat{x}(t|t+m)$ for $m > 0$) using a calculation dual to the one in Lecture 6.

Exercise 7

1. Run the Matlab-demo `per/tex/undervisning/FFU/lqg/examples.m`. If you are interested in LMIs read the file lmid.m also.

2. Experiment with Riccati-solvers in Matlab. Try to figure out what the achievable performance is today. How large problems can we typically solve in reasonable time without too much numerical difficulties. What kind of problems are the hardest?

3. Implement the matrix sign algorithm

$$Z_{k+1} = \frac{1}{2c}(Z_k + c^2 Z_k^{-1})$$

where $Z_0 = H \in R^{q \times q}$ and $\text{sign}(H) := \lim Z_k$. Try both $c = 1$ and $c = |\text{det} Z_k|^{1/q}$. Use it to find the Riccati solution to

$$A^T S + SA + C^T C - SBB^T S = 0$$

by putting

$$Z = \text{sign} \begin{pmatrix} A & -BB^T \\ -C^T C & -A^T \end{pmatrix}$$

and solving

$$\begin{pmatrix} Z_{12} \\ Z_{22} + I \end{pmatrix} S = - \begin{pmatrix} Z_{11} + I \\ Z_{21} \end{pmatrix}.$$

Check your algorithm on some example. How is the performance on large problems? Does the algorithm seem numerically reliable?
(Research problem: Has anyone tried the matrix sign algorithm on the H_∞ Riccati equations? How is the performance?)

We will use the rest of this exercise to discuss any open questions from the lectures, handins and old exercises. I will also talk about recommended literature.

Handin 1 – Due Oct 31 10.15

1. The internal report TFRT-7456 (see the Mosaic-page for the LQG-course) describes an LQG design for the so called flexible servo. This is used as a demo in the digital control course. All macros are available under the `/home/kursdr/demo` directory.
Use the variable λ to find a state-feedback controller giving a reasonable tension in the spring (state x_3). Hand in plot(s) showing how x_3 reacts on a reference change, load disturbance and some measurement noise. Make sure you read and understand the matlab-files defmod, deflqg, dolqg, doeval.

```
Hints :
Start matlab3 and simnon in two different windows
add  ~kursdr/demo/discrete to your matlab and simon paths
(or copy all files)
Some usefule commands:
Matlab
>>lqgbox
>>defmod
>>rho=5;lambda=0;v1=10000;v2=1e7;
>>deflqg
>>dolqg
>>frbox
>>evpl(gp,gfb,gff,gl,gn,gz)

Simnon
>syst servo aafilt dsf sfcon
>evaxes
>get sfpar
>doeval
```

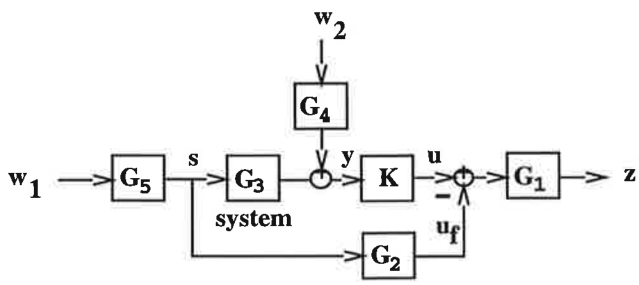
2. Compute the H_2 norm of the aircraft example on page 152 (with sideslip angle β as output). *Hint: lyap or h2norm in matlab*

3. Assume $G(s) = C(sI - A)^{-1}B$ is stable. Show that the continuous time H_2-norm equals the square root of $\mathrm{tr}(BSB^T)$ where S is the unique solution to

$$SA + A^T S + C^T C = 0.$$

What are the corresponding formulas for the discrete time case?

4. Fig 1 shows an input estimation problem. w_1 and w_2 are white noise and the goal is to design the filter K such that z has minimal variance. Rewrite this as a so called standard problem.



Handin 2 – Due Nov 7 13.15

No cross-terms in these problems. Here is the Riccati equation (DRE)

$$-\dot{S} = SA + A'S + C_1'C_1 - SB_2B_2'S.$$

The algebraic Riccati equation (ARE) is obtained by putting $\dot{S} = 0$. That S is "stabilizing" will mean that $A - B_2B_2'S$ is Hurwitz (all eigenvalues in open left half plane). In problem 2 you might need the fact that if A is Hurwitz then

$$SA + A'S \leq 0 \quad \Longrightarrow \quad S \geq 0.$$

1. Solve the finite time horizon problem

$$\min \int_0^T u^2(t)\,dt + q_T x^2(T)$$
$$\dot{x} = x + u, \quad x(0) = x_0.$$

Show that there are two solutions $S \geq 0$ two the algebraic Riccati equation. Which one, S_{opt}, should be used for the infinite time horizon problem where stability is reqiured? For which q_T does $S(0,T)$ approach this S_{opt} when $T \to \infty$?

