Sum rules and physical bounds for a particulate slab

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1 Coherent transmitted field by a slab

In recent years, many useful sum rules and physical bounds have been developed for electromagnetic applications [1, 3]. In this paper, we develop two sum rules and corresponding physical bounds for the coherent transmitted electromagnetic field by a particulate slab, and we predict the bandwidth for a given transmission level.

The geometry of the problem is depicted in Figure 1, and for the application in this paper, we specialize to an incident plane wave at normal incidence polarized in the x-direction. For simplicity and to fix ideas, we assume all particles are identical, homogeneous spheres of radius a. The incident and the transmitted coherent (average) field of the slab are

$$\boldsymbol{E}_{\rm i}(z) = \hat{\boldsymbol{x}}\mathrm{e}^{\mathrm{i}kz}, \qquad \langle \boldsymbol{E}_{\rm t}(z)\rangle = \hat{\boldsymbol{x}}t(k)\mathrm{e}^{\mathrm{i}kz}, \quad z > z_2 + a$$

The transmission coefficient $t(k)$ of the slab is [2]

$$t(k) = 1 + \frac{3fkD}{2(ka)^3 d}\sum_{\tau,l} \mathrm{i}^{-l+\tau-1}\sqrt{\frac{2l+1}{8\pi}}\int_{z_1}^{z_2} \mathrm{e}^{-\mathrm{i}kz'} f_{\tau l}(z')\,\mathrm{d}z'$$

The volume fraction of the particles is denoted f. Under the assumption of the Quasi Crystalline Approximation, the coefficients $f_{\tau l}(z)$ satisfy the following system of integral equations [2]:

$$f_{\tau l}(z) = t_{\tau l}\mathrm{i}^{l-\tau+1}\sqrt{2\pi(2l+1)}\mathrm{e}^{\mathrm{i}kz} + \frac{3fkDt_{\tau l}}{4\pi(ka)^3 d}\sum_{\tau' l'}\int_{z_1}^{z_2} K_{\tau l\tau' l'}(z-z') f_{\tau' l'}(z')\,\mathrm{d}z', \quad z\in[z_1, z_2]$$

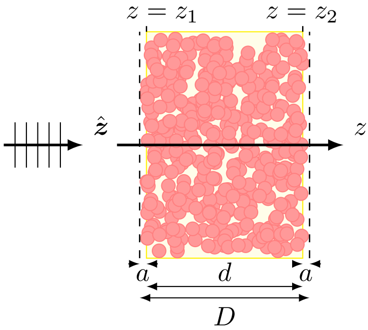

Figure 1: The geometry of the material region $z \in [z_1 - a, z_2 + a]$. The yellow region denotes the region of possible locations of local origins, *i.e.*, the interval $[z_1, z_2]$.

where the transition matrix of the particles are denoted $t_{\tau l}$, and the kernel $K_{\tau l \tau' l'}(z)$ has a closed form solution for the hole correction (HC), *i.e.*, the pair correlation function $g(r) = \mathrm{H(r - 2a)}$, for details see [2].

Under the assumption of hole correction (HC) and for non-magnetic spherical particles of radius a with a permittivity ϵ_1, the low-frequency expression of the transmission coefficient is [2]

$$t(k) = 1 + \frac{3\mathrm{i}fkD}{2} \frac{y}{1 - fy\frac{D}{d}} = 1 + \mathrm{i}kH, \quad y = \frac{\epsilon_1 - \epsilon}{\epsilon_1 + 2\epsilon}$$

2 Analytic properties

In a time-domain setting, a general incident wave of fixed polarization along $\hat{\boldsymbol{x}}$ impinges on the slab. The background wave velocity is c and the wavenumber is denoted $k = \omega/c$. We have

$$\begin{cases} \boldsymbol{E}_\mathrm{i}(z_1 - a, t) = \hat{\boldsymbol{x}} \int_{-\infty}^{\infty} A(k) \mathrm{e}^{-\mathrm{i}kct} \, \mathrm{d}k \\ \boldsymbol{E}_\mathrm{t}(z_2 + a, t + D/c) = \hat{\boldsymbol{x}} \int_{-\infty}^{\infty} t(k) A(k) \mathrm{e}^{-\mathrm{i}kct} \, \mathrm{d}k \end{cases}$$

