On Performance Guarantees for Systems with Conic Constraints

Vladu, Emil

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LUND UNIVERSITY

PO Box 117
221 00 Lund
+46 46-222 00 00

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EMIL VLADU
DEPARTMENT OF AUTOMATIC CONTROL | LUND UNIVERSITY

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LUND UNIVERSITY

Department of Automatic Control
P.O. Box 118, 221 00 Lund, Sweden
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Abstract

In this thesis, we provide a number of novel algebraic means of certifying stability and performance for linear systems constrained in various ways by cones. The purpose is mainly threefold: to provide mathematical statements with applicative potential, to unify seemingly dissimilar results in the literature and thereby increase understanding, and to advance the state of the art on dynamical systems with conic constraints, an area of control still in its infancy. The main contributions of the five papers contained in the thesis are as follows. Paper I provides an analytical upper bound on the deviation from H-infinity optimality of a certain controller class as a function of the deviation from symmetry in the state matrix. Paper II goes on to establish a diagonal solution to a Riccati inequality which certifies H-infinity optimality of a particular controller both when the open-loop state matrix is symmetric and when the closed-loop system is positive. In Paper III, a necessary and sufficient condition is given in the form of a stable coefficient matrix for a nonsymmetric Riccati equation to admit a stabilizing cone-preserving solution. This result is subsequently applied in Paper IV to obtain a nonsymmetric variant of the bounded real lemma in H-infinity control on self-dual cones. Finally, Paper V establishes an equivalence between the existence of a bounded linear functional satisfying a conic inequality and the satisfaction of certain integral linear constraints on trajectories confined to a cone. This result in turn yields a non-strict version of the Kalman-Yakubovich-Popov Lemma when the cone is taken as the positive semidefinite cone, thereby serving to further bring together linear-cone theory with the dominating linear-quadratic paradigm in control.

Acknowledgments

There are many people to whom I owe a great deal for helping me in various ways to complete this thesis. First and foremost, I would like to thank my main supervisor Anders Rantzer for his many brilliant insights and invaluable opinions on my results, and for always being positive and encouraging with me. Thank you in particular for all the laughter and the many glimmers in the eyes shared over various exciting ideas! To Carolina Bergeling, my co-supervisor during the beginning of my studies, I am grateful in particular for all those talks that instilled confidence in me when I needed it the most. To Dongjun Wu, my co-supervisor during the latter part of my studies, thank you for your meticulous proofreading of my manuscripts and for always cheering me on. Finally, a very special thank you to Pontus Giselsson for what may have been the single most important meeting throughout my studies, in which my view on research changed completely.

I also consider myself very fortunate to have been part of the Department of Automatic Control at Lund University. This is a place with a heart, and it houses not only great competence within its walls, but also laughter and compassion and warmth and humanity. A heartfelt thank you to everyone with whom I have had the pleasure to share a coffee at fika: it really does wonders on a daily basis! A special thanks to Bo Bernhardsson and Richard Pates for their ability to make me laugh. Thank you also to Frida Heskebeck and Pex Tufvesson for having been such amazing office mates for such a long time. Similarly, I am grateful to Eva Andersson and Susanne Mårtensson for the many lovely hallway chats over the years, and thank you to the entire cleaning staff for all the hard work in making our facilities look as fresh and clean as they do. I am also grateful to the TA team for their seemingly infinite patience with me and my many technical failings. A very special thank you to its head, Eva Westin, for her wisdom and for all our amazing discussions.

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I am to have crossed paths with you in life. Finally, I am so very grateful to my parents, my role models and heroes, Livia Vladu and Gheorghe Vladu, for their unconditional love and support.

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Contents

Nomenclature

Notation	**Description**
$\mathbb{R}(\mathbb{C})$	Set of real (complex) numbers.
$\mathbb{R}^n(\mathbb{C}^n)$	Set of vectors with entries in $\mathbb{R}(\mathbb{C})$.
$\mathbb{R}^{n\times m}(\mathbb{C}^{n\times m})$	Set of $n \times m$ matrices with entries in $\mathbb{R}(\mathbb{C})$.
$\mathbb{R}^n_+$	Set of entrywise nonnegative vectors in $\mathbb{R}^n$.
$\mathcal{S}^n$	Set of $n \times n$ symmetric matrices.
$\mathcal{S}^n_+$	Set of $n \times n$ positive semidefinite matrices.
$\mathrm{Int}(M)$	Interior of a set M.
$I(I_n)$	Identity matrix (of dimension n).
$\\|\cdot\\|$	Matrix spectral norm.
$\geq$ $(>)$	Entrywise inequality for scalars, vectors and matrices.
$\succeq$ $(\succ)$	Partial order induced by $\mathcal{S}^n_+$.
$\succeq_K$ $(\succ_K)$	Partial order induced by a proper cone $K \subseteq \mathbb{R}^n$ i.e., $x \succeq_K y \Leftrightarrow x - y \in K$ and $x \succ_K y \Leftrightarrow x - y \in \mathrm{Int}(K)$.
K^*	Dual of a cone $K \subseteq \mathbb{R}^n$.
z^*	Denotes bounded linear functionals in general.
Z^*	Denotes the dual space corresponding to a normed vector space Z in general.
$B(Z, X)$	Set of bounded linear transformations between the normed vector spaces Z and X.
$\\|\cdot\\|_Z$	Norm corresponding to a normed vector space Z.
$L^n_2(-\infty, \infty)$	Set of square-integrable $\mathbb{C}^n$-valued functions of time.
$\\|\cdot\\|_\infty$	H_∞ norm of a stable transfer function.

1

Introduction

Control technology is ubiquitous in society. It is concerned with achieving a desired system behavior without human intervention. The applications in modern society are innumerable: transportation networks, autonomous vehicles, automatic flight control, power grid stabilization, district heating control, robotics, automatic control in process industry and router protocols are all a testament to the great societal use of control [Åström and Murray, 2021]. What is remarkable is that there is a common structure to the control of all the aforementioned applications. At the heart lies the phenomenon called feedback, which happens when several interconnected systems influence one another. It occurs naturally in biological systems, such as when maintaining a constant body temperature, and when it occurs in engineering, one or more of these interconnected systems can be a so-called controller. This is essentially a device that takes data from sensors as an input and subsequently issues corrective actions so as to achieve a desired behavior. In doing this, the device obeys a control law, i.e., a formula designed by an engineer to map sensory inputs to corrective outputs. As a step in the design process, being able to analyze the behavior of the current and desired system is fundamental. Importantly, experience in terms of not only intuition but also knowledge is crucial for performing this task properly.

Motivation

As it happens, many common denominators in control applications, such as those outlined above, can be captured by the language of mathematics. For instance, the essence of a system can be modeled using differential equations obtained from physical laws. Certainly, far from all aspects of a real world system can be captured in this way (imagine accounting for every tiny speck of dust!), but if sufficiently many dominant features are taken into account, then, perhaps surprisingly, a remarkable correspondence with reality is often observed. By removing so-called higher order (and therefore less important) terms in a Taylor expansion, linear systems aim to achieve precisely this in a controlled way. And astoundingly, for such systems, intuitive notions relevant to control, when properly formalized, can be verified algebraically and numerically by means of computers. For instance, by Lyapunov's well-known theorem [Lyapunov, 1992] for unforced linear systems

$$\dot{x} = Ax,$$

stability is equivalent to the existence of a positive definite matrix P such that

$$A^T P + PA$$

is negative definite. As a result, if an engineer fails to find such a matrix P numerically, s/he can suspect on reasonable grounds that the system under scrutiny will be ill-behaved. Similarly, other kinds of beneficial constraints on the system dynamics can be established numerically or even graphically in the frequency domain by means of such analytical results. Thus, an engineer well-versed in theory is capable of making quality predictions on the behavior of a system in a controlled fashion. This is especially important since many notions fundamental to control, such as feedback and causality, can be unintuitive and treacherous and therefore lead to disaster in safety-critical systems if not treated properly.

In summary, the mathematical theory associated with control offers sophisticated means of making informed predictions about the behavior of a system, as well as synthesizing useful control laws. Thus, new results of this character can both advance the state of the art and open doors to new creative applications in the spirit of the previous paragraph. Further, theory also allows for a unified treatment of various seemingly dissimilar phenomena. As a result, understanding is gained for what truly transpires, i.e., what the essence of the matter is. Moreover, the results which are generated in the process can also be exploited to solve novel problems. This constitutes the main motivation and justification for the contributions presented in this thesis.

Thesis Topic

In one way or another, the main contributions of this thesis involve an algebraic solution to an equation or inequality in order to verify some dynamical property such as stability or some performance criterion. The latter will almost exclusively be an L_2-gain bound, associated usually with worst-case disturbance rejection. A surprising variety of such means of verification can be found in the domain called positive systems theory, a fairly recent addition to control with applications in areas such as biology, economy, Markov models and large-scale control. As such, positivity or more generally conic constraints on the trajectories of a given dynamical system is a recurring theme, and one purpose of the work presented here is to blur the line between traditional linear-quadratic control and the niche but appealing positive systems theory. Thus, the subject of this thesis lies mainly at the intersection of cone theory and control, with methods of the former leveraged for the benefit of the latter.

Statement of Contribution

In this section follows a list of the author's work, as well as a description of the corresponding contributions for the papers included in the thesis.

Paper I

Vladu, E., and A. Rantzer (2022). "H-infinity control with nearly symmetric state matrix". *IEEE Control Syst. Lett. 6*, pp. 3026–3031.

Abstract: In this letter, we give an upper bound on the deviation from H-infinity optimality of a class of controllers as a function of the deviation from symmetry in the state matrix. We further suggest a scalar measure of symmetry which is shown to be directly relevant for estimating nearness to optimality. In connection to this, we give a simple analytical solution to a class of Lyapunov equations for two dimensional state matrices. Finally, we demonstrate how a well-chosen symmetric part for nearly symmetric state matrices may lead not only to near-optimality, but also to controller sparsity, a desirable property for large-scale systems. In the special case that the state matrix is symmetric and Hurwitz, our main result simplifies to give an H-infinity optimal controller with several benefits, a result which has recently appeared in the literature. In this sense, the above is a significant generalization which considers a much wider class of systems, yet allows one to retain the benefits of symmetric state matrices, while offering means of quantifying the effect of this on the H-infinity norm.

Authors' Contributions: E. Vladu conceived of, formulated and proved all the results and prepared the manuscript. A. Rantzer reviewed the manuscript and suggested modifications to the theorem formulations.

Paper II

Vladu, E. (2024). "A unifying statement for an H-infinity optimal controller with positivity properties". In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 3686–3691.

Abstract: In this paper, we unify two already published results on state feedback H-infinity optimality. Previously, optimality has been shown for a particular controller structure in the case that the open-loop state matrix is symmetric, as well as in the case that the closed-loop system is internally positive. By contrast, the main result of the present paper gives optimality based on neither of these two properties. As a result, when applied to a class of buffer networks, it succeeds not only in showing optimality when the system parameters are chosen so as to give open-loop symmetry and closed-loop positivity, respectively, but also when both of these properties are absent.

Author's Contributions: E. Vladu conceived of, formulated and proved all the results and prepared the manuscript.