Remark: In this problem we have $[C_1, A]$ not detectable but an optimal stabilizing controller still exists. Note that this controller will not be found as $T \to \infty$ if $S(T,T) = 0$ is used as final condition (as in the book). The problem gives a counterexample to the statement on page 48 10 lines from the bottom in the book. This is why we did not follow the book in the lecture on the point of choice of $S(T,T)$.

2. Assume that S_1 solves the ARE and $A - B_2B_2'S_1$ is asymptotically stable. Show that $S_1 \geq 0$. Show also that

$$(S_1 - S_2)(A - B_2B_2'S_1) + (A - B_2B_2'S_1)'(S_1 - S_2) + (S_1 - S_2)B_2B_2'(S_1 - S_2) = 0$$

where S_2 is any other solution to the ARE. Conclude that $S_1 \geq S_2$. Can there be more than one stabilizing solution of the ARE? (No assumptions on $[C_1, A]$ needed in this exercise.)

Remark As we saw in problem 1 there might be several $S \geq 0$ solving the ARE and one must be careful in checking that one chooses a stabilizing one. It can be shown that with the stronger conditions of the book (stabilizability of $[A, B_2]$ and detectability of $[C_1, A]$) there is only one $S \geq 0$, in which case it is the stabilizing solution.

Handin 3 – Due Nov 18 13.15

1. Prove that in discrete time (SISO system with cross-terms in the loss)

$$q^2 P(z^{-1})P(z) = B_1^T(z^{-1})B_1(z) + [B_2(z^{-1}) + \rho A(z^{-1})][B_2(z) + \rho A(z)]$$

see TFRT-7475 p. 7 for the definitions.

2. Consider the design in Ch. 6.2. Check the calculations of v_i^0 and x_i^0, $i = 1, \ldots, 4$. Also check the calculation of D, R and K_ρ. Do not bother about that we have not proven (6.2.5-13). If you are interested in this design technique read more in [Harvey and Stein AC 78, pp. 378-387].

Handin 4 – Due Dec 2 10.15

Choose TWO of the following three problems.

1. Consider the system on observer form

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} w_1$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x + \sigma w_2$$
$$\text{i.e.} \quad y = \frac{s+1}{s^2} w_1 + \sigma w_2.$$

Here w_1 and w_2 are independent white noise with zero mean and variance 1. Calculate the optimal stationary Kalman filter and write it on the form

$$\begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} b_1(s)/a_1(s) \\ b_2(s)/a_2(s) \end{pmatrix} y$$

What happens when $\sigma \to 0$? Explain! (8 p)

2. Consider the "random-walk" vector

$$x(t+1) = x(t) + bv(t) \qquad x_0 \in N(0, \sigma_0 I)$$

where $x, b \in R^n$. Determine the one-step predictor $\hat{x}(t|t-1)$ and the covariance $P(t)$ when we have the observations

$$y(t) = x(t) + e(t)$$

Show that

$$P_\infty = \lim_{t\to\infty} P(t)$$

exists and is singular. Explain ! (5 extra points for those who obtain an explicit formula for P_∞).
The white noise signals v and e, with covariance 1 and I, are independent and independent of x_0.
Hint: It might be a good idea to use a transformation of the form $z = Ux$ to simplify things. Experiment in Matlab to guess P_∞. (10 p)

3. Consider the system

$$\dot{x} = Ax + B_1 w$$
$$y = C_2 x + D_{21} w$$

To simplify the calculations we assume that $B_1 D_{21}^T = 0$ and $D_{21} D_{21}^T = I$. Show that among all linear filters

$$\dot{\xi} = F\xi + Gy$$
$$\hat{z} = H\xi + Jy$$

one that minimizes the stationary variance

$$\mathrm{E}(z^T z) \qquad z = \hat{z} - Lx$$

is given by the Kalman filter

$$\begin{pmatrix} F & G \\ H & J \end{pmatrix} = \begin{pmatrix} A - KC_2 & K \\ L & 0 \end{pmatrix}$$

where

$$0 = AP + PA^T + B_1B_1^T - PC_2^TC_2P$$

and $A - PC_2^TC_2$ is stable.
Hint Check that the closed loop is given by

$$\begin{pmatrix} \dot{x} \\ \dot{\xi} \end{pmatrix} = \begin{pmatrix} A & 0 \\ GC_2 & F \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + \begin{pmatrix} B_1 \\ GD_{21} \end{pmatrix} w$$

$$z = \begin{pmatrix} -L + JC_2 & H \end{pmatrix} \begin{pmatrix} x \\ \xi \end{pmatrix} + JD_{21}w$$