We assume $\boldsymbol{E}_{\mathrm{i}}(z_1 - a, t) = 0$, $t < 0$ and that $A(k) \in L^2(\mathbb{R})$. If we assume causality, $\boldsymbol{E}_{\mathrm{t}}(z_2 + a, t + D/c) = 0$, $t < 0$, Titchmarsh's theorem implies that $A(k)$ and $t(k)$ are analytic in the upper complex plane $\mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. The assumption of passive material, implies $|t(k)| \leq 1$, which can be analytically continued to $|t(k)| \leq 1$ in $\mathbb{C}_+ \cup \mathbb{R}$.

Sum rule with the logarithm: Construct a function $h_1(k)$ from the transmission coefficient $t(k)$.

$$h_1(k) = -\mathrm{i} \ln \left(t(k) \prod_{n=1}^{N} \frac{1 - k/k_n^*}{1 - k/k_n} \right)$$

where k_n, $n = 1, 2, \ldots, N$, are the zeros of the transmission coefficient $t(k)$ in $\mathbb{C}_+$. Note that $\operatorname{Im} h_1(k) = -\ln |t(k)|$, $k \in \mathbb{R}$.

The low-frequency behavior of $h_1(k)$ is

$$h_1(k) = kH + 2k \sum_{n=1}^{N} \operatorname{Im} \frac{1}{k_n} + O(k^2)$$

Consequently, the sum rule is [1]

$$-\int_0^\infty \ln |t(\lambda)| \, \mathrm{d}\lambda = \pi^2 \left(H + 2 \sum_{n=1}^{N} \operatorname{Im} \frac{1}{k_n} \right) \leq \pi^2 H$$

Let $I(t_0)$ (length $|I(t_0)|$) denote the wavelength interval defined by $|t(\lambda)| \leq t_0 \in (0, 1)$. Then, a crude estimate of the sum rule implies

$$|I(t_0)| \ln \frac{1}{t_0} \leq \int_{I_0} \ln \frac{1}{|t(\lambda)|} \, \mathrm{d}\lambda \leq \int_0^\infty \ln \frac{1}{|t(\lambda)|} \, \mathrm{d}\lambda \leq \pi^2 H$$

Sum rule with pulse Herglotz function: A more elaborate function is the pulse Herglotz function [1].

$$h_\Delta(z) = -\frac{1}{\pi} \int_{-\Delta}^{\Delta} \frac{1}{z - t} \, \mathrm{d}t = \frac{1}{\pi} \ln \frac{z - \Delta}{z + \Delta}, \quad \operatorname{Im} z > 0, \ \Delta > 0$$

The imaginary part of $h_\Delta(z)$ is non-negative and bounded by unity in $\mathbb{C}_+$. Specifically, the inner part of the circle $|z| = \Delta$ in $\mathbb{C}_+$ maps to $1/2 < \operatorname{Im} h_\Delta(z) < 1$.

The Möbius transformation $w(z) = \mathrm{i}(1+z)/(1-z)$ maps the unit circle to $\mathbb{C}_+$. Define the function $h_2(k)$ with asymptotes

$$h_2(k) \stackrel{\text{def}}{=} h_\Delta(w(t(k))) = \frac{k\Delta H}{\pi} + O(|k|^2), \quad k \to 0$$

The parameter $\Delta = \Delta(t_0)$ and the threshold $|t| = t_0 \leq 1$ is set to $\Delta(t_0) = (1 + t_0)/(1 - t_0) > 0$, and the sum rule reads [1]

$$\int_0^\infty \operatorname{Im} h_2(\lambda)\, \mathrm{d}\lambda = \pi\Delta H$$

This is an exact sum rule and serves as an independent check on the numerical precision in the numerical calculations.

With the wavelength interval $I(t_0)$ defined above, the integrand is larger than 1/2 by construction, and we get

$$\frac{|I(t_0)|}{2} \leq \int_{I(t_0)} \operatorname{Im} h(\lambda)\, \mathrm{d}\lambda \leq \int_0^\infty \operatorname{Im} h(\lambda)\, \mathrm{d}\lambda = \pi\Delta H$$

References

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