



LUND UNIVERSITY

Dynamic Epistemic Logic of Diffusion and Prediction in Threshold Models

Baltag, Alexandru; Christoff, Zoé; Rendsvig, Rasmus Kraemmer; Smets, Sonja

2016

[Link to publication](#)

Citation for published version (APA):

Baltag, A., Christoff, Z., Rendsvig, R. K., & Smets, S. (2016). *Dynamic Epistemic Logic of Diffusion and Prediction in Threshold Models*. Paper presented at Twelfth Conference on Logic and the Foundations of Game and Decision Theory, Maastricht, Netherlands.

Total number of authors:

4

Creative Commons License:

Unspecified

General rights

Unless other specific re-use rights are stated the following general rights apply:

Copyright and moral rights for the publications made accessible in the public portal are retained by the authors and/or other copyright owners and it is a condition of accessing publications that users recognise and abide by the legal requirements associated with these rights.

- Users may download and print one copy of any publication from the public portal for the purpose of private study or research.
- You may not further distribute the material or use it for any profit-making activity or commercial gain
- You may freely distribute the URL identifying the publication in the public portal

Read more about Creative commons licenses: <https://creativecommons.org/licenses/>

Take down policy

If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

Dynamic Epistemic Logics of Diffusion and Prediction in Social Networks

(Extended Abstract)

Alexandru Baltag¹, Zoé Christoff², Rasmus K. Rendsvig³, and Sonja Smets¹

¹ Institute for Logic, Language and Computation, University of Amsterdam

A.Baltag@uva.nl, S.J.L.Smets@uva.nl

² Department of Computer Science, University of Liverpool

zoe.christoff@gmail.com,

³ Theoretical Philosophy, Lund University, and Center for Information and Bubble Studies, University of Copenhagen

rendsvig@gmail.com

Keywords: social network theory, threshold models, diffusion in networks, social epistemology, formal epistemology, dynamic epistemic logic, opinion dynamics, opinion dynamics under uncertainty

An individual's actions or opinions are often influenced by the actions of people around her. The way a new product or fashion gets adopted by a population depends on how agents are influenced by others, which in turn depends both on the way the population is structured and on how influenceable agents are.

This paper focuses on one particular account of social influence, “threshold influence”, as presented in e.g. [10,27], relying on an imitation or conformity pressure effect: *agents adopt a behavior/product/opinion/fashion whenever a critical fraction of their “friends” (neighbors in the network) have adopted it already*. In this sense, diffusion in social networks can be seen as a study of local influence, triggering agents to adopt a similar behavior/opinion/product as their neighbors [28,13]. The so-called *threshold models*, first introduced in [12,24], are used precisely to represent the dynamics of diffusion under threshold-limited influence. This type of models has received a lot of attention in the recent literature [10,15,19,26,1,11,16,18], also within the logic community [25,29,23,17,7,8,5,20,21].

The contributions of this paper are three-fold: 1) We introduce an *epistemic* dimension to threshold models, thus taking into account the real-life limitations posed by the agents' limited access to information; for this, we propose an epistemic variant of the above adoption rule: agents adopt a behavior only they *know* that enough many of their neighbors have adopted it. 2) We investigate *foresighted agents*, who look ahead trying to *predict* the others' behavior based on their current information, with the aim of coordinating with their friends by adopting the new behavior whenever they know that enough many of their neighbors *will* adopt it at the same time. 3) We provide logical formalisms for reasoning about threshold models, for each of three variants: standard, epistemic and predictive adoption rules.

In this extended abstract, most proofs are omitted. They may be found in [3], the full version of this paper.

1 Threshold Models

Threshold Models. A *threshold model* is a tuple $\mathcal{M} = (\mathcal{A}, N, B, \theta)$ where \mathcal{A} is a finite set of *agents*, $N \subseteq \mathcal{A} \times \mathcal{A}$ is a *network* (described as a relation of “neighborhood” or “friendship” between agents), $B \subseteq \mathcal{A}$ is a *behavior* (identified with the set of agents who have adopted this behavior), and $\theta \in [0, 1]$ is an adoption *threshold*.⁴ N is assumed irreflexive, symmetric and serial.

A threshold model includes a network N of agents \mathcal{A} and a behavior B distributed over the agents. As such, it represents the current spread of B through the network. An adoption threshold prescribes how the state will evolve: agents adopt B when the proportion of their neighbors who have already adopted it meets the threshold.

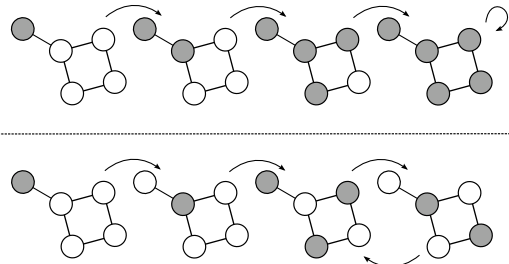


Fig. 1. A threshold model with $\theta = \frac{1}{4}$ and B marked by gray, updated according to (1) (top) and (2) (bottom).

Threshold Model Update. Several update policies for the behavior set B exist. The most popular [15,10] is given by the following threshold adoption rule:

$$B' = B \cup \left\{ a : \frac{|N(a) \cap B|}{|N(a)|} \geq \theta \right\}. \quad (1)$$

⁴ The literature contains several variations, including infinite networks [19], non-inflating behavior [19], agent-specific threshold [15], weighted links [15], and multiple behaviors [1].

(1) states that a adopts B at time t_{n+1} iff either a adopted B at time t_n , **or** the proportion of a 's neighbors who have adopted B at t_n is *larger or equal* than the threshold θ .

The former disjunct makes this update *inflationary* w.r.t. B , i.e. $B \subseteq B'$. This guarantees that repeated updates according to (1) will eventually reach a fixpoint. The ‘or equal to’ clause embeds a tie-breaking rule favoring B .

Another popular option uses an alternative policy, that drops inflation and invokes a conservative tie-breaking rule:

$$B' = \left\{ a : \frac{|N(a) \cap B|}{|N(a)|} > \theta \right\} \cup \left\{ a : \frac{|N(a) \cap B|}{|N(a)|} = \theta \text{ and } a \in B \right\}. \quad (2)$$

Since (2) does not cause B to inflate, this alternative rule allows the possibility of *loops* in behavior, i.e. where $B = B'' \neq B'$. Thereby repeated updates according to (2) do *not* necessarily reach a fixpoint. For this and other reasons, in this paper we focus on the adoption policy given by (1).