Conclude that $J = 0$ and that the resulting norm is given by

$$||T_{zw}||_2^2 = tr \begin{pmatrix} -L & H \end{pmatrix} \tilde{P} \begin{pmatrix} -L^T \\ H^T \end{pmatrix}$$

Show that

$$\tilde{P} - \begin{pmatrix} P & 0 \\ 0 & 0 \end{pmatrix} \geq 0$$

which gives the bound tr LPL^T. Show that equality is obtained by the estimator above. (10 p)

Handin 5 – Due Fri Dec 9 10.15

Do one of the following two alternatives

1. Make a reasonable LQG design for the 5th order example below. Evaluate your design in many several ways. Comment!

```
% System description for "Flexible Transmission" benchmark
%
%        /Micke

% Real pole

omega_0 = 0.5;

% Complexe conjugate pair 1

omega_1 = 12;
zeta_1 = 0.04;

% Complexe conjugate pair 2

omega_2 = 33;
zeta_2 = 0.02;

% Form transfer function G(s) = b(s)/a(s) with static gain 1.

a = conv([1 omega_0], ...
    conv([1 2*omega_1*zeta_1 omega_1^2], [1 2*omega_2*zeta_2 omega_2^2]));

b = a(length(a));

% Do some plotting

fr = frc(b,a,0,-2,2,1000);
figure;
pzpl(b,a);
pzgrid;
title('Pole-Zero Plot for Flexible Transmission');
figure;
bopl(fr);
bogrid;
title('Bode Plot for Flexible Transmission');
figure;
nypl(fr);
nygrid;
title('Nyquist Plot for Flexible Transmission');
```

2. The real control engineer of course want real stuff. We have a system in the lab with almost the same dynamics as above. Do an LQG-design on this system. Talk with me about this problem. This problem involves more work since an identification is also needed. You can also use this example on the last handin.

Handin 6 – Due Fri Dec 16 10.15

1. LQG-balanced realizations You have probably heard about the "balanced realization" algorithm to obtain a reduced order system that models the "most important part" of a system. This is achieved in matlab with the command `balreal`. What balreal does is to find a state space transformation to a realization where both the controllability and observability gramians are diagonal and equal, i.e. $W_c = W_o = \Sigma$. Small elements in in Σ corresponds to states that are very little controllable and observable, i.e. they do not influence the open loop transfer function very much. It is then natural to use a reduced order model where theses states are skipped. See the command `modred` that achieves this. This is taught in the system identification course.

It is however natural to argue that one should not consider which states are important for the open loop, since it is the closed loop that is important. (That the balreal command above only works for open loop stable systems is a direct consequence of this.) Of course one should insted consider how important different states are for the closed loop behavior.

This is achieved with "LQG-balanced realization". This is based on the assumption that the loop is going to be closed with the LQG-optimal controller. One then finds a state space tranformation T such that $S = P = \Sigma$, are digaonal, where S and P are solutions to the two LQG-Riccati equations.

a. Show that after a state space transformation $A \rightarrow T^{-1}AT$, $B \rightarrow T^{-1}B$ etc we have $T^{-1}PT^{-T}$ and T^TST as new solution to the Riccati equations. Conclude that the eigenvalues of SP are invariant under state space transformations.

b. The file

`/home/fulqg/lqg94/matlab/lqgbalreal.m`

gives the state space transformation needed to diagonalize S and P as above. (5 extra points to those who explain why the algorithm works). Run the demo

`/home/fulqg/lqg94/matlab/handin6.m`

that describes LQG-control of an integrator with a resonances. The demo shows what happens when the bandwith of the closed loop system is increased (by punishing states more and more and decreasing the assumed measurement noise).

Explain, based on the the LQG-balanced values, for which bandwidths a first order model (and hence first order controller) suffices for the LQG-design?

Handin 7 – Due Fri Dec 19 24.00

1. Do a good LQG-design on an interesting process. Suggestions for processes:
   - Landau Benchmark Problem (can be combined with Handin 5)
   - JAS 39 Gripen (Ask BoB)
   - Hot Rolling Mill (Lars Malcolm has details)
   - Horizontal Axis Wind Power Plants (Sven Erik's Thesis)
   - IFAC Benchmark Problems 1990 (several different suggestions, ask BoB)
   - Velocity control of horizontal axis in our inverted pendulum (good model exists)
   - Our lab helicopter (need identification)
   - Your own favorite process