Paper III

Vladu, E. and A. Rantzer (2025). "A cone-preserving solution to a nonsymmetric Riccati equation". *Linear Algebra Appl.* **709**, pp. 449–459.

Abstract: In this paper, we provide the following simple equivalent condition for a nonsymmetric Algebraic Riccati Equation to admit a stabilizing cone-preserving solution: an associated coefficient matrix must be stable. The result holds under the assumption that said matrix be cross-positive on a proper cone, and it both extends and completes a corresponding sufficient condition for nonnegative matrices in the literature. Further, key to showing the above is the following result which we also provide: in order for a monotonically increasing sequence of cone-preserving matrices to converge, it is sufficient to be bounded above in a single vectorial direction.

Authors' Contribution: E. Vladu conceived of, formulated and proved all the results and prepared the manuscript. A. Rantzer was behind an idea to significantly shorten the proof of Theorem 2. Both authors contributed to reviewing the manuscript.

Paper IV

Vladu, E. (2024). *Stability and performance analysis on self-dual cones*, arXiv: 2411.12100 [math.OC]

Abstract: In this paper, we consider nonsymmetric solutions to certain Lyapunov and Riccati equations and inequalities with coefficient matrices corresponding to cone-preserving dynamical systems. Most results presented here appear to be novel even in the special case of positive systems. First, we provide a simple eigenvalue criterion for a Sylvester equation to admit a cone-preserving solution. For a single system preserving a self-dual cone, this reduces to stability. Further, we provide a set of conditions equivalent to

testing a given H-infinity norm bound, as in the bounded real lemma. These feature the stability of a coefficient matrix similar to the Hamiltonian, a solution to two conic inequalities, and a stabilizing cone-preserving solution to a nonsymmetric Riccati equation. Finally, we show that the H-infinity norm is attained at zero frequency.

Author's Contribution: E. Vladu conceived of, formulated and proved all the results and prepared the manuscript. In introducing the notion of the Lyapunov-like function of two variables, he drew inspiration from a discussion with A. Rantzer.

Paper V

Vladu, E., A. Megretski, and A. Rantzer (2025). "On integral linear constraints on convex cones".

Abstract: In this paper, we consider integral linear constraints and the dissipation inequality with linear supply rates for certain sets of trajectories confined pointwise in time to a convex cone which belongs to a finite-dimensional normed vector space. Such constraints are then shown to be satisfied if and only if a bounded linear functional exists which satisfies a conic inequality. This is analogous to the typical situation in which a quadratic integrand over the entire space is related to a linear matrix inequality. A connection is subsequently drawn precisely to linear-quadratic control: by proper choice of cone, the main results can be applied to produce a known L_1-gain analog to the bounded real lemma in positive systems theory, as well as a non-strict version of the Kalman-Yakubovich-Popov Lemma in linear-quadratic control.

Authors' Contribution: E. Vladu conceived of, formulated and proved the main results in their original forms. Based on this, he also produced the idea behind the dynamics version proof of the KYP Lemma presented in the paper. He further introduced the K-analogs to certain standard system theoretic concepts. A. Rantzer contributed through recurrent and inspiring discussions with E. Vladu during this process. A. Megretski suggested separating hyperplanes as a proof technique in order to remove an initially troublesome assumption in the main result, recognized the useful equilibrium condition, noticed the independence of dynamics in some results as well as several redundant assumptions, and contributed with the parallel algebraic proof.

Additional Publications

In addition to the above papers, the author has produced the following publications.

Vladu, E., and A. Rantzer (2022) "On decentralized H-infinity optimal positive systems”. *IEEE Control Syst. Lett. 7,* pp. 391–394.

Vladu, E., C. Bergeling, and A. Rantzer (2021). “Global solution to an H-infinity control problem with input nonlinearity”. In: *Proc. IEEE Conf. Decis. Control (CDC),* pp. 3237–3242.

Vladu, E., and A. Rantzer (2022). "Global solution to an H-infinity control problem for control-affine systems”. In: *Proc. 25th Int. Symp. Math. Theory Netw. Syst. (MTNS); IFAC-PapersOnLine 55.30*, pp. 388–393.

2

Background

In this section, we summarize past results relevant to the contributions of the present thesis. The field of control is immeasurably large and consists of many subdomains. As such, there is no hope of including even a sliver of all the research that deserves to be recognized, nor is it the aim. Instead, we recall a set of classical results along with some recent contributions with direct relevance to the material in Chapter 4. The overarching goal is to highlight both the contrasts and the curious parallels between standard linear-quadratic control and the more recent theory of positive systems.

The outline of this section is as follows: in the first subsection, we summarize a set of cornerstones in linear-quadratic theory as well as some recent results. The next subsection subsequently recalls some central results in positive systems theory, with the third subsection doing the same for cone theory. Unless otherwise stated, we consider linear systems on the form

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the regulated output. For illustrative purposes, we consider only the case with zero direct term, but the rule rather than the exception is that all cited results hold also when such a term is included. No contributions of the author appear in this chapter.

Linear-Quadratic Control

Linear-quadratic control is arguably a dominating paradigm in control theory. It has a long history, and can roughly be said to concern problems with quadratic performance objectives for linear dynamics. Typical features of such problems include explicit solutions in terms of linear matrix inequalities (LMIs), algebraic Riccati equations (AREs) or the satisfaction of frequency inequalities. This in turn allows for numerical means of performing either analysis or controller synthesis as needed.

LQR and the KYP Lemma

At its inception in the 1960s was the linear-quadratic regulator problem (LQR), a variational problem concerned with finding a feedback law such that a quadratic cost in state and control is minimized for each initial value [Kalman et al., 1960]. In particular, the feedback law and the minimizing value were shown to be tied to an algebraic (differential) Riccati equation in the infinite (finite) horizon case. From an application point of view, the interest may lie for instance in returning to the origin swiftly under reasonable actuator effort.

Around the same time, the Lur'e (absolute stability) problem in nonlinear control was being addressed: for an interconnection consisting of a linear part and a memoryless nonlinearity, find conditions on both parts to ensure global asymptotic stability of the origin of the closed-loop system, e.g., [Khalil, 1992]. Among other things, these efforts gave rise to the Popov criterion [Popov, 1961]. In connection to this, Kalman [Kalman, 1963] and Yakubovich [Yakubovich, 1962] introduced a version of what is now known as the Kalman-Yakubovich-Popov (KYP) Lemma in order to prove Popov's criterion, i.e., to pass from a set of frequency inequalities to an LMI which in turn produces a Lyapunov function certifying stability. In time, however, its use shifted in the other direction, i.e., verifying a frequency inequality by means of an LMI. This shift was stimulated by new interior-point methods for solving certain convex optimizations problems in polynomial time [Nesterov and Nemirovskii, 1994] and the subsequent application of convex optimization to solve control problems with numerical efficiency [Boyd et al., 1994].

Following its inception, the KYP Lemma underwent many generalizations, e.g., [Anderson, 1967] to a multivariable setting, and one version of the lemma is the following [Willems, 1971][Rantzer, 1996]:

Proposition 1
Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$ *with* (A, B) *controllable and* $\det(i\omega I - A) \neq 0$ *for all* $\omega \in \mathbb{R}$, *the following statements are equivalent:*

(i) For all $\omega \in \mathbb{R} \cup \{\infty\}$,

$$\begin{pmatrix} (i\omega I - A)^{-1}B \\ I \end{pmatrix}^* M \begin{pmatrix} (i\omega I - A)^{-1}B \\ I \end{pmatrix} \preceq 0.$$

(ii) There exists a matrix $P \in \mathbb{R}^{n\times n}$ such that $P = P^T$ and

$$M + \begin{pmatrix} A^T P + PA & PB \\ B^T P & 0 \end{pmatrix} \preceq 0.$$

The corresponding equivalence for strict inequalities holds even if (A, B) is not controllable.

It is important to note that the above is far from the only version of the KYP Lemma. For instance, a version in which the frequency inequality is required to hold only on a compact interval can be found in [Iwasaki et al., 2000][Iwasaki and Hara, 2005], which exploits a generalization of the S-procedure [Yakubovich, 1977] as a basis for the proof. Other versions can also be found in [Megretski, 2010]. As for proofs, the first versions exploited spectral factorizations and value functions, whereas subsequent proofs were completely algebraic and exploited separating hyperplanes [Rantzer, 1996].

Linear-Quadratic Minimization and Dissipativity

Around a decade later, the variational interpretation of the above stability criteria was explored in [Willems, 1971]. In particular, a number of dynamical minimization problems was posed of the type

$$\inf_{u\in L_{2e}} \int_0^\infty \begin{pmatrix} y \\ u \end{pmatrix}^T M \begin{pmatrix} y \\ u \end{pmatrix} \, \mathrm{dt},$$

where M no longer had to satisfy the typical definiteness conditions seen in LQR, and the terminal state could be either free or constrained to the origin. Under the assumption of controllability, the boundedness of the infima were related precisely to the above frequency inequality and LMI in Proposition 1. Further, the actual value attained for the minimum was shown to be connected to the maximal solution of an ARE.

Yet another criterion that was supplied for the boundedness of the infimum was the satisfaction of the dissipation inequality, i.e., the existence of a function $V : \mathbb{R}^n \to \mathbb{R}$, called a storage function, such that

$$V(x(t_0)) + \int_{t_0}^{t_1} \begin{pmatrix} x \\ u \end{pmatrix}^T M \begin{pmatrix} x \\ u \end{pmatrix} \, \mathrm{dt} \geq V(x(t_1)) \tag{2.1}$$

for all $t_1 \geq t_0$, all initial conditions $x(t_0)$ and all trajectories (x, u) satisfying linear dynamics. When in particular $V \geq 0$ and the integrand is a general function $w(y, u)$, called the supply rate, the system (possibly nonlinear) is said to be dissipative w.r.t. $w(y, u)$. A physical interpretation of the storage function is the energy stored in the system, and the dissipation inequality may then be interpreted as saying that the power supplied to the system is greater than the change in energy level $V(x(t_1)) - V(x(t_0))$ that it causes, i.e., the system is dissipating.

The notion of dissipativity was thoroughly examined in [Willems, 1972a] for general (possibly nonlinear) systems, and it can be thought of as an open-system analog to the concept of stability. While initially inspired by passivity and its beneficial properties in the context of interconnected systems, by proper choice of the supply rate $w(y, u)$, the dissipativity framework can capture also additional concepts, e.g., upper bounds on the L_2-gain [Van Der Schaft, 1992][Van der Schaft, 2017]. Finally, a continuation of [Willems, 1972a] for linear systems with the special supply rate $w(y, u) = y^T u$ corresponding to passivity led to [Willems, 1972b] in which characterizations of dissipativity are given both in terms of AREs and frequency inequalities, analogous to the above.

H_∞ Control and Worst-case Disturbance Rejection

Yet another decade later, in the 1980s, much effort was expended on the so-called H_∞ control problem: minimize the H_∞ norm (L_2-gain) of a closed-loop transfer function G_K from disturbances to regulated outputs over the set of stabilizing controllers K. Since for stable transfer functions G

$$\|G\|_\infty := \sup_{\omega \in \mathbb{R}} \|G(i\omega)\| = \sup_{u \neq 0} \frac{\|y\|_2}{\|u\|_2},$$

one interpretation of this problem is to control the system so as to minimize the worst-case impact of unit norm disturbances on the output.