Dynamics as Induced by Game Play. (1) and (2) correspond to the best response dynamics of agents playing an instance of a coordination game

	B	¬B
B	x, x	$0, 0$
¬B	$0, 0$	y, y

with each of their neighbors at each timestep, under the constraint that at each timestep, each agent may pick only one strategy. The utility of a play round for an agent a is the sum of utilities of the individual coordination games played by a in that round. With B the set of agents currently playing B, B is thus a best response for agent a iff

$$x \cdot \frac{|N(a) \cap B|}{|N(a)|} \geq y \cdot \frac{|N(a) \cap \neg B|}{|N(a)|} \iff \frac{|N(a) \cap B|}{|N(a)|} \geq \frac{y}{x+y} =: \theta.$$

Specifically, (2) captures such plays’ best response dynamics with conservative tie-breaking [19]: B' as given by (2) is exactly the set of agents for whom B is a best response. Hence the diffusion dynamics arising from updating a network using best response analysis is step-wise equivalent with those given by (2). Moreover, for any $\theta \in [0, 1]$, there exists coordination game payoffs that yield best response dynamics equivalent to those of (2) instantiated with the given θ .

Equation (1) captures the same dynamics with discriminating tie-breaking, but also the added assumption of a (possibly irrational) ‘seed’ of agents always playing B (see [19, 10] for game theoretic details and [20] for an action model approach).

This paper focuses on the dynamics given by (1) with a game-theoretic interpretation of the choice to adopt, i.e. when agents choose to play B over ¬B. As agents cannot unadopt, this is the only rationality consideration in play.

2 Epistemic Threshold Models

In the update policies described by (1) and (2), agents *react to their actual environment*: they are always influenced by the actual behavior of their direct neighbors. In situations of imperfect information, this “nomothetic” update style seems unrealistic: it requires agents to act based on information that they may not actually possess! This is exemplified in Fig. 2.

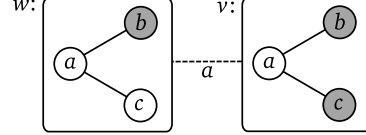


Fig. 2. A situation of uncertainty. Agent a cannot tell whether world w or world v is the actual one, as indicated by the dashed line (when representing indistinguishability relations we omit reflexive and transitive links). Hence, a does not know whether c has adopted or not. Assume that the threshold is $\theta > 1/2$ and that v is the actual world. Then, according to the ‘threshold model update’, a should adopt – but a does not know that!

To accommodate this shortcoming, we extend the standard threshold models with an epistemic dimension and define a refined adoption policy where agents’ behavioral change depends on their knowledge of others’ behavior. To this end, we follow current practice in Logic, Economics and Computer Science, by adopting a possible-worlds’ semantics where each agent is endowed with an epistemic indistinguishability relation over worlds [14], as illustrated in Fig. 2. This induces an “information partition” [2] of the set of worlds. Each partition cell captures the uncertainty of the agent: i.e. the worlds she cannot tell apart.

Definition 1 (ETM). An epistemic threshold model (ETM) is a tuple $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$ where \mathcal{W} is a finite, non-empty set of possible worlds/states, \mathcal{A} is a finite non-empty set of agents,

$\sim_a \subseteq \mathcal{W} \times \mathcal{W}$ is an equivalence relation, for each $a \in \mathcal{A}$,

$N : \mathcal{W} \rightarrow (\mathcal{A} \rightarrow \mathcal{P}(\mathcal{A}))$ assigns a to each $a \in \mathcal{A}$ and each $w \in \mathcal{W}$ a neighborhood $N(w)(a)$ such that:

- $a \notin N(w)(a)$ (Irreflexivity)
- $b \in N(w)(a) \iff a \in N(w)(b)$ (Symmetry)
- $N(w)(a) \neq \emptyset$ (Seriality)

$B : \mathcal{W} \rightarrow \mathcal{P}(\mathcal{A})$ gives each $w \in \mathcal{W}$ an adoption set $B(w)$.

$\theta \in [0, 1]$ is a uniform adoption threshold.

To reason about the impact of knowledge on diffusion in network situations, we want to impose limiting assumptions regarding the agents’ uncertainty. It is for example natural to assume that agents know how many direct neighbors they have, and know their neighbors’ behavior. But cases exist where it is natural that agents know more about the network: they may know how many neighbors their neighbors have, or maybe the entire network structure is even common knowledge; they may know the behavior of their neighbors’ neighbors, etc.

One way to impose restrictions on uncertainty is by giving agents an ego-centric “sphere of sight”, corresponding to how far they can “see” in the network, assuming that if they can see further, they can see closer. We will say that an agent has *sight* n when she can “see” at least n agents away, i.e., when she knows at least both the network structure and the behavior of all agents within n -distance.

Definition 2 (n -distant, n -sight). Let $n \in \mathbb{N}$ and let $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$ be an ETM. Define $N^n : \mathcal{W} \rightarrow \mathcal{A} \rightarrow \mathcal{P}(\mathcal{A})$ by $\forall w \in \mathcal{W}$ and $\forall a, b, c \in \mathcal{A}$:

$$N^0(w)(a) = \{a\}$$

$$N^{n+1}(w)(a) = N^n(w)(a) \cup$$

$$\{b \in \mathcal{A} : \exists c \in N^n(w)(a) \text{ and } b \in N(w)(c)\}$$

If $b \in N^n(w)(a)$, then b belongs to the set of agents that a has within her sight at world w . Moreover, if $b \in N^n(w)(a)$ we say that b is n -distant from a in w . An ETM $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$ has sight n if, for all $a, b \in \mathcal{A}$ and all $w, v \in \mathcal{W}$, we have that:

- i) If $w \sim_a v$ and $b \in N^n(w)(a)$, then $b \in B(w)$ iff $b \in B(v)$, and
- ii) If $w \sim_a v$ and $b \in N^{n-1}(w)(a)$, then $N(w)(b) = N(v)(b)$.

In other words: in an ETM of sight n , the structure of the network and the others’ behavior are known at least up to distance n , and this is common knowledge.

Knowledge-Dependent Diffusion. To remedy the problem of agents acting on information they may not possess, we introduce a knowledge-dependent adoption policy, that captures the intuitive idea that an agent should only be influenced by what she knows about other agents around him. More precisely, the agents adopt whenever they know that enough of their neighbors have adopted already:

Definition 3 (Informed Update). Let $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$ be an ETM of sight n . the informed adoption update of \mathcal{M} results in an ETM $\mathcal{M}^i = (\mathcal{W}, \mathcal{A}, N, B^i, \theta, \{\sim_a^i\}_{a \in \mathcal{A}})$ where, for all $a \in \mathcal{A}$ and all $w, w' \in \mathcal{W}$, we put:

$$B^i(w) = B(w) \cup \{a \in \mathcal{A} : \forall v \sim_a w \frac{|N(v)(a) \cap B(v)|}{|N(v)(a)|} \geq \theta\}, \text{ and}$$

$$w \sim_a^i w' \text{ iff } i) \ w \sim_a w' \text{ and}$$

$$ii) \ \forall b \in N^n(w)(a) : b \in B^i(w) \Leftrightarrow b \in B^i(w').$$

The first condition tells us that the new set of adopters at world w includes the previous set of adopters $B(w)$ (hence agents do not give up their previously adopted behavior) and it includes also all agents who, as far as they know, are certain of the fact that enough influential neighbors (given by θ) have adopted already. The second condition ensures that the informed update of an ETM with sight n is again an ETM with sight n , i.e., agents can still see the (new) behavior of n -distant neighbors after the update.

Implicit Information and Redundant Knowledge. Under some epistemic conditions, the epistemic and non-

epistemic diffusion policies are equivalent. If each agent always knows at least who her immediate neighbors are and how they are behaving, then the two policies give rise to the same diffusion dynamics, in the following sense: the diffusion dynamics resulting from the informed update on an ETM reduces to the diffusion dynamics under the initial (non-epistemic) update applied to each possible world of the ETM. This is the content of Proposition 1 below.

Proposition 1 relates two important insights. The first is that standard threshold models make the *implicit epistemic assumption* that agents know their neighborhood and its behavior. The second is that *knowledge about more distant agents is redundant* as it will not affect behavior.

To prove the result, we first define how to generate a (non-epistemic) threshold model from a possible state of an epistemic threshold model:

Definition 4 (State-Generated Model). With an ETM $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$, let $w \in \mathcal{W}$ and $a \in \mathcal{A}$. The state-generated threshold model $\mathcal{M}(w) = (\mathcal{A}, N_{\mathcal{M}(w)}, B_{\mathcal{M}(w)}, \theta)$ is given by:

$$N_{\mathcal{M}(w)}(a) = N(w)(a), \text{ and } a \in B_{\mathcal{M}(w)} \Leftrightarrow a \in B(w).$$

Proposition 1. Let $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$ be an ETM and $w \in \mathcal{W}$. Let \mathcal{M}^i and $\mathcal{M}(w)$ be respectively the informed update and state-generated models of \mathcal{M} . Let $\mathcal{M}^i(w)$ be the state-generated model of \mathcal{M}^i and let $\mathcal{M}(w)'$ be standard (non-epistemic) threshold update of $\mathcal{M}(w)$ according to the rule (1). Then:

$$\text{If } \mathcal{M} \text{ has sight } n \geq 1, \text{ then } \mathcal{M}^i(w) = \mathcal{M}(w)'.$$

Proposition 1 provides a precise, but partial, interpretation of the dynamics of non-epistemic threshold models as a process of information-dependent behavior diffusion. As witnessed by its proof, only the immediate neighborhood of agents matters for the adoption behavior in a threshold model. A next step is to investigate how this changes when agents are equipped with predictive abilities; see Section 3.

The interpretation is partial, since the restriction to the case of sight $n \geq 1$ does not fully characterize the standard threshold dynamics (1). In the case of *no sight* ($n = 0$), the agent may have uncertainty about some neighbor b ’s behavior, and might not even know exactly who are all her neighbors; but she might still know that a large enough proportion of these neighbors have adopted B : in which case she will still update according to the standard threshold dynamics!

Situations in which agents lack knowledge of some of their immediate neighbors’ behavior are interesting in that they may cause the diffusion process to *slow down* compared to the standard update policy:

Proposition 2. *There exists an ETM with sight $n = 0$, $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$, such that*

$$B_{\mathcal{M}^i(w)} \subset B_{\mathcal{M}(w)'},$$

where \mathcal{M}^i and $\mathcal{M}(w)$ are resp. the informed update and state-generated models of \mathcal{M} , and $\mathcal{M}^i(w)$ is the state-generated model of \mathcal{M}^i and $\mathcal{M}(w)'$ is the Eq. (1) update of $\mathcal{M}(w)$.

Fig. 3 illustrates this “slower” diffusion process.

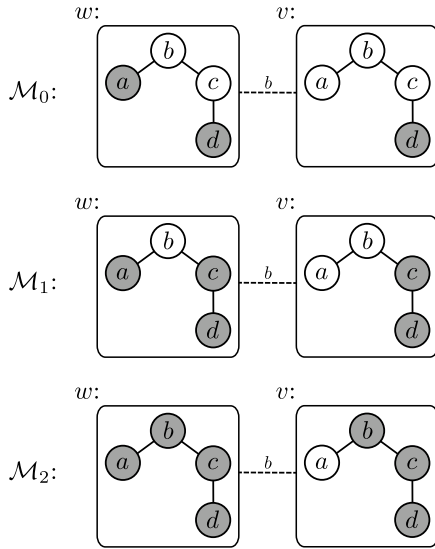


Fig. 3. A diffusion process slowed down by the uncertainty of b . Threshold: $\theta = \frac{1}{2}$. Consider w in \mathcal{M}_0 : a has adopted, but b does not know it. Therefore, b will not adopt immediately. The diffusion according to the informed update policy in state w will only stabilize after updating *twice*. Under the non-epistemic Eq. (1) update, or if agent b knew whether a has adopted, the situation depicted in w would stabilize after one step, hence the slow-down.

3 Prediction Update

In defining our informed update rule based on epistemic threshold models, we ensure that agents do not act on information they do not possess. Such agents are however still limited, in that they do not take *all* their available information into account. This section investigates effects of agents that are allowed to reason about more than only the *present* behavior of the network. In particular, we focus on providing agents with *predictive power*.

Consider the “no uncertainty” ETM illustrated in Fig. 4, in which all the network and behaviors are assumed to be common knowledge, so the informed dynamics coincides with the standard, non-epistemic dynamics.

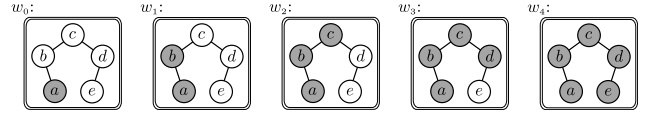


Fig. 4. An ETM with no uncertainty about the actual state w , developing according to informed update. B is marked by gray, and a threshold $\theta < 1/2$ is assumed. At time 0 (w_0), only a has adopted. According to informed adoption, b adopts at time 1. At time 2, c also adopts the behavior, etc.