In the case of analysis, a central result is arguably the so-called bounded real lemma which relates a number of equivalent conditions to the γ-suboptimality of the H_∞ norm of a stable transfer function G. When the direct term is set to zero so that $G = C(sI - A)^{-1}B$, one version is as follows.

Proposition 2
[Zhou and Doyle, 1998, Corollary 12.3] Let $\gamma > 0$ and suppose that A is Hurwitz. Define now

$$H = \begin{pmatrix} A & \frac{1}{\gamma^2} BB^T \\ -C^T C & -A^T \end{pmatrix}.$$

The following conditions are equivalent:

(i) $\|G\|_\infty < \gamma$

(ii) *H has no eigenvalues on the imaginary axis.*

(iii) There exists a $P \succeq 0$ *such that*

$$\frac{1}{\gamma^2} PBB^T P + A^T P + PA + C^T C = 0 \tag{2.2}$$

and $A + \frac{1}{\gamma^2} BB^T P$ *has no imaginary axis eigenvalues.*

(iv) There exists a $P \succ 0$ *such that*

$$\frac{1}{\gamma^2} PBB^T P + A^T P + PA + C^T C \prec 0. \tag{2.3}$$

Here, the matrix H in condition (ii) is called the Hamiltonian, the equation in condition (iii) is an ARE and the inequality in condition (iv) is an ARI. In order to relate Proposition 2 to the material in the previous subsections, invoke the Schur complement lemma [Zhang, 2006] on the ARI to obtain an LMI on the one hand, and note that condition (i) follows by choosing

$$M = \begin{pmatrix} C^T C & 0 \\ 0 & -\gamma^2 I \end{pmatrix}$$

in the strict version of the KYP Lemma. Similarly, in a dissipativity setting, the supply rate $w(y,u) = y^T y - \gamma^2 u^T u$ can be chosen for a finite-time analog, e.g., [Van Der Schaft, 1992].

A solution to the corresponding output feedback synthesis problem and its connection to AREs was provided by [Doyle et al., 1988]. The controller obtained thus, called the central controller, could subsequently be exploited to construct other viable controllers resulting from a linear fractional transformation and a free parameter. By contrast, an alternative approach to the synthesis problem by means of ARIs/LMIs came in [Gahinet and Apkarian, 1994], and it relied heavily on the bounded real lemma. For more on H_∞ and robust control in general, see e.g., [Zhou and Doyle, 1998][Dullerud and Paganini, 2013][Kwakernaak, 1993], and [Khargonekar et al., 1988][Scherer, 1989] for the state feedback problem in particular.

Positive Systems

In this section we consider the much younger subfield in control called positive systems theory. It is concerned with the so-called positive systems, i.e., systems for which

$$x(0) \geq 0 \text{ and } u(t) \geq 0$$

for all $t \geq 0$ imply

$$x(t) \geq 0 \text{ and } y(t) \geq 0$$

for all $t \geq 0$. Such systems exhibit curious properties, and stability and performance verification as well as controller synthesis simplify greatly as compared to the case for general systems in linear-quadratic control, e.g., [Farina and Rinaldi, 2000][Rantzer and Valcher, 2018] and the references therein. The basis for the theory exploits Perron-Frobenius theory for nonnegative matrices to a great extent [Perron, 1907][Frobenius, 1912][Berman and Plemmons, 1994], and among its many applications of the theory are biological systems, economics and Markov models. Technically, attention was called to this area already in the 1970s [Luenberger, 1979], and in the 1990s efforts were directed mainly toward the positive realization problem, e.g., [Farina, 1996][Anderson et al., 1996][Benvenuti and Farina, 2004], as well as the relation between controllability and reachability on the nonnegative orthant on the one hand, and the corresponding standard notions on the other, e.g., [Ohta et al., 1984][Coxson and Shapiro, 1987][Fanti et al., 1990][Valcher, 1996][Valcher, 2009]. Related classes of systems have also been treated extensively, e.g., those whose linearization along any trajectory is positive [Forni and Sepulchre, 2015].

Stability

In the last decade, however, focus has shifted to applications for large-scale systems, for which positive systems theory appears to be eminently suited. The reason is mainly twofold: scalable stability and performance certificates, and structured controller synthesis by means of linear programming. An immediate example is given by the following equivalent condition for the stability of a Metzler matrix $A \in \mathbb{R}^{n \times n}$

$$\exists \; p > 0 \text{ such that } Ap < 0 \tag{2.4}$$

and another by

$$\exists \text{ diagonal } D \succ 0 \text{ such that } A^T D + DA \prec 0. \tag{2.5}$$

Relevant in this context is also the following condition.

$$\exists \; P \text{ with } P + P^T \succ 0 \text{ such that } A^T P^T + PA \prec 0. \tag{2.6}$$

These conditions should be compared to the standard way in which to verify stability for general A, i.e., positive semidefinite solutions to a Lyapunov equation. The two first stability certificates scale linearly with the dimension of the system; by contrast, a non-diagonal solution scales quadratically with the dimension. Note also that the first condition applied to A^T gives rise to a linear Lyapunov function $V(x) = p^T x$ for the system $\dot{x} = Ax$ which is nonnegative and decreasing on the nonnegative orthant.

Performance

As for performance analysis, there are results which may be thought of as analogs to the bounded real lemma in Proposition 2 above, e.g., the following result on the γ-suboptimality of the L_1-gain [Ebihara et al., 2011][Briat, 2013] (and its L_∞-gain analog), quoted here for the standard L_1-norm cost.

Proposition 3
[Ebihara et al., 2011, Theorem 3] Let $\gamma > 0$ be given along with the LTI system $\mathcal{G} = (A, B, C)$ with A Metzler and $B, C \geq 0$. Then the following conditions are equivalent:

(i) A is Hurwitz and

$$\|\mathcal{G}\|_{L_1} < \gamma.$$

(ii) There exists $p > 0$ such that

$$\begin{aligned} A^T p + C^T \mathbb{1}_p &< 0 \\ B^T p - \gamma \mathbb{1}_m &< 0. \end{aligned} \tag{2.7}$$

The above result can perhaps be compared to the stability test (2.4). By contrast, an analog to (2.5) exists in the form of a diagonal solution to the LMI in Proposition 2 as an equivalent condition to the γ-suboptimality of the H_∞ norm [Tanaka and Langbort, 2011]. This was later generalized in [Rantzer, 2015a] to a result very much reminiscent of the KYP Lemma in Proposition 1, in which a diagonal solution to the LMI is necessary and sufficient for a corresponding frequency inequality to hold. It is relevant to mention also the additional vectorial condition which was added, quite reminiscent of (2.7). Finally, an analog to (2.6) was obtained in [Ebihara et al., 2014] and was used to obtain less conservative performance estimates when parametric uncertainty is included.

Controller Synthesis

In the context of positive systems, there are some remarkable results in the area of controller synthesis which we mention for the sake of context. The main appeal is that for any desired sparsity structure on a controller, the existence of a stabilizing static controller which results in a closed-loop system which is positive and γ-suboptimal can be established by solving a linear program. The corresponding results with an L_1/L_∞-gain performance objective was obtained in [Rantzer, 2015b]; the case with an H_∞-norm performance objective was examined in [Tanaka and Langbort, 2011]. Given the context that optimizing over structured controllers is in general non-convex [Lessard and Lall, 2011], the fact that it can be reduced to a linear program is striking.

Cone-preserving Systems

It is clear from the previous section that positive systems offer a surprising amount of ways in which to verify both stability and performance. In order to better understand these phenomena and to increase the pool of potential applications, researchers took to studying the conic structure of the nonnegative orthant. The theory for cone-preserving matrices is well-developed, e.g., [Berman and Plemmons, 1994][Schneider and Tam, 2006][Barker, 1981] and the references therein, but the corresponding theory for systems theory and control is arguably still at its infancy. Below we attempt to summarize the progress made so far.

Cones and Linear-Quadratic Control

In the field of control, linear-quadratic control plays a central role. By contrast, although very appealing, positive systems theory is held back by the fundamental assumption of positivity and as a result can be applied only to a niche set of systems. From a mathematical point of view, however, matters are quite different. This can be seen by considering the ubiquitous Lyapunov and Riccati equations in linear-quadratic control, e.g.,

$$\exists\, P \succ 0 \text{ such that } A^T P + PA \prec 0$$

to certify stability. Symmetry is the characteristic feature of these equations and inequalities not only in their form but also in the solutions that are sought after. It turns out, however, that the more fundamental and basic stability test is the one exploited for Metzler matrices A in positive systems (2.4), i.e.,

$$\exists\, p > 0 \text{ such that } Ap < 0.$$

In fact, the relevant structure that generates both of these phenomena is the conic structure of the positive semidefinite cone $\mathcal{S}_+^n$ and the nonnegative orthant $\mathbb{R}_+^n$ in $\mathcal{S}^n$ and $\mathbb{R}^n$, respectively. More specifically, given a finite-dimensional Hilbert space V and a proper cone $K \subseteq V$, the following result was formalized in [Gowda and Tao, 2009]: if a linear operator $L : V \to V$ with the associated system $\dot{x} = L(x)$ preserves the cone K in the sense that

$$x(0) \in K \Rightarrow x(t) \in K$$

for all $t \geq 0$, then $\dot{x} = L(x)$ is asymptotically stable if and only if

$$\exists\, P \succ_K 0 \text{ such that } L(P) \prec_K 0.$$

Operators which satisfy this cone-preserving property are known by various names in the literature, but suffice it to say that they were examined thoroughly for finite-dimensional spaces already in the 1970s under the name of cross-positivity [Schneider and Vidyasagar, 1970]. In the case of linear-quadratic control, the Lyapunov operator $L(P) = A^T P + PA$ preserves $\mathcal{S}_+^n$ and the existence of a $P \succ 0$ shows that $\dot{X} = L(X)$ is asymptotically stable, which in turn is equivalent to the stability of $\dot{x} = Ax$. Thus, the same mechanism lies at the heart of both ways of verifying stability. It is in this sense that the conic structure of the nonnegative orthant for positive systems can in fact be said to be more basic than its symmetric and quadratic counterpart over all of $\mathbb{R}^n$.

Very recently, various other results in linear-quadratic control were considered through the unifying lens offered by the above connection between cones and linear dynamical systems. In particular, [Bamieh, 2024] considers classical finite-horizon results on LQR and integral quadratic constraints and demonstrates the connection to differential Riccati equations by means of a general duality result for cones on Banach spaces. By contrast, [Pates and Rantzer, 2024] exploits the Bellman equation in discrete time in order to draw parallels between the recent [Rantzer, 2022] and LQR. Specifically, both pass through the matrix dynamical system

$$\dot{X} = A^T X + XA + BU + U^T B^T \tag{2.8}$$

for which quadratic performance objectives are rendered linear. However, as a result two additional constraints are incurred: a conic constraint and a rank 1 constraint. This follows as the dynamics of the original system $\dot{x} = Ax + Bu$ is known to be represented by rank 1 matrix trajectories existing on the boundary of the cone $\mathcal{S}_+^{n+m}$. Note that system (2.8) is an extension of the above system $\dot{X} = L(X)$, where L is the Lyapunov operator, and that it has been leveraged similarly in the past in order to reduce linear-quadratic problems to semidefinite programs, e.g., [Gattami, 2009] in the context of stochastic control.