Given a game-theoretic interpretation of agents’ choices, the dynamics in Fig. 4 seems “irrational”. In particular, as the network and behavior distribution are known to c , and as $\theta < 1/2$ means that the payoff of successfully coordinating with a neighbor on B is higher than that for coordinating on $\neg B$, c seems to lack foresight when choosing *not* to adopt during the first update. As c knows that a has adopted, she also knows that b will adopt during round 1. Hence c would successfully coordinate with more neighbors and thus be better off in round 1 if she, too, has chosen to adopt. To represent this “predictive rationality” we define a new update mechanism.

Prediction Update as the Least Fixed Point. In defining “prediction update”, we make use of the notion of *least fixed point*. This is necessary because of the circular character of prediction update: an agent adopts based on the predicted behavior of her neighbors, but that behavior is in its turn based on their predictions about the first agent’s behavior (among others), etc. This fixed point may be approximated using a chain of lower level predictions. The intuitive idea of the approximation may be illustrated by reference to Fig. 4: assume agent a predicts that his only neighbor b will adopt B in round 2, even if b follows the (non-predictive) informed update policy. Then a may act preemptively, by also adopting B in round 2, rather than in round 3 as the informed update prescribes.⁵ In this case, a may be thought of as a *level 1 predictor*: he assumes no-one else makes predictions, that the others are of level 0. However, a may come to think that b is as smart as he is, i.e., that also b is a level 1 predictor. Assuming this, a now foresees that b will not wait till round 2 to adopt, but instead she adopts B already in round 1; based on this prediction about b ’s predictions, a may now also adopt in round 1. In this case, a is a *level 2 predictor*, etc. If this reasoning is pushed to its limit, it will “catch up with itself”: in the fixed point, every agent will be a level ω predictor, predicting under the assumption that all others are the same.

Common Knowledge of Predictive Rationality and of Complete Information Use. Prediction update incorporates two epistemic assumptions. One is that it is common knowledge that all agents try to foresee the behavior of their neighbors. The second assumption is that it is common knowledge that predictors use all their available information (about the network structure, the current be-

⁵ If a acted according to the informed update policy, he must first see b adopt before he is influenced by b ’s choice

havior spread and information available to others when determining their next action).

Before defining the prediction update, a few preliminaries are required.

Definition 5 (The Lattice of Behaviors and the Informed-Update Map). For a given ETM $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$ let $\mathcal{P}(\mathcal{A})^{\mathcal{W}}$ be the set of all possible “behaviors”, i.e. all functions $f : \mathcal{W} \rightarrow \mathcal{P}(\mathcal{A})$. We can convert this set into a lattice, by defining a partial order \leq on $\mathcal{P}(\mathcal{A})^{\mathcal{W}}$, given by:

$$f \leq g \iff f(w) \subseteq g(w).$$

The informed-update map is a function

$$\Gamma_B : \mathcal{P}(\mathcal{A})^{\mathcal{W}} \longrightarrow \mathcal{P}(\mathcal{A})^{\mathcal{W}}$$

, mapping any behavior $f \in \mathcal{P}(\mathcal{A})^{\mathcal{W}}$ to some behavior $\Gamma_B(f)$, given by, for all $w \in \mathcal{W}$,

$$\Gamma_B(f)(w) = B(w) \cup \left\{ a \in \mathcal{A} : \forall v \sim_a w, \frac{|N(v)(a) \cap f(v)|}{|N(v)(a)|} \geq \theta \right\}.$$

Lemma 1. Let \mathcal{M} , $\mathcal{P}(\mathcal{A})^{\mathcal{W}}$, \leq and Γ_B be as in Definition 5. Then 1) $(\mathcal{P}(\mathcal{A})^{\mathcal{W}}, \leq)$ is a finite, and hence complete, lattice. 2) Informed update Γ_B is an order-preserving (monotonic) map.

Definition 6 (Least Fixed Point). Let $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$ be an ETM, and $\mathcal{P}(\mathcal{A})^{\mathcal{W}}, \leq, \Gamma_B$ be as in Definition 5. The least fixed point of Γ_B , denoted by $\text{lfp}(\Gamma_B)$, is the unique behavior $x \in \mathcal{P}(\mathcal{A})^{\mathcal{W}}$ such that

$$\Gamma_B(x) = x, \text{ and}$$

$$\forall y \in \mathcal{P}(\mathcal{A})^{\mathcal{W}}, \text{ if } \Gamma_B(y) = y, \text{ then } x \leq y$$

Theorem 1 (lfp Existence, Uniqueness and Approximation). Let \mathcal{M} , $(\mathcal{P}(\mathcal{A})^{\mathcal{W}}, \leq)$ and Γ_B be as in Definition 6. Then $\text{lfp}(\Gamma_B)$ exists. Moreover, this least fixed point is unique, and it can actually be reached by finite iterations of the informed-update map starting on the bottom element of the lattice. More precisely: if we put

$$\begin{aligned} \Gamma_B^0 &= \perp, \text{ where } \perp(w) = \emptyset \text{ for all } w \in \mathcal{W}, \\ \Gamma_B^{n+1} &= \Gamma_B(\Gamma_B^n), \text{ for all } n \geq 1, \end{aligned}$$

then there exists some $N \in \mathbb{N}$, such that the sequence stabilizes at stage N , and we have: $\text{lfp}(\Gamma_B)(w) = \Gamma_B^N(w) = \Gamma_B^{N+1}(w)$.

Proof. By Lemma 1, Γ_B is a monotonic map on the complete lattice $(\mathcal{P}(\mathcal{A})^{\mathcal{W}}, \leq)$. Hence, the least fixed point $\text{lfp}(\Gamma_B)$ exists by the Knaster-Tarski Fixed Point Theorem (see e.g. [9, p. 50]). Moreover, since our lattice is finite, the proof of that theorem shows in fact that $\text{lfp}(\Gamma_B)$ is reached at some finite iteration Γ_B^N . \square

Defining Prediction Update. Given the previous paragraph, we may now define prediction update as follows:

Definition 7 (Prediction Update). Let $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}})$ be an ETM of sight n , and let $(\mathcal{P}(\mathcal{A})^{\mathcal{W}}, \leq)$ be as in Lemma 1. Let $\Gamma_B : \mathcal{P}(\mathcal{A})^{\mathcal{W}} \rightarrow \mathcal{P}(\mathcal{A})^{\mathcal{W}}$ be given as in Definition 5. The prediction update of \mathcal{M} results in the ETM $\mathcal{M}^p = (\mathcal{W}, \mathcal{A}, N, B^p, \theta, \{\sim_a^p\}_{a \in \mathcal{A}})$ where $\forall w, w' \in \mathcal{W}$, $B^p(w) = \text{lfp}(\Gamma_B)(w)$, and

- i) $w \sim_a^p w' \iff w \sim_a w'$ and
- ii) $\forall b \in N^{\leq n}(w)(a) : b \in B^p(w) \iff b \in B^p(w')$.