Monotone Linear Systems

As seen in the previous section, stability for Metzler (cross-positive) matrices A can be verified either directly by exploiting the fact that $\dot{x} = Ax$ preserves the nonnegative orthant (a proper cone), or by resorting to the roundabout way applicable to general A enabled instead by the cone-preservance of the Lyapunov operator. What is perhaps more curious is that for Metzler matrices, benefits arise also when verifying stability in the latter (Lyapunov) way, as evidenced by the necessary and sufficient diagonal solution for stability in (2.5). We now consider to which extent this and some other remarkable properties mentioned in the previous section can be generalized to cone-preserving systems.

The standard definition in the literature for a cone-preserving or monotone LTI system given a proper cone triple $(K_u, K_x, K_y) \subseteq (\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p)$ is the following:

$$x(0) \succeq_{K_x} 0 \text{ and } u(t) \succeq_{K_u} 0$$

for all $t \geq 0$ imply

$$x(t) \succeq_{K_x} 0 \text{ and } y(t) \succeq_{K_y} 0$$

for all $t \geq 0$. This definition was introduced in [Angeli and Sontag, 2003], one of the early comprehensive works studying (possibly nonlinear) generalizations of positive systems, but it is used also in a linear setting, see e.g., [Shen and Lam, 2017]. The intuition is that for any state initialized in K_x and any input u belonging to K_u pointwise in time, both x and y will belong similarly to K_x and K_y, respectively. It is known to be equivalent to the fact that A is cross-positive on K_x, $B(K_u) \subseteq K_x$ and $C(K_x) \subseteq K_y$, as for positive systems on the nonnegative orthant.

Generalizations to proper cones are the exception rather than the rule. A successful example is given by [Shen and Lam, 2017], in which the result in Proposition 3 is extended in the natural way. Generally, however, such extensions are not straightforward. For instance, it was shown in [Tanaka, 2012, Chapter 4] that unlike the case for positive systems, the H_∞ norm of a monotone system is not always determined by the zero frequency, although this appears to hold when the spectral norm is exchanged for the spectral radius [Tanaka et al., 2013]. Similarly, the celebrated necessary and sufficient condition for stability and H_∞ γ-suboptimality in terms of a diagonal solution to a Lyapunov equation and LMI, respectively, also fail to hold in general beyond the nonnegative orthant. The natural setting for this property appears instead to be so-called symmetric cones, a subset of the self-dual cones for which the literature is formidable, e.g., [Faraut and Korányi, 1994]. For such cones, the desired LMI solution will no longer be a diagonal matrix but will instead be generated by the quadratic representation of a Jordan Algebra associated with the symmetric cone when applied to a vector obtained through a conic program [Shen and Lam, 2016], see [Dalin et al., 2024] for a Lie-algebraic approach in the stability case. An extension analogous to [Rantzer, 2015a] was provided very recently in [Lu et al., 2024], which includes in addition controller synthesis performed w.r.t. the spectral radius as performance objective. Examples of symmetric cones include the Lorentz cone, see e.g., [Papusha and Murray, 2015] for an application to a transportation network.

3

Contributions

In this section, the main contributions of the papers included in this thesis are highlighted. In general, the work in this thesis belongs to the intersection between cone theory and control. Among other things, it leverages cone theoretic methods in order to establish novel algebraic means of verifying stability and performance for linear systems satisfying conic constraints. The assumptions are motivated by applications. The main contributions and a discussion thereof follow below.

Paper I: H_∞ Control with Nearly Symmetric State Matrix

The paper [Vladu and Rantzer, 2022b] provides an upper bound on the deviation from H_∞ optimality of a particular controller class as a function of the deviation from symmetry in the state matrix. This follows from the following main result.

Theorem 1
Consider an LTI system and suppose

$$A^T P + PA + I \prec 0$$

for some $P \succ 0$. Then for the choice $K_P = -B^T P$ the closed-loop system G_{K_P} is stable and its H_∞ norm is bounded above and below as

$$\begin{aligned}\|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}} \leq \inf_K \|G_K\|_\infty \leq \|G_{K_P}\|_\infty \leq \\ \|(AA^T + BB^T - (P^{-1} + A)(P^{-1} + A)^T)^{-1}\|^{\frac{1}{2}}.\end{aligned}$$

Given a decomposition $A = S + \Delta$ for some $S \prec 0$ and some sufficiently small Δ, the corresponding closed-loop system G_S with the controller $u = B^T S^{-1} x$ after taking $P = -S^{-1}$ must satisfy

$$\|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}} \leq \|G_S\|_\infty \leq \|(AA^T + BB^T - \Delta\Delta^T)^{-1}\|^{\frac{1}{2}}.$$

As a result, a perturbation on the symmetry of A translates into a perturbation of the lower bound into an upper bound. In particular, nearly symmetric matrices thus generate near-optimal controllers in a straightforward way. Moreover, there is a freedom to choose controller so as to gain additional benefits, e.g., controller sparsity. This can happen if the input matrix B is sparse due to the network topology and S is chosen to be diagonal. The consequence of this controller choice on the H_∞ norm is quite transparent using the above relations and can thus be estimated quickly and intuitively. By contrast, the suggested procedure should be compared to the standard means of obtaining a not necessarily sparse H_∞ optimal or γ-suboptimal controller by solving Riccati equations [Doyle et al., 1988] or LMIs [Gahinet and Apkarian, 1994]. Note finally that Theorem 1 can be seen as a generalization of a result in [Lidström and Rantzer, 2016] which establishes that the controller $u = B^T A^{-1} x$ is optimal when A is symmetric and Hurwitz: choose simply $S = A$. For more on optimal and sparse controllers obtained on closed-form and their relations to the vast literature on decentralized and distributed control, see e.g., [Bergeling, 2019][Heyden, 2021] and the references therein.

Paper II: An H_∞ Optimal Controller for Open-loop Symmetric and Closed-loop Positive Systems

This paper [Vladu, 2024b] considers the controller $K_* = B^T A^{-T}$. It was conceived of and shown to be H_∞ optimal in [Lidström and Rantzer, 2016] under a symmetry and stability assumption on A, and generalized in various directions afterwards, e.g., [Bergeling, 2019][Vladu et al., 2021][Vladu and Rantzer, 2022a]. In particular, attention was drawn to its favorable sparsity properties, an observation made first in [Lidström and Rantzer, 2016]. Moreover, the question of departing from symmetry was further investigated in [Bergeling et al., 2020] and in [Vladu and Rantzer, 2022c], with an analytical condition for optimality in the form of closed-loop positivity provided only in the latter. To illustrate these conditions, consider as in [Vladu, 2024b] the following very simple model of an irrigation network in which one pool is partially leaking into the other:

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ \alpha & -1 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u + w$$

To illustrate the above conditions for optimality, note that the controller K_* is H_∞ optimal for the system from w to (x, u) by open-loop symmetry when $\alpha = 0$ [Lidström and Rantzer, 2016] and by closed-loop positivity when $\alpha = 1$ [Vladu and Rantzer, 2022c]. The following main result of Paper II improves on this and gives optimality in the entire range $0 \leq \alpha \leq 1$, outside of which optimality is soon lost.

Theorem 2
Consider an LTI system and suppose that A and $A_{cl} = A + BK_*$ are Metzler Hurwitz and that

$$K_*^T K_* + A_{cl}^T \mathrm{diag}(-A)^{-1} + \mathrm{diag}(-A)^{-1} A_{cl} \tag{3.1}$$

is Metzler. Then $u = K_* x$ is an H_∞ optimal controller, i.e.,

$$\min_K \|G_K\|_\infty = \|G_{K_*}\|_\infty = \|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}}.$$

This result essentially trades off closed-loop positivity for a more natural open-loop positivity and an additional assumption. However, although this third assumption (3.1) seems cumbersome, the two latter terms will in fact always be Metzler and therefore only ever help the entire expression towards becoming Metzler. Thus, this assumption can be considered as margin for the controller term $K_*^T K_*$ to be entrywise negative.

From a mathematical perspective, it is perhaps worthwhile to note that at the heart of Theorem 2 lies in fact a diagonal solution to a Riccati inequality. At the same time, positive systems are known to exhibit not only diagonal stability certifiers, but also an H_∞ norm governed by the static gain, the case also with this controller structure [Bergeling et al., 2020]. It may therefore be a surprise to find out that despite being constrained by various positivity assumptions, when $\alpha = 0$, the above closed-loop system fails to be even externally positive. Further, monotone linear systems in which the nonnegative orthant can be any proper cone fail in general to exhibit diagonal solutions, even on symmetric cones where they instead manifest as a function of a vector [Shen and Lam, 2016]. As such, this diagonal solution may be a contribution of substance, especially given that square matrices M which admit diagonal Lyapunov solutions are notoriously difficult to characterize explicitly. A well-known implicit characterization is in terms of the existence of a positive diagonal entry of PM for every positive semidefinite matrix P [Barker et al., 1978]. For some explicit algebraic conditions for 2- and 3-dimensional matrices and for the many applications of diagonal solutions, see [Oleng and Narendra, 2003] and the references therein.

Paper III: A Cone-preserving Solution to a Nonsymmetric Riccati Equation

This paper [Vladu and Rantzer, 2025] characterizes the existence of stabilizing cone-preserving solutions of nonsymmetric algebraic Riccati equations in terms of the stability of an associated coefficient matrix as follows.

THEOREM 3
Let the proper cone $K \subseteq \mathbb{R}^n$ and the matrices $A, B, C, D \in \mathbb{R}^{n \times n}$ be given. Suppose now that

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is cross-positive on $K \times K$. Then L is stable if and only if

$$XBX + DX + XA + C = 0 \tag{3.2}$$

has a solution $X_* \succeq_K 0$ such that $A + BX_*$ and $D + X_*B$ are stable and cross-positive on K.

The direction corresponding to sufficiency in this statement already exists on the nonnegative orthant for various degrees of stability imposed on L, e.g., [Guo, 2001][Guo and Higham, 2007][Guo and Lu, 2016] and the references therein. However, the generalization to a conic setting, as well as the completion with necessity into an equivalence, is novel to the best of the author's knowledge, as is the application of nonsymmetric AREs to certain control-theoretical problems, see [Vladu, 2024a].

The theory on algebraic Riccati equations is vast but mostly focused on the standard symmetric formulation arising to a great extent in the applications, e.g., [Lancaster and Rodman, 1995][Willems, 1971]. Thus, Theorem 3 is perhaps best compared to results for such symmetric formulations, such as those providing a unique stabilizing symmetric solution under the assumption of stabilizability and detectability in certain settings [Kučera, 1973] or the well-known imaginary axis eigenvalue condition on the related Hamiltonian matrix [Zhou and Doyle, 1998].