Theorem 1 is important, since it ensures first, that our prediction update is well-defined, and second that, when engaged in prediction update agents do *not* run the risk of falling into infinite chains of reasoning about each other (which presumably would take an infinite time): they can compute the resulting prediction (and update) in finitely many steps.

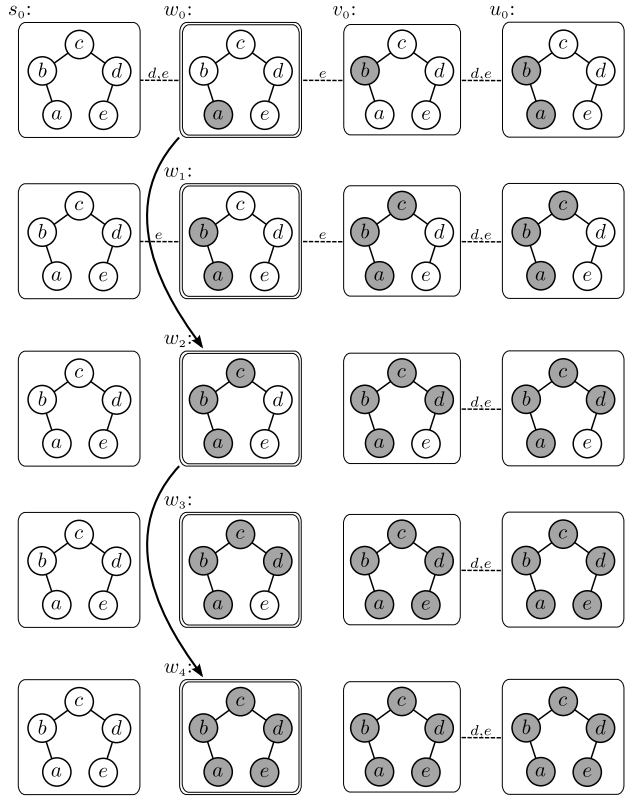


Fig. 5. The prediction update of a finite sight 2 ETM with actual state w , $\theta < 1/2$. Agents a, b, c know the actual state; d, e are uncertain. The development follows informed adoption; states w_0 – w_4 are from Fig. 4. The arrow shows the prediction update dynamics of the actual world. With informed update, w reaches a fixed point after 4 updates; with prediction update, it reaches the same fixed point after only 2 steps. Due to uncertainty, the prediction update does not jump to the fixed point in 1 step: as d does not know whether a has adopted at time 0, she does not know that c will adopt under the prediction update. Hence, she will refrain herself from adopting until w_3 . Similar considerations goes for e .

Iterated Dynamics, Fixed Point, Cascades, Speed of Convergence. When any of our adoption updates is iterated, a long-term dynamics is produced, in the form of an infinite sequence of models $\mathcal{M}, \mathcal{M}^{(1)}, \mathcal{M}^{(2)}, \dots, \mathcal{M}^{(n)}$. Since all the update rules considered in this paper are inflationary, a fixed point is always eventually reached: some stage N such that $\mathcal{M}^{(N)} = \mathcal{M}^{(N+1)}$. The extent of the cascade produced by each update type on an initial model \mathcal{M} is given by the behavior $B^{(N)}$ in the fixed point $\mathcal{M}^{(N)}$, which comprises the set of all agents who will eventually adopt B (in a given world). A *full cascade* is produced if all agents will eventually adopt B , i.e. $B^{(N)}(w) = \mathcal{A}$. It is easy to see that *prediction update accelerates the cascading behavior in comparison to informed update*: the fixed point of the adoption process is typically reached earlier if the agents use prediction update than if they use informed update. A full analysis of the relationship between the three types of update is left for future work. But a concrete example in this sense is given below.

Example, Sanity Check and Proof of Concept. The “irrational” behavior illustrated in Fig. 4 is solved by prediction update. The dynamics are illustrated in Fig. 5. Notice that now c adopts B as soon as she knows B is preferred.

Bounded Rationality. Prediction assumes that agents have unbounded rationality (maximal predictive and reasoning power given the available information). A *bounded rationality* version of prediction update could be defined, in which agents can only compute a fixed finite number n of steps of the prediction chain. A natural way of doing this would be by defining an update that uses Γ_B^n instead of $\text{lfp}(\Gamma_B)$. When n is low enough, the dynamics of bounded-rationality update would differ from the dynamics of unbounded prediction update. We leave the exploration of bounded-rationality updates for future work.

4 Logics

Definition 8 (Languages $\mathcal{L}_{K[]}$ and \mathcal{L}_K). Let the set of atomic propositions be given by $\{N_{ab} : a, b \in \mathcal{A}\} \cup \{\beta_a : a \in \mathcal{A}\}$ for a finite set \mathcal{A} . Where $a, b \in \mathcal{A}$, the formulas of $\mathcal{L}_{K[]}$ are given by

$$\varphi := N_{ab} \mid \beta_a \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_a\varphi \mid [\text{adopt}]\varphi$$

The formulas of that “static” fragment \mathcal{L}_K are those of $\mathcal{L}_{K[]}$ that do not involve the dynamic $[\text{adopt}]$ modality.

Intuitively, N_{ab} means that agent b is a neighbor of a , β_a means that agent a has adopted behavior B , $K_a\varphi$ means the agent a knows φ , and $[\text{adopt}]\varphi$ means that φ will hold after the next adoption update. The other Boolean operators (disjunction, implication) can be defined as abbreviations using negation and conjunction, in the usual way. We will consider two different interpretations for the language $\mathcal{L}_{K[]}$: the logic of informed update, and the logic of prediction update.

Definition 9 (Class: $\mathcal{C}_{\theta n}$). For $\theta \in [0, 1]$ and $n \in \mathbb{N}$, the class $\mathcal{C}_{\theta n}$ consists of all ETM’s with threshold θ and sight n .

Definition 10 (Informed Update Semantics). Let $\theta \in [0, 1]$ and $n \in \mathbb{N}$. Given any ETM $\mathcal{M} = (\mathcal{W}, \mathcal{A}, N, B, \theta, \{\sim_a\}_{a \in \mathcal{A}}) \in \mathcal{C}_{\theta n}$ of threshold θ and sight n , we recursively define a satisfaction relation \models between worlds $w \in \mathcal{W}$ in model \mathcal{M} and formulas $\varphi \in \mathcal{L}_{K[]}$:

$$\begin{aligned} \mathcal{M}, w \models \beta_a & \quad \text{iff} \quad a \in B(w) \\ \mathcal{M}, w \models N_{ab} & \quad \text{iff} \quad b \in N(w)(a) \\ \mathcal{M}, w \models \neg\varphi & \quad \text{iff} \quad \mathcal{M}, w \not\models \varphi \\ \mathcal{M}, w \models \varphi \wedge \psi & \quad \text{iff} \quad \mathcal{M}, w \models \varphi \text{ and } \mathcal{M}, w \models \psi \\ \mathcal{M}, w \models K_a\varphi & \quad \text{iff} \quad \forall v \in \mathcal{W} : v \sim_a w \Rightarrow \mathcal{M}, v \models \varphi \\ \mathcal{M}, w \models [\text{adopt}]\varphi & \quad \text{iff} \quad \mathcal{M}^i, w \models \varphi, \text{ with } \mathcal{M}^i \text{ the informed update of } \mathcal{M} \text{ (Def. 3).} \end{aligned}$$

Validity: as usual, we write $\models_{\mathcal{C}_{\theta n}} \varphi$ if φ is true at all worlds in all models $\mathcal{M} \in \mathcal{C}_{\theta n}$.

Axiomatization. In the specification of the epistemic reduction axioms, the syntactic shorthand in Table 1 are used. Using these shorthands, the axioms for Epistemic Threshold Models and the dynamics of Informed Update are given in Table 2.

The reduction law $Ep.Red.Ax.\beta$ states that a has adopted β after the update just in case she had already adopted it before the update, or *she knew that* she had a large enough proportion of neighbors who had already adopted it before the update. $Ep.Red.Ax.K.sight.n$ captures that an agent knows that φ will be the case after the update if, and only if, *she knows that*, if those very agents who actually are going to adopt do adopt, then φ will hold after the update.

Definition 11 (Logic of Informed Update). The logic $L_{\theta n}$ consists of the axioms and rules of Table 2, together with any complete set of axioms and rules for propositional logic. We write $\vdash_{L_{\theta n}} \varphi$ if φ is a theorem in the logic $L_{\theta n}$.

Theorem 2. (Soundness, Completeness, Expressivity and Decidability of $L_{\theta n}$) Let $\theta \in [0, 1]$, $n \in \mathbb{N}$ and $\varphi \in \mathcal{L}_{K[]}$. Then:

$$\models_{\mathcal{C}_{\theta n}} \varphi \text{ iff } \vdash_{L_{\theta n}} \varphi.$$

In fact, the reduction axioms can be used to show that the language $\mathcal{L}_{K[]}$, endowed with the informed update semantics, has the same expressivity as its static counterpart \mathcal{L}_K . Moreover, $L_{\theta n}$ is decidable.

Definition 12 (Prediction Update Semantics). Given $\theta \in [0, 1]$, $n \in \mathbb{N}$ and any ETM $\mathcal{M} \in \mathcal{C}_{\theta n}$, the satisfaction relation for the prediction update semantics can be defined using the same truth clauses as in Def. 10, except for the formulas of the form $[\text{adopt}]\varphi$, for which we put:

$$\mathcal{M}, w \models [\text{adopt}]\varphi \text{ iff } \mathcal{M}^p, w \models \varphi, \text{ with } \mathcal{M}^p \text{ the prediction update of } \mathcal{M}.$$

- 1)
$$\beta_{N(a)} \geq \theta := \bigvee_{\{g \subseteq \mathcal{N} \subseteq \mathcal{A} : \frac{|g|}{|\mathcal{N}|} \geq \theta\}} \left(\bigwedge_{b \in \mathcal{N}} N_{ab} \wedge \bigwedge_{b \notin \mathcal{N}} \neg N_{ab} \wedge \bigwedge_{b \in g} \beta_b \right)$$
- 2)
$$N_{ab}^1 := N_{ab}$$
$$N_{ab}^{k+1} := N_{ab}^k \vee \bigvee_{c \in \mathcal{A}} (N_{ac}^k \wedge N_{cb})$$
- 3)
$$(\mathcal{B} = N_a^k \beta^+) := \bigwedge_{b \in \mathcal{B}} (N_{ab}^k \wedge [\text{adopt}] \beta_b) \wedge \bigwedge_{b \in \mathcal{A} \setminus \mathcal{B}} (N_{ab}^k \rightarrow [\text{adopt}] \neg \beta_b)$$

Table 1 (Abbreviations). 1) $\beta_{N(a)} \geq \theta$ means that the proportion of agent a 's neighbors who have adopted is equal to or above the threshold θ . 2) For any $k \in \mathbb{N} \geq 1$, the formula N_{ab}^k means that b is k -distant from a . 3) For $\mathcal{B} \subseteq \mathcal{A}$, the formula $\mathcal{B} = N_a^k \beta^+$ says that \mathcal{B} is the set of agents who are k -distant from a and will have adopted after the next update.

Network Axioms		Knowledge Axioms		Inference Rules	
$\neg N_{aa}$	Irreflexivity	$K_a \varphi \rightarrow \varphi$	Ax.T	From φ and $\varphi \rightarrow \psi$, infer ψ	Modus Ponens
$N_{ab} \leftrightarrow N_{ba}$	Symmetry	$K_a \varphi \rightarrow K_a K_a \varphi$	Ax.4	From φ , infer $K_a \varphi$ for any $a \in \mathcal{A}$	Nec. K_a
$\bigvee_{b \in \mathcal{A}} N_{ab}$	Seriality	$\neg K_a \varphi \rightarrow K_a \neg K_a \varphi$	Ax.5	From φ , infer $[\text{adopt}] \varphi$	Nec. $[\text{adopt}]$
Knowledge-Network Axioms		Boolean Reduction Axioms			
$(N_{ab}^n \wedge \beta_b) \rightarrow K_a \beta_b$	Known Behavior	$[\text{adopt}] N_{ab} \leftrightarrow N_{ab}$			Red.Ax.N
$(N_{ab}^{n-1} \wedge N_{bc}) \rightarrow K_a N_{bc}$	Known Neighbors	$[\text{adopt}] \neg \varphi \leftrightarrow \neg [\text{adopt}] \varphi$			Red.Ax. \neg
		$[\text{adopt}] \varphi \wedge \psi \leftrightarrow [\text{adopt}] \varphi \wedge [\text{adopt}] \psi$			Red.Ax. \wedge
Epistemic Reduction Axioms					
$[\text{adopt}] \beta_a \leftrightarrow \beta_a \vee K_a (\beta_{N(a)} \geq \theta)$		Ep.Red.Ax. β			
$[\text{adopt}] K_a \varphi \leftrightarrow \bigvee_{\mathcal{B} \subseteq \mathcal{A}} (\mathcal{B} = N_a^n \beta^+ \wedge K_a (\mathcal{B} = N_a \beta^+ \rightarrow [\text{adopt}] \varphi))$		Ep.Red.Ax.K.sight.n			

Table 2: Axioms and inference rules for the logic of sight n ETMs and informed update. Subscripts a, b are arbitrary over \mathcal{A} .

Axiomatization. We present an axiomatic system that is *sound* for the logic of prediction update, although *completeness remains an open question*. Note that in this section, the $[\text{adopt}]$ modality is a fixed point operator and hence may no longer be reduced away. In contrast to the informed update logic, the prediction update logic appears to be strictly more expressive than its static counterpart.

To state the proof system, we need a more general version of the abbreviation $\beta_{N(a)} \geq \theta$ that was introduced in Table 1. For this, we make a new abbreviation $K_a(\varphi_{N(a)} \geq \theta)$, introduced in Table 3, whose intended meaning is that a knows that a fraction of at least θ of her neighbors has the property φ (where for instance φ_b can stand for $N_{ab} \wedge \beta_b$). In particular, $K_a([\text{adopt}] \beta_{N(a)} \geq \theta)$ expresses that a knows that a fraction of at least θ of her neighbors will have adopted β after the next prediction update step.

Definition 13 (The Logic of Prediction Update). The logic $L_{\theta n}^{\text{predict}}$ consists of the same axioms and rules as the logic $L_{\theta n}$ above, except for two changes: the axiom $\text{Ep.Red.Ax.}\beta$ is replaced by the Fixed Point Axiom in Table 4, and the set of rules is extended with the Least Fixed point Inference Rule in Table 4.

As before, we write $\vdash_{L_{\theta n}^{\text{predict}}} \varphi$ if φ is a theorem in the logic $L_{\theta n}^{\text{predict}}$.

The Fixed Point axiom of Table 4 is almost identical to $\text{Ep.Red.Ax.}\beta$ of Table 2, except for the inclusion of the $[\text{adopt}]$ modality on the right-hand side. This states that a will adopt after the prediction update iff she has already adopted, or if she knows that enough of her neighbors *will* have adopted after the next update step (in which they will apply the same predictive reasoning that she uses). The Least Fixed Point Inference rule reflects the fact that prediction update was defined as a *least* fixed point operator.

Proposition 3. (Soundness of $L_{\theta n}^{\text{predict}}$) Let $\theta \in [0, 1]$, $n \in \mathbb{N}$ and $\varphi \in \mathcal{L}_K[\cdot]$. Then:

$$\vdash_{L_{\theta n}} \varphi \text{ implies } \models_{\mathcal{G}_{\theta n}}.$$

We were unable to prove completeness, but we have reasons to make the following

Conjecture: The system $L_{\theta n}^{\text{predict}}$ is a complete axiomatization of predictive update logic over the class $\mathcal{G}_{\theta n}$.

5 Conclusions and Future Work

The paper has focused on two intertwined objectives. On the one hand, we have developed models for the diffusion dynamics under uncertainty, based on two natural epistemic variants of the standard threshold adoption rule:

$$4) \quad K_a(\varphi_{N(a)} \geq \theta) := K_a \left(\bigvee_{\{\mathcal{G} \subseteq \mathcal{N} \subseteq \mathcal{A} : \frac{|\mathcal{G}|}{|\mathcal{N}|} \geq \theta\}} \left(\bigwedge_{b \in \mathcal{N}} N_{ab} \wedge \bigwedge_{b \notin \mathcal{N}} \neg N_{ab} \wedge \bigwedge_{b \in \mathcal{G}} \varphi_b \right) \right)$$

Table 3 (Abbreviations). 4) The expression $K_a(\varphi_{N(a)} \geq \theta)$ is defined for tuples for $(\varphi_b)_{b \in \mathcal{A}}$, one for each agent $a \in \mathcal{A}$.

Fixed Point Axiom	Least Fixed Point Inference Rule
$[adopt]\beta_a \leftrightarrow \beta_a \vee K_a([adopt]\beta_{N(a)} \geq \theta)$	$\frac{\vdash \{\varphi_a \leftrightarrow \beta_a \vee K_a(\varphi_{N(a)} \geq \theta)\}_{a \in \mathcal{A}}}{\vdash \{\varphi_a \rightarrow [adopt]\beta_a\}_{a \in \mathcal{A}}}$

Table 4 (Fixed Point Laws). Fixed point laws of prediction update logic $L_{\theta_n}^{predict}$. The fixed point axiom takes the place of the informed update reduction axiom and the least fixed point inference rule is added.

the informed update, and the prediction update. On the other hand, we presented logical frameworks for reasoning about diffusion dynamics. We proved soundness and completeness for the logic of informed update, and proposed a sound system for the logic of prediction update. The problem of completeness for the later logic is an open question.

In epistemic threshold models, if agents' behavior is dictated by that of their direct neighbors, then knowledge of more distant agents is redundant. To act as under the standard threshold model dynamics, knowledge of neighbors' behavior is however required. If this information is not available, the diffusion speed decreases. In the limit case where no information is available, the diffusion process stops. Taken together, the most economical epistemic interpretation of standard threshold models is that their dynamics embodies an *implicit* epistemic assumption that exactly the network structure and behavior of agents in distance 1 is known.

Prediction update allows agents to better coordinate with their neighbors in adopting a spreading behavior, by using their information about the others' future behavior. As a result, prediction-update agents increase a network's speed of convergence. In the extreme case when the network and behavior distribution are common knowledge, the prediction update jumps in one step to the fixed point of the standard threshold model update. But in general, even describing the one-step dynamics of prediction update requires a dynamic fixed point operator, which is atypical of dynamic epistemic logic. As a consequence, the logic of prediction update does not have full reduction axioms: the dynamic modality seems to genuinely add expressivity in this case. This poses technical challenges to obtaining a completeness proof.

Future work. We plan to tackle in another paper the open problem about completeness of the logic of prediction update. Besides this question, there are four other main directions for further research: (A) explore the dynamics induced by boundedly-rational versions of predictive up-

date; (B) explore the game theoretic perspectives of game play on networks under uncertainty and in particular the game structure underpinning the intuitive rationality of prediction update; (C) develop a full comparative analysis of the different update processes that we have outlined in this paper, in particular the differences with respect to limit behavior and speed of possible stabilization; (D) investigate the epistemic and predictive versions of the non-inflationary adoption rules, such as the policy given by (2) above. Such rules, that allow agents to *unadopt* an already adopted behavior, can lead to very different limit behavior, e.g. to a cyclic dynamics. Understanding the epistemic aspects of such oscillating behavior will require logical tools going beyond the fixed point theory used in this paper.⁶

Acknowledgements. Zoé Christoff acknowledges support for this research by the Engineering and Physical Sciences Research Council (EPSRC) grant EP/M015815/1 “Foundations of Opinion Formation in Autonomous Systems”.

The contribution of R.K. Rendsvig to the research reported in this article was funded by the Swedish Research Council through the framework project ‘Knowledge in a Digital World’ (Erik J. Olsson, PI). The Center for Information and Bubble Studies is sponsored by Carlsberg Foundation.

The contribution of S. Smets to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC Grant agreement no. 283963.

References

1. K. R. Apt and E. Markakis. Diffusion in Social Networks with Competing Products. In G. Persiano, editor, *SAGT 2011*, LNCS 6982, pages 212–223. Springer, 2011. [1](#), [4](#)
2. R. J. Aumann. Agreeing to Disagree. *Annals of Statistics*, 4:1236–1239, 1976. [2](#)

⁶ Some very recent work goes towards this direction: [4] investigates the logic of oscillations, [22] defines model transformers for DEL dynamical systems with limit cycles, while [6] proposes an epistemic logic for unadoptable behavior.

3. A. Baltag, Z. Christoff, R. K. Rendsvig, and S. Smets. Dynamic Epistemic Logic of Diffusion and Prediction in Social Networks. *ILLC Prepublication Series*, PP-2015-22, 2015. 1, 2
4. J. Benthem. Oscillations, Logic, and Dynamical Systems. In S. Ghosh and J. Szymanik, editors, *The Facts Matter: Essays on Logic and Cognition in Honour of Rineke Verbrugge*, pages 9–22. College Publications, 2015. 6
5. Z. Christoff. *Dynamic Logics of Networks: Information Flow and the Spread of Opinion*. PhD thesis, Institute for logic, Language and Computation, University of Amsterdam, Amsterdam, The Netherlands, 2016. ILLC Dissertation Series DS-2016-02. 1
6. Z. Christoff, J. Hansen, and C. Proietti. Reflecting on social influence in networks. *Journal of Logic, Language and Information*, 2016. To appear. 6
7. Z. Christoff and J. U. Hansen. A two-tiered formalization of social influence. In D. Grossi, O. Roy, and H. Huang, editors, *Logic, Rationality, and Interaction*, volume 8196 of *Lecture Notes in Computer Science*, pages 68–81. Springer Berlin Heidelberg, 2013. 1
8. Z. Christoff and J. U. Hansen. A logic for diffusion in social networks. *Journal of Applied Logic*, 13(1):48–77, 2015. 1
9. B. A. Davey and H. A. Priestley. *Introduction to Lattices and Order*. Cambridge University Press, second edition, 2002. 3
10. D. Easley and J. Kleinberg. *Networks, Crowds, and Markets*. Cambridge University Press, 2010. 1, 1, 1
11. S. Goyal. Interaction Structure and Social Change. *Journal of Institutional and Theoretical Economics*, 152(3):472–494, 1996. 1
12. M. Granovetter. Threshold Models of Collective Behavior. *American Journal of Sociology*, 83(6):1420–1443, 1978. 1
13. P. G. Hansen, V. F. Hendricks, and R. K. Rendsvig. Infotorms. *Metaphilosophy*, 44(3):301–326, 2013. 1
14. J. Hintikka. *Knowledge and Belief*. Cornell University Press, 1962. 2
15. D. Kempe, J. Kleinberg, and E. Tardos. Maximizing the spread of influence through a social network. In *KDD 03: Proceedings of the ninth ACM SIGKDD international conference on Knowledge discovery and data mining*, pages 137–146. ACM, New York, NY, USA, 2003. 1, 1, 4
16. F.-H. Li, C.-T. Li, and M.-K. Shan. Labeled Influence Maximization in Social Networks for Target Marketing. *2011 IEEE Third Int’l Conference on Privacy, Security, Risk and Trust and 2011 IEEE Third Int’l Conference on Social Computing*, pages 560–563, Oct. 2011. 1
17. F. Liu, J. Seligman, and P. Girard. Logical dynamics of belief change in the community. *Synthese*, 191(11):2403–2431, 2014. 1
18. C. Long and R. C.-W. Wong. Minimizing Seed Set for Viral Marketing Paper. *2011 IEEE 11th International Conference on Data Mining*, pages 427–436, Dec. 2011. 1
19. S. Morris. Contagion. *Review of Economic Studies*, 67:57–78, 2000. 1, 4, 1
20. R. K. Rendsvig. Diffusion, Influence and Best-Response Dynamics in Networks: An Action Model Approach. In R. d. Haan, editor, *Proceedings of the ESSLLI 2014 Student Session*, pages 63–75, Tübingen, 2014. 1, 1
21. R. K. Rendsvig. Pluralistic ignorance in the bystander effect: Informational dynamics of unresponsive witnesses in situations calling for intervention. *Synthese*, 191(11):2471–2498, 2014. 1
22. R. K. Rendsvig. Model Transformers for Dynamical Systems of Dynamic Epistemic Logic. In W. van der Hoek, W. H. Holliday, and W.-f. Wang, editors, *Logic, Rationality, and Interaction (LORI 2015, Taipei)*, LNCS, pages 316–327. Springer, 2015. 6
23. J. Ruan and M. Thielscher. A logic for knowledge flow in social networks. In D. Wang and M. Reynolds, editors, *AI 2011: Advances in Artificial Intelligence*, volume 7106 of *Lecture Notes in Computer Science*, pages 511–520. Springer Berlin Heidelberg, 2011. 1
24. T. C. Schelling. *Micromotives and Macrobehavior*. Norton, New York, 1978. 1
25. J. Seligman, F. Liu, and P. Girard. Logic in the community. In M. Banerjee and A. Seth, editors, *Logic and Its Applications*, volume 6521 of *Lecture Notes in Computer Science*, pages 178–188. Springer Berlin Heidelberg, 2011. 1
26. P. Shakarian, S. Eyre, and D. Paulo. A Scalable Heuristic for Viral Marketing Under the Tipping Model. 2013. 1
27. P. Singh, S. Sreenivasan, B. K. Szymanski, and G. Korniss. Threshold-limited spreading in social networks with multiple initiators. *Scientific Reports*, 3(2330), 2013. 1
28. T. W. Valente. Social network thresholds in the diffusion of innovations. *Social Networks*, 18(1):69–89, Jan. 1996. 1
29. L. Zhen and J. Seligman. A logical model of the dynamics of peer pressure. *Electronic Notes in Theoretical Computer Science*, 278(0):275 – 288, 2011. 1