Finally, Paper III also provides the following somewhat surprising result, which is instrumental in bringing about Theorem 3.

THEOREM 4
Let the proper cone $K \subseteq \mathbb{R}^n$ and the sequence $\{X_i\}_{i=1}^{\infty}$ in $\mathbb{R}^{n \times n}$ be given. Suppose now that $0 \preceq_K X_i \preceq_K X_{i+1}$ and that there exist $s, r \in \mathbb{R}^n$ with $r \succ_K 0$ such that $X_i r \preceq_K s$ for all positive integers i. Then $\{X_i\}_{i=1}^{\infty}$ converges.

The result essentially says that a sequence of monotonically increasing matrices bounded above in a single vectorial direction converges. This is hardly expected for standard matrices, but is perhaps in keeping with the general flavor of positive systems in control. For example, one needs only the zero frequency in order to determine the H_∞ norm for a transfer function, and the dominant eigenvalue of a Metzler matrix is all that is required to determine stability. In other words, it is consistent with the intuition in positive systems theory that less is required to establish more.

Paper IV: Stability and Performance Analysis on Self-dual Cones

In this paper [Vladu, 2024a], the results in Paper III are exploited for the benefit of control. Denoting by $\mu(A)$ the greatest real part of all eigenvalues of a square matrix A, the first result is the following.

THEOREM 5
Suppose $A, D \in \mathbb{R}^{n\times n}$ are cross-positive on a proper cone K. Then there exists $P \succ_K 0$ such that

$$DP + PA \prec_K 0$$

if and only if

$$\mu(A) + \mu(D) < 0.$$

This result is reminiscent of Lyapunov's theorem for verifying stability. However, the partial order is now induced by the cone of cone-preserving matrices, two systems instead of one are considered, and one can be unstable. Further, the above conditions are equivalent to the existence of a Lyapunov-like function $V(x, y) = y^T Px$ which is nonnegative and decreasing along the trajectories of the two systems $\dot{x} = Ax$ and $\dot{y} = D^T y$ given that they are confined to K and K^*, respectively. This kind of Lyapunov-like function is novel to the best of the author's knowledge. Note finally that when K is self-dual and $D = A^T$, then a new stability condition for Metzler matrices A is obtained, cf. (2.5) and (2.6).

The main result of the paper is the following analog to the bounded real lemma for monotone systems on self-dual cones.

THEOREM 6
Consider an LTI system with zero direct term and A Hurwitz, and let $\gamma > 0$ and the three self-dual proper cones $(K_u, K_x, K_z) \subseteq (\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p)$ be given. Suppose now that the system is monotone with respect to (K_u, K_x, K_z). Then the following conditions are equivalent:

(i) $\|G\|_\infty < \gamma$.

(ii) *L is Hurwitz, where*

$$L = \begin{pmatrix} A & \frac{1}{\gamma^2}BB^T \\ C^TC & A^T \end{pmatrix}.$$

(iii) *There exists $P \succeq_{K_x} 0$ such that*

$$\frac{1}{\gamma^2}PBB^TP + A^TP + PA + C^TC = 0 \qquad (3.3)$$

and $A + \frac{1}{\gamma^2}BB^TP$ is Hurwitz.

(iv) There exists $P \in \mathbb{R}^{n\times n}$ *with* $P + P^T \succ 0$ *such that*

$$\frac{1}{\gamma^2} PBB^TP^T + A^TP^T + PA + C^TC \prec 0. \tag{3.4}$$

(v) There exists $p, q \succ_{K_x} 0$ *such that*

$$\begin{aligned} Ap + \frac{1}{\gamma^2} BB^T q &\prec_{K_x} 0 \\ C^TCp + A^Tq &\prec_{K_x} 0. \end{aligned}$$

This result is best compared to the main result in [Shen and Lam, 2016], which also provides equivalent conditions for condition (i). However, this is done in the context of symmetric cones, a subset of the self-dual ones, and conditions (ii), (iii) and (iv) are absent in exchange for a generalization of the diagonal solution to an LMI in [Tanaka and Langbort, 2011]. Further, the two vectors that satisfy the conic inequality in (v) appear to be an improvement from the three required in similar results such as [Rantzer, 2015a][Lu et al., 2024] and the four in [Shen and Lam, 2016]. Finally, condition (ii) and (iii) appear to be novel even for positive systems and are perhaps best compared to the Hamiltonian and the Riccati equation conditions in the standard bounded real lemma, see Proposition 2.

As a last contribution, we mention also the following.

THEOREM 7
Consider an LTI system with zero direct term and A Hurwitz and suppose that the system is monotone with respect to the self-dual proper cones $(K_u, K_x, K_z) \subseteq (\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p)$. *Then*

$$\|G\|_\infty = \|G(0)\|.$$

This result essentially states that the curious phenomenon for positive systems in which the H_∞ norm of a transfer function attains its value at zero frequency appears in fact to be rooted in the self-duality of the nonnegative orthant. This improves on [Shen and Lam, 2016], which establishes this for the symmetric cones; by contrast, [Tanaka, 2012] provides a counterexample to the phenomenon for general proper cones.

Paper V: Integral Linear Constraints on Cones and the KYP Lemma

This paper seeks to recognize the common structure in some fundamental results in linear-quadratic control and positive systems theory in the same vein as [Gowda and Tao, 2009], [Bamieh, 2024] and [Pates and Rantzer, 2024]. More specifically, the KYP Lemma (Proposition 1) and an L_1-gain analog to the bounded real lemma (Proposition 2.7) are targeted. Such considerations lead to the following general result.

THEOREM 8
Let the finite-dimensional normed spaces Z, X and the convex cone $K \subseteq Z$ be given. Suppose now that $L(K) = X$. Then for any given $m^* \in Z^*$ and $L \in B(Z, X)$, the following conditions are equivalent:

(i) There exists $p^* \in X^*$ such that

$$L^*(p^*) - m^* \preceq_{K^*} 0. \tag{3.5}$$

(ii) $m^*(z_0) \geq 0$ for any $z_0 \in K$ such that $L(z_0) = 0$.

Theorem 8 lies at the heart of the development in the paper, and a version in which all inequalities are strict is also presented. In this case, K is additionally required to be closed and pointed in favor of removing the assumption $L(K) = X$. A separating hyperplane argument is central to the proof of both statements, as in [Rantzer, 1996].

Although finite-dimensional in nature, Theorem 8 does have bearing on trajectories subject to dynamics, as the following result illustrates.

COROLLARY 1
Let the finite-dimensional normed spaces Z, X and the convex cone $K \subseteq Z$ be given. Suppose now that $L(K) = X$. Then for any $m^* \in Z^*$ and $E, L \in B(Z, X)$, the following conditions are equivalent:

(i) There exists $p^* \in X^*$ such that

$$L^*(p^*) - m^* \preceq_{K^*} 0.$$

(ii) (E, L) satisfies the dissipation inequality on K w.r.t. m^*.

In addition, if the dissipation inequality holds for some function V, then it also holds for some function in X^*.

Here, condition (ii) means simply that there exists a storage function V, not necessarily nonnegative, such that the dissipation inequality is satisfied for all trajectories $z(t) \in K$ such that $\frac{d}{dt}E(z) = L(z)$ with linear supply rate. The theorem may thus be viewed as a linear-cone analog to the linear-quadratic version in [Willems, 1971] in which the supply rate is quadratic over all of $\mathbb{R}^{n+m}$ and the conic inequality corresponds to an LMI. Note that dissipation with linear supply rate has previously been considered on the nonnegative orthant for positive systems [Haddad and Chellaboina, 2005].

Theorem 8 additionally also gives the following linear-cone analog to the non-strict KYP Lemma (Proposition 1).

THEOREM 9
Let the finite-dimensional normed spaces Z, X and the convex cone $K \subseteq Z$ be given. Suppose now that the pair $E, L \in B(Z, X)$ is controllable on K and that $E(K)$ has nonempty interior. Then for any given $m^* \in Z^*$, the following conditions are equivalent:

(i) There exists $p^* \in X^*$ such that

$$L^*(p^*) - m^* \preceq_{K^*} 0.$$

(ii) For all $z(t) \in K$ such that $\frac{d}{dt}E(z) = L(z)$ and $\int_{-\infty}^{\infty} \|z(t)\|_Z \mathrm{dt} < \infty$,

$$\int_{-\infty}^{\infty} m^*(z(t))\, \mathrm{dt} \geq 0.$$

(iii) $m^*(z_0) \geq 0$ for every $z_0 \in K$ such that $L(z_0) = 0$.

Here, the trajectories z are sufficiently regular, and K-controllability is defined in the natural way in keeping with [Valcher, 1996] as the possibility of transferring any element in $E(K)$ to any other element there in finite time by a trajectory $z(t) \in K$ such that $\frac{d}{dt}E(z) = L(z)$.

Theorem 9 is a more primitive and elementary statement compared to the standard KYP Lemma. This brings with it a great advantage, namely the opportunity to better understand the desired connection in the KYP Lemma between the LMI on the one hand and dynamical constraints which are useful in a control setting on the other. It just so happens that the latter appears as an integral linear constraint in the linear-cone version, rather than as a frequency inequality. However, for better comparison, a simple application of Parseval's theorem brings the frequency inequality over to an integral quadratic constraint, see the paper for more details. Such a formulation of the KYP Lemma would thus be more consistent with its linear-conic structure.

Similar to the Lyapunov case [Gowda and Tao, 2009], an additional step is needed to bring Theorem 9 over to the KYP Lemma.

THEOREM 10
For every $Q : \mathbb{R} \to \mathcal{S}_+^{n+m}$ the following conditions are equivalent:

(i) Q satisfies

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} I & 0 \end{pmatrix} Q \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix} Q \begin{pmatrix} I \\ 0 \end{pmatrix} + \begin{pmatrix} I & 0 \end{pmatrix} Q \begin{pmatrix} A^T \\ B^T \end{pmatrix}. \tag{3.6}$$

(ii) There exist $n+m$ functions $x_i : \mathbb{R} \to \mathbb{R}^n$ and $u_i : \mathbb{R} \to \mathbb{R}^m$ such that

$$Q = \sum_{i=1}^{n+m} \begin{pmatrix} x_i \\ u_i \end{pmatrix} \begin{pmatrix} x_i^T & u_i^T \end{pmatrix}$$

and either $x_i = 0$ on $\mathbb{R}$ or $\dot{x}_i = Ax_i + Bu_i$ on $\mathbb{R}$ for all i.

The contribution of Theorem 10 is the non-trivial direction $(i) \Rightarrow (ii)$, which can be interpreted as saying that in fact nothing new happens in the interior of $\mathcal{S}_+^{n+m}$. As such, Theorem 10 can be thought of as a bridge between linear-quadratic and positive systems intuition. For example, it neatly transfers normal controllability onto its K-analog as noted in the paper, and it turns linear functionals into the familiar quadratic ones from linear-quadratic control. For context, the matrix dynamical system (3.6) is important in many works, and the dynamics on $\dot{x} = Ax + Bu$ is previously known to correspond to rank 1 trajectories on the boundary of $\mathcal{S}_+^{n+m}$, e.g., [Bamieh, 2024].

An intuitive outline of the proof of the nontrivial direction in the KYP Lemma is now as follows: take K as $\mathcal{S}_+^{n+m}$ and E, L as in (3.6) and note that by controllability, Theorem 9 can be invoked to transfer the LMI in Proposition 1 to an integral linear constraint on $\mathcal{S}_+^{n+m}$ with integrand $m^*(Q) = \operatorname{tr}(-MQ)$. Theorem 10 now allows for the crucial necessary and sufficient weakening of matrix trajectories on $\mathcal{S}_+^{n+m}$ into rank 1 trajectories so that the linear integrand becomes quadratic

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{tr}(MQ(t))\,\mathrm{dt} &= \sum_{i=1}^{n+m} \int_{-\infty}^{\infty} \operatorname{tr}\left(M \begin{pmatrix} x_i(t) \\ u_i(t) \end{pmatrix} (x_i(t)^T u_i(t)^T) \right) \mathrm{dt} \\ &= \sum_{i=1}^{n+m} \int_{-\infty}^{\infty} \begin{pmatrix} x_i(t) \\ u_i(t) \end{pmatrix}^T M \begin{pmatrix} x_i(t) \\ u_i(t) \end{pmatrix} \mathrm{dt} \le 0\end{aligned}$$

and the frequency inequality follows by an application of Parseval's theorem.

Although there already exist several proofs of various versions of the KYP Lemma, the author argues that the conic framework offers a neat decomposition of the proof into two natural units: Theorem 9 and Theorem 10, both of which are useful also for other purposes. Further, although more straightforward and relying on more elementary mathematics, algebraic proofs such as [Rantzer, 1996] postpone the desired connection between the LMI and dynamical constraints until the end, thereby obscuring with algebra the simpler and more elementary connection between algebraic solutions and dynamics, i.e., between bounded linear functionals and conic inequalities on the one hand and integral linear constraints on trajectories confined to a cone on the other, as observed in Theorem 9. Finally, as to proofs relying on the value function or linear-conic duality as in [Bamieh, 2024], the author argues that there is no need to invoke such machinery, as the KYP Lemma is not about optimization at its core. For example, optimization on general cones cannot be taken for granted and appears to require additional minimality assumptions on the cone, e.g., [Pates and Rantzer, 2024]. Figuratively, this is similar to how distance maximization to hyperplanes in duality is preceded by the more basic existence of a separating hyperplane, the heart of Theorem 8 and therefore Theorem 9.

Conclusions

As mentioned previously, the main contributions of this thesis are all mainly algebraic means of verifying beneficial behaviors for dynamical systems constrained by cones. Such constraints can be perceived in the Lyapunov inequality assumption in Paper I, a constraint related to the positive semidefinite cone, in Paper II with the Metzler assumption related to the nonnegative orthant, and certainly in the remaining ones varying from general convex cones (Paper V) to proper cones (Paper III) to self-dual cones (Paper IV). At the same time, even when there appears to be no constraint, there could in fact be one dormant, as in Lyapunov's theorem for stability, in which the conic constraint appears instead on the associated Lyapunov operator rather than the state matrix [Gowda and Tao, 2009]. In other words, the choice of dynamical behavior can imply a conic constraint. As such, both the main contributions in this thesis and the well-known appealing results for positive systems in linear-quadratic control, such as diagonal stability certifiers, arise perhaps more generally as the combination of several conic constraints, as linear-quadratic problems and positive systems both seem to feature them inherently. It would be interesting to find out in the future to which extent this idea can be formalized and exploited.

As for additional future works, there are many exciting avenues to pursue from here on. For instance, it would be interesting to pursue an analog theory of the so-called K-analogs to standard control-theoretical concepts such as dissipativity or controllability, which were introduced in Paper V. Second, pursuing the proof idea in Paper V for the KYP Lemma in a nonlinear setting would also be interesting. Third, it would be exciting to find a natural class of systems accounting for the typical positive behavior of the non-positive closed-loop system in Paper II, namely diagonal solutions to Riccati inequalities and the zero frequency governing the H_∞ norm. Fourth, the main result in Theorem 6 could be incomplete, with for example a Riccati inequality in the cone sense potentially missing. Additionally, a control interpretation of the matrix L analogous to existing ones for the related Hamiltonian matrix would be interesting to explore. In summary, even without pursuing extensions of the above results, many unanswered questions remain.

Bibliography

Anderson, B. D. (1967). “A system theory criterion for positive real matrices”. *SIAM J. Control* **5**:2, pp. 171–182.

Anderson, B. D., M. Deistler, L. Farina, and L. Benvenuti (1996). “Nonnegative realization of a linear system with nonnegative impulse response”. *IEEE Trans. Circ. Syst. I: Fundam. Theory Appl.* **43**:2, pp. 134–142.

Angeli, D. and E. D. Sontag (2003). “Monotone control systems”. *IEEE Trans. Autom. Control* **48**:10, pp. 1684–1698.

Åström, K. J. and R. Murray (2021). *Feedback Systems: an Introduction for Scientists and Engineers.* Princeton University Press. ISBN: 9780691193984.

Bamieh, B. (2024). *Linear-quadratic problems in systems and controls via covariance representations and linear-conic duality: finite-horizon case,* arXiv:2401.01422 [eess.SY].

Barker, G. P. (1981). “Theory of cones”. *Linear Algebra Appl.* **39**, pp. 263–291.

Barker, G. P., A. Berman, and R. J. Plemmons (1978). “Positive diagonal solutions to the Lyapunov equations”. *Linear Multilinear Algebra* **5**:4, pp. 249–256.

Benvenuti, L. and L. Farina (2004). “A tutorial on the positive realization problem”. *IEEE Trans. Autom. Control* **49**:5, pp. 651–664.

Bergeling, C. (2019). *On H-infinity Control and Large-Scale Systems.* PhD thesis. Lund University. ISBN: 978-91-7895-095-9.

Bergeling, C., R. Pates, and A. Rantzer (2020). “H-infinity optimal control for systems with a bottleneck frequency”. *IEEE Trans. Autom. Control* **66**:6, pp. 2732–2738.

Berman, A. and R. J. Plemmons (1994). *Nonnegative Matrices in the Mathematical Sciences.* SIAM. ISBN: 9780120922505.

Boyd, S., L. El Ghaoui, E. Feron, and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory.* Vol. 15. SIAM. ISBN: 9780898713343.

Briat, C. (2013). “Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: L_1-gain and L_∞-gain characterization”. *Int. J. Robust Nonlinear Control* **23**:17, pp. 1932–1954.

Coxson, P. G. and H. Shapiro (1987). “Positive input reachability and controllability of positive systems”. *Linear Algebra Appl.* **94**, pp. 35–53.

Dalin, O., A. Ovseevich, and M. Margaliot (2024). “On special quadratic lyapunov functions for linear dynamical systems with an invariant cone”. *IEEE Trans. Autom. Control* **69**:9, pp. 6435–6441.

Doyle, J., K. Glover, P. Khargonekar, and B. Francis (1988). “State-space solutions to standard H_2 and H_∞ control problems”. In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 1691–1696.

Dullerud, G. E. and F. Paganini (2013). *A Course in Robust Control Theory: a Convex Approach.* Vol. 36. Springer Science & Business Media. ISBN: 9780387989457.

Ebihara, Y., D. Peaucelle, and D. Arzelier (2011). “L_1 gain analysis of linear positive systems and its application”. In: *Proc. 50th IEEE Conf. Decis. Control (CDC)*, pp. 4029–4034.

Ebihara, Y., D. Peaucelle, and D. Arzelier (2014). “LMI approach to linear positive system analysis and synthesis”. *Syst. & Control Lett.* **63**, pp. 50–56.

Fanti, M., B. Maione, and B. Turchiano (1990). “Controllability of multi-input positive discrete-time systems”. *Int. J. Control* **51**:6, pp. 1295–1308.

Faraut, J. and A. Korányi (1994). *Analysis on Symmetric Cones.* Oxford university press. ISBN: 9780198534778.

Farina, L. (1996). “On the existence of a positive realization”. *Syst. & Control Lett.* **28**:4, pp. 219–226.

Farina, L. and S. Rinaldi (2000). *Positive Linear Systems: Theory and Applications.* Vol. 50. John Wiley & Sons. ISBN: 9780471384564.

Forni, F. and R. Sepulchre (2015). “Differentially positive systems”. *IEEE Trans. Autom. Control* **61**:2, pp. 346–359.

Frobenius, G. F. (1912). “Über matrizen aus nicht negativen elementen”. *Sitzungsber. Kon. Preuss. Akad. Wiss.*, pp. 456–477.

Gahinet, P. and P. Apkarian (1994). “A linear matrix inequality approach to H_∞ control”. *Int. J. Robust Nonlinear Control* **4**:4, pp. 421–448.

Gattami, A. (2009). “Generalized linear quadratic control”. *IEEE Trans. Autom. Control* **55**:1, pp. 131–136.

Gowda, M. S. and J. Tao (2009). "Z-transformations on proper and symmetric cones: Z-transformations". *Math. Program.* **117**:1, pp. 195–221.

Guo, C.-H. (2001). "Nonsymmetric algebraic Riccati equations and Wiener–Hopf factorization for M-matrices". *SIAM J. Matrix Anal. Appl.* **23**:1, pp. 225–242.

Guo, C.-H. and N. J. Higham (2007). "Iterative solution of a nonsymmetric algebraic Riccati equation". *SIAM J. Matrix Anal. Appl.* **29**:2, pp. 396–412.

Guo, C.-H. and D. Lu (2016). "On algebraic Riccati equations associated with regular singular M-matrices". *Linear Algebra Appl.* **493**, pp. 108–119.

Haddad, W. M. and V. Chellaboina (2005). "Stability and dissipativity theory for nonnegative dynamical systems: a unified analysis framework for biological and physiological systems". *Nonlinear Anal.: Real World Appl.* **6**:1, pp. 35–65.

Heyden, M. (2021). *On the Control of Transportation Networks with Delays.* PhD thesis. Lund University. ISBN: 978-91-8039-117-7.

Iwasaki, T. and S. Hara (2005). "Generalized kyp lemma: unified frequency domain inequalities with design applications". *IEEE Trans. Autom. Control* **50**:1, pp. 41–59.

Iwasaki, T., G. Meinsma, M. Fu, et al. (2000). "Generalized S-procedure and finite frequency kyp lemma". *Math. Probl. Eng.* **6**, pp. 305–320.

Kalman, R. E. (1963). "Lyapunov functions for the problem of lur'e in automatic control". *Proc. Natl. Acad. Sci.* **49**:2, pp. 201–205.

Kalman, R. E. et al. (1960). "Contributions to the theory of optimal control". *Bol. soc. mat. mexicana* **5**:2, pp. 102–119.

Khalil, H. K. (1992). *Nonlinear Systems.* NY, USA: Macmillan Publishing Company. ISBN: 9780130673893.

Khargonekar, P. P., I. R. Petersen, and M. A. Rotea (1988). "H-infinity optimal control with state-feedback". *IEEE Trans. Autom. Control* **33**:8, pp. 786–788.

Kučera, V. (1973). "A review of the matrix Riccati equation". *Kybernetika* **9**:1, pp. 42–61.

Kwakernaak, H. (1993). "Robust control and H_∞-optimization—tutorial paper". *Automatica* **29**:2, pp. 255–273.

Lancaster, P. and L. Rodman (1995). *Algebraic Riccati Equations.* Clarendon Press. ISBN: 9780198537953.

Lessard, L. and S. Lall (2011). "Quadratic invariance is necessary and sufficient for convexity". In: *Proc. 2011 IEEE Amer. Control Conf. (ACC)*, pp. 5360–5362.

Lidström, C. and A. Rantzer (2016). "Optimal H_∞ state feedback for systems with symmetric and Hurwitz state matrix". In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 3366–3371.

Lu, X., Y. Chen, B. Zhu, J. Shen, B. Du, Y. Chen, and J. Lam (2024). "Kyp lemma for cone-preserving systems and its applications to controller design". *IEEE Trans. Autom. Control* **69**:12, pp. 8812–8819.

Luenberger, D. G. (1979). *Introduction to Dynamic Systems: Theory, Models, and Applications.* J. Wiley Sons. ISBN: 9780471025948.

Lyapunov, A. M. (1992). "The general problem of the stability of motion (english translation)". *Int. J. Control* **55**:3, pp. 531–534.

Megretski, A. (2010). *KYP lemma for non-strict inequalities and the associated minimax theorem* arXiv:1008.2552 [math.OC].

Nesterov, Y. and A. Nemirovskii (1994). *Interior-point Polynomial Algorithms in Convex Programming.* SIAM. ISBN: 9780898713190.

Ohta, Y., H. Maeda, and S. Kodama (1984). "Reachability, observability, and realizability of continuous-time positive systems". *SIAM J. Control Optim.* **22**:2, pp. 171–180.

Oleng, N. O. and K. S. Narendra (2003). "On the existence of diagonal solutions to the lyapunov equation for a third order system". In: *Proc. IEEE Amer. Control Conf. (ACC).* Vol. 3, pp. 2761–2766.

Papusha, I. and R. M. Murray (2015). "Analysis of control systems on symmetric cones". In: *Proc. 54th IEEE Conf. Decis. Control (CDC)*, pp. 3971–3976.

Pates, R. and A. Rantzer (2024). *Optimal control on positive cones* arXiv:2407.18774 [math.OC].

Perron, O. (1907). "Zur theorie der matrices". *Math. Annal.* **64**:2, pp. 248–263.

Popov, V.-M. (1961). "Absolute stability of nonlinear systems of automatic control". *Avtomatika Telemekhanika* **22**:8, pp. 961–979.

Rantzer, A. (1996). "On the kalman—yakubovich—popov lemma". *Syst. & Control Lett.* **28**:1, pp. 7–10.

Rantzer, A. (2015a). "On the kalman-yakubovich-popov lemma for positive systems". *IEEE Trans. Autom. Control* **61**:5, pp. 1346–1349.

Rantzer, A. (2015b). "Scalable control of positive systems". *Eur. J. Control* **24**, pp. 72–80.

Rantzer, A. (2022). "Explicit solution to bellman equation for positive systems with linear cost". In: *Proc. 61st IEEE Conf. Decis. Control (CDC)*, pp. 6154–6155.

Rantzer, A. and M. E. Valcher (2018). "A tutorial on positive systems and large scale control". In: *Proc. IEEE Conf. Decis. Control (CDC)*, pp. 3686–3697.

Scherer, C. (1989). "H_∞ control by state feedback: an iterative algorithm and characterization of high-gain occurence". *Syst. & Control Lett.* **12**:5, pp. 383–391.

Schneider, H. and B.-S. Tam (2006). "Matrices leaving a cone invariant". *Handb. Linear Algebra, ed. L. Hogben, Chapman & Hall.*

Schneider, H. and M. Vidyasagar (1970). "Cross-positive matrices". *SIAM J. Numer. Anal.* **7**:4, pp. 508–519.

Shen, J. and J. Lam (2016). "Some extensions on the bounded real lemma for positive systems". *IEEE Trans. Autom. Control* **62**:6, pp. 3034–3038.

Shen, J. and J. Lam (2017). "Input–output gain analysis for linear systems on cones". *Automatica* **77**, pp. 44–50.

Tanaka, T. (2012). *Symmetric formulation of the Kalman-Yakubovich-Popov lemma and its application to distributed control of positive systems.* PhD thesis. University of Illinois at Urbana-Champaign.

Tanaka, T. and C. Langbort (2011). "The bounded real lemma for internally positive systems and H-infinity structured static state feedback". *IEEE Trans. Autom. Control* **56**:9, pp. 2218–2223.

Tanaka, T., C. Langbort, and V. Ugrinovskii (2013). "DC-dominant property of cone-preserving transfer functions". *Syst. & Control Lett.* **62**:8, pp. 699–707.

Valcher, M. E. (1996). "Controllability and reachability criteria for discrete time positive systems". *Int. J. Control* **65**:3, pp. 511–536.

Valcher, M. E. (2009). "Reachability properties of continuous-time positive systems". *IEEE Trans. Autom. Control* **54**:7, pp. 1586–1590.

Van der Schaft, A. (2017). *L_2-gain and Passivity Techniques in Nonlinear Control.* Springer, Cham. ISBN: 9781852330736.

Van Der Schaft, A. J. (1992). "L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control". *IEEE Trans. Autom. Control* **37**:6, pp. 770–784.

Vladu, E. (2024a). *Stability and performance analysis on self-dual cones,* arXiv: 2411.12100 [math.OC].

Vladu, E. (2024b). "A unifying statement for an H-infinity optimal controller with positivity properties". In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 3686–3691.

Vladu, E., C. Bergeling, and A. Rantzer (2021). "Global solution to an H-infinity control problem with input nonlinearity". In: *Proc. 60th IEEE Conf. Decis. Control (CDC)*, pp. 3237–3242.

Vladu, E. and A. Rantzer (2022a). "Global solution to an H-infinity control problem for control-affine systems". *Proc. 25th Int. Symp. Math. Theory Netw. Syst. (MTNS); IFAC-PapersOnLine* **55**:30, pp. 388–393.

Vladu, E. and A. Rantzer (2022b). "H-infinity control with nearly symmetric state matrix". *IEEE Control Syst. Lett.* **6**, pp. 3026–3031.

Vladu, E. and A. Rantzer (2022c). "On decentralized H-infinity optimal positive systems". *IEEE Control Syst. Lett.* **7**, pp. 391–394.

Vladu, E. and A. Rantzer (2025). "A cone-preserving solution to a nonsymmetric Riccati equation". *Linear Algebra Appl.* **709**, pp. 449–459. DOI: 10.1016/j.laa.2025.01.020.

Willems, J. (1971). "Least squares stationary optimal control and the algebraic Riccati equation". *IEEE Trans. Autom. Control* **16**:6, pp. 621–634.

Willems, J. C. (1972a). "Dissipative dynamical systems part I: general theory". *Arch. Ration. Mech. Anal.* **45**:5, pp. 321–351.

Willems, J. C. (1972b). "Dissipative dynamical systems part II: linear systems with quadratic supply rates". *Arch. Ration. Mech. Anal.* **45**, pp. 352–393.

Yakubovich, V. A. (1962). "Solution of certain matrix inequalities in theory of automatic control". *Doklady Akademii Nauk SSSR* **143**:6, pp. 1304–+.

Yakubovich, V. A. (1977). "The s-procedure in non-linear control theory". *Vestnik Leningrad Univ. Mathe.* **4**, pp. 73–93.

Zhang, F. (2006). *The Schur Complement and its Applications.* Vol. 4. New York, NY, USA: Springer Science & Business Media. ISBN: 9780387242712.

Zhou, K. and J. C. Doyle (1998). *Essentials of Robust Control.* Vol. 104. Upper Saddle River, NJ, USA: Prentice Hall. ISBN: 9780135258330.

Paper I

H-infinity Control with Nearly Symmetric State Matrix

Emil Vladu Anders Rantzer

Abstract

In this letter, we give an upper bound on the deviation from H-infinity optimality of a class of controllers as a function of the deviation from symmetry in the state matrix. We further suggest a scalar measure of symmetry which is shown to be directly relevant for estimating nearness to optimality. In connection to this, we give a simple analytical solution to a class of Lyapunov equations for two dimensional state matrices. Finally, we demonstrate how a well-chosen symmetric part for nearly symmetric state matrices may lead not only to near-optimality, but also to controller sparsity, a desirable property for large-scale systems. In the special case that the state matrix is symmetric and Hurwitz, our main result simplifies to give an H-infinity optimal controller with several benefits, a result which has recently appeared in the literature. In this sense, the above is a significant generalization which considers a much wider class of systems, yet allows one to retain the benefits of symmetric state matrices, while offering means of quantifying the effect of this on the H-infinity norm.

1. Introduction

In many applications, it is of great importance to suppress the impact of disturbances on some desired output behavior. For example, maintaining a given course in spite of wind and turbulence is crucial for airplanes. The area within control engineering responsible for this problem along with many others, such as dealing with model uncertainties, is called robust control. There, the H_∞ norm of a linear system is a central object of study. This is due to its connection with matters such as robust stability, robust performance and the worst-case deviation from a desired output as caused by disturbances, see e.g., [Zhou et al., 1996]. A central problem is to minimize the H_∞ norm over a set of admissible controllers, or at least to upper bound it.

For linear time-invariant systems, there exist numerical tools for constructing controllers which generate closed-loop systems with H_∞ norm within some desired ϵ-tolerance from optimality. These include solving Riccati equations [Doyle et al., 1988] or LMIs [Gahinet and Apkarian, 1994] iteratively by computer. However, a significant problem with these controllers is that they are often unnecessarily complex, where in applications simplicity and transparency is desired. For instance, static controllers $u = Kx$ obtained through the procedure outlined above will in general be dense. This means that every control signal will require full state information, something which may be computationally unfeasible in the context of large-scale systems.

By contrast, a result which appeared recently in the literature [Lidström and Rantzer, 2016] gives the simple H_∞ optimal controller $u = B^T A^{-1} x$, where A is the state matrix and B is the input matrix. In addition, this controller may potentially be sparse if A and B are sparse, a common occurrence. Unfortunately, optimality is guaranteed only when A is symmetric and Hurwitz. Efforts have thus been made to extend the class of systems for which the benefits of this controller structure may be reaped. Extensions include systems with bottleneck frequency [Bergeling et al., 2020] and systems with input nonlinearity [Vladu et al., 2021]. Optimality has also been shown for other controller structures, such as PI controllers [Rantzer et al., 2017].

In this letter we make yet another such extension, this time targeting the class of systems with nearly symmetric state matrix. The aim is to enlarge the class of admissible systems while preserving the benefits of the above optimal controller; in exchange, we settle for near-optimality. From an application point of view, strict optimality for its own sake is often uninteresting, hence the motivation behind this letter.

An important contribution of this letter is an upper bound on the H_∞ norm of the closed-loop systems generated by a class of controllers on the form $u = -B^T P x$, where P is a positive definite matrix which solves a Lyapunov inequality. A key observation is then the similarity between this upper bound and a particular lower bound over the set of all stabilizing controllers known previously in the literature [Bergeling et al., 2020]. The significance of this becomes apparent when P is chosen as a particular function of the symmetric part S of A, given some decomposition $A = S+\Delta$. The upper bound will then simplify to appear identical to the lower bound, except for a perturbation term $\Delta\Delta^T$. The deviation Δ from symmetry in A thus translates directly into a deviation from optimality. This is a strong continuity statement allowing us to quantify the maximum deviation from H_∞ optimality by direct use of the deviation from symmetry in A. This is the main contribution of this letter, see subsection 2.1.

The remaining results in subsection 2.2 explore the effects of alternative choices of P on the upper bound. We find in particular that for 2×2 Hurwitz state matrices, P may be chosen in such a way as to make the upper bound identical to the lower bound, perturbed this time by a scalar. This is a consequence of the simple analytical solution to a particular Lyapunov equation that we provide, which was previously unheard of to the best of the authors' knowledge. The scalar, dependent on A, is bounded between 0 and 2 and becomes 2 if and only if A is symmetric. Hence, it may be interpreted as a measure of symmetry with direct relevance for estimating the nearness to optimality for the corresponding controller.

Finally, we consider a numerical example of a buffer network system in Section 3. Two different controllers corresponding to two different decompositions of A are investigated. We find in particular that although both are close to optimality, one preserves the sparsity structure of the incidence matrix B and would thus become sparse in a large-scale setting. This latter phenomenon is a consequence of the symmetric part being chosen as the diagonal of A. The near-optimality is then made possible due to the weak dynamical coupling between the buffers. More generally, state matrices with large diagonal entries are abundant in applications, see e.g., diagonally dominant matrices [Meyer, 2000]. Now, neglecting the inter-state coupling allows us to treat the system as though the state matrix were symmetric and receive the same sparsity benefits. This at the expense of a slight deviation from optimality, which, crucially, we can bound by our main result. This demonstrates one of the important aspirations of this letter, namely to work with nearly symmetric state matrices as though they were symmetric according to [Lidström and Rantzer, 2016], in exchange for a modest cost incurred by the H_∞ norm.

1.1 Outline

The outline of this letter is as follows. In Section 2, we give the main result (Subsection 2.1) along with some supplementary results (Subsection 2.2).

Section 3 then gives a numerical example to illustrate the main result of Section 2, and Section 4 subsequently concludes the letter. Finally, in Section 5 we offer proofs of the results in Section 2.

1.2 Mathematical notation and preliminaries

Let $\mathbb{R}$ denote the set of real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ shall denote the set of n-dimensional vectors and $n \times m$ matrices, respectively, with elements in $\mathbb{R}$. A square matrix A is said to be Hurwitz if all its eigenvalues have negative real part. The transpose of a matrix A is denoted by A^T. For symmetric matrices A, we mean by $A \succ (\succeq)0$ and $A \prec (\preceq)0$ that A is positive (semi)definite and negative (semi)definite, respectively. The spectral norm (2-induced norm) shall be denoted by $\|A\|$. The identity matrix is written I, where the dimension should be clear from the context. The trace and determinant of a square matrix A are denoted by $\mathrm{tr}(A)$ and $\det(A)$, respectively. We mean by $\mathrm{diag}(c_1, \dots, c_n)$ an $n \times n$ diagonal matrix with diagonal entries $c_1, \dots, c_n$.

Given an input-output system $\mathcal{S}$, its L_2 gain is defined as the supremum of the set $\left\{ \frac{\|\mathcal{S}(w)\|}{\|w\|} \;\middle|\; w \in L_2[0,\infty),\ w \neq 0 \right\}$. Here, $\|w\|$ refers to the $L_2[0,\infty)$ norm of w. The H_∞ norm of a stable transfer function G, denoted by $\|G\|_\infty$, is defined by $\sup_\omega \|G(i\omega)\|$. It is well known that the L_2 gain of a stable linear time-invariant system is equal to the H_∞ norm of the corresponding transfer function.

2. Results

In this section we give the results of this letter. Subsection 2.1 gives the main result followed by an important discussion on its significance and its ramifications for the purpose of this letter. Subsection 2.2 gives additional results further probing the main result.

2.1 The Main Result

Consider the system

$$\begin{aligned} \dot{x} &= Ax + Bu + w, \quad x(0) = 0 \\ z &= \begin{pmatrix} x \\ u \end{pmatrix} \end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $w(t) \in \mathbb{R}^n$ and $z(t) \in \mathbb{R}^{n+m}$ denote the state, the control input, the disturbance, and the regulated output, respectively, at time t. Let G_K be the closed-loop transfer function from w to z given the static control law $u = Kx$ with $K \in \mathbb{R}^{m\times n}$.

The main result is as follows.

Theorem 11
Consider system (1) and suppose

$$A^T P + PA + I \prec 0 \tag{2}$$

for some $P \succ 0$. Then for the choice $K_P = -B^T P$ the closed-loop system G_{K_P} is stable and its H_∞ norm is bounded above and below as

$$\begin{aligned} &\|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}} \le \inf_K \|G_K\|_\infty \le \|G_{K_P}\|_\infty \le \\ &\|(AA^T + BB^T - (P^{-1} + A)(P^{-1} + A)^T)^{-1}\|^{\frac{1}{2}}. \end{aligned} \tag{3}$$

Proof. See Appendix. □

The significance of Theorem 11 lies mainly in its connection with symmetric matrices. In order to see this, suppose

$$A = S + \Delta \tag{4}$$

for some $S \prec 0$ and some sufficiently small perturbation Δ. Choosing $P_S = -S^{-1}$, the upper bound in (3) collapses to

$$\|(AA^T + BB^T - \Delta\Delta^T)^{-1}\|^{\frac{1}{2}}. \tag{5}$$

A perturbation Δ on the symmetry of A thus translates into a perturbation of the lower bound $\|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}}$. In other words, if A is nearly symmetric, then the controller $u = -B^T P_S x$ is nearly H_∞ optimal. Moreover, the maximum deviation from optimality may be quantified by the deviation from symmetry according to (5). Theorem 11 thus embodies a statement more powerful than mere continuity.

Below follow some remarks on Theorem 11.

Remark 1
Theorem 11 considers a specific class of controllers parameterized by $P \succ 0$. It then supplies a P-dependent upper bound on the H_∞ norm of the corresponding closed-loop system. This upper bound may in turn be compared to the similar-looking lower bound $\|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}}$, which is in fact known from the literature [Bergeling et al., 2020] to lower bound not only the considered controller class, but all stabilizing controllers.

Remark 2
Theorem 11 is readily extended to enable tuning by means of weighting matrices. For this purpose, consider the weighted output $z = \big((W_x x)^T, (W_u u)^T\big)^T$ in (1), where W_x and W_u are invertible. We further allow a factor $H \in \mathbb{R}^{n\times p}$ in front of $w \in \mathbb{R}^p$ in (1). Then Theorem 11 reads exactly as above with the

symbolical substitutions $A \to AW_x^{-1}$, $B \to BW_u^{-1}$ *and* $P \to W_x^{-T}P$ *in (3). Further,* $A^TP + PA + W_x^TW_x \prec 0$ *replaces (2) and the controller considered is* $K_P = -(W_u^TW_u)^{-1}B^TP$*. Finally,* H^T *and* H *enter only as factors to the left and right of the inverse, respectively, in both bounds. In order to prove this, proceed exactly as in the proof of Theorem 11. Further, for the discussion following Theorem 11, consider the weighted state matrix* $A(W_x^TW_x)^{-1}$ *instead of* A *in (4).*

Remark 3
The assumption (2) in Theorem 11 may be interpreted as a stability condition on the open-loop system $\dot{x} = Ax$*. It is well known that asymptotic stability is equivalent to a solution* $P \succ 0$ *to the Lyapunov equation* $A^TP + PA = Q$ *for all* $Q \prec 0$*. Thus, if* A *is Hurwitz, then there always exists* $P \succ 0$ *so that Theorem 11 may be employed, as any* $Q \prec -I$ *gives a* P *which satisfies (2).*

Remark 4
Failure to satisfy assumption (2) in Theorem 11 can have dramatic effects on the result. It is not difficult to construct counterexamples in which (2) fails and the upper bound in (3) ends up below the H_∞ *norm.*

2.2 Supplementary Results

We begin by considering the important special case which occurs by assuming $\Delta = 0$ in (4). The upper bound in (3) then reduces to the lower bound and Theorem 11 collapses to the following well-known optimality result [Lidström and Rantzer, 2016].

Corollary 2
Consider system (1) and suppose that A *is symmetric and Hurwitz. Then*

$$\inf_K \|G_K\|_\infty = \|\left(A^2 + BB^T\right)^{-1}\|^{\frac{1}{2}}$$

with the infimum attained by $K = B^TA^{-1}$*.*

Proof. Take $P = -A^{-1}$ and apply Theorem 11. □

Corollary 2 gives the H_∞ optimal controller $u = B^TA^{-1}x$ if A is symmetric and Hurwitz. This controller may be compared with the controller obtained from the following natural decomposition in (4)

$$A = \underbrace{\frac{A + A^T}{2}}_{=S} + \underbrace{\frac{A - A^T}{2}}_{=\Delta}. \tag{6}$$

The corresponding controller with $P = -S^{-1}$ is

$$u = 2B^T(A + A^T)^{-1}x.$$

This controller may be employed even if A is not symmetric, so long as Δ is small enough that $\Delta^T P + P\Delta \prec I$ and thereby (2) holds. Note, however, that other decompositions (4) may be more beneficial for various purposes. For instance, a diagonal S may lead to greater controller sparsity, a desirable property for large-scale systems, see Section 3.

Thus far we have emphasized those P generated by the decomposition in (4) through $P = -S^{-1}$. However, alternative decompositions may bring additional benefits. We hint at these benefits by considering an important special case which takes the ideas outlined in (4)-(5) one step further. The following remarkable fact is crucial for this purpose.

Theorem 12
Suppose that $A \in \mathbb{R}^{2\times 2}$ is Hurwitz and set $P_* = \sqrt{(AA^T)^{-1}}$. Then there is a $\rho \in [0,2]$ such that

$$A^T P_* + P_* A = -\rho I.$$

In particular, $\rho = -\mathrm{tr}(P_* A) = \frac{-2\,\mathrm{tr}(A)}{\sqrt{\mathrm{tr}(AA^T)+2\det(A)}}$.

Proof. See Appendix. □

Although Theorem 12 is of independent interest, for our purposes it suffices to note that Theorem 11 together with Theorem 12 gives the simplified upper bound in (3)

$$\left\| \left((\rho-1)AA^T + BB^T\right)^{-1} \right\|^{\frac{1}{2}} \tag{7}$$

for the controller $u = K_{P_*}x = -B^T\sqrt{(AA^T)^{-1}}x$ if $\rho > 1$. Noting that A Hurwitz is symmetric if and only if $\rho = 2$ (to see this, singular value decompose A), the function

$$\rho(A) = \frac{-2\,\mathrm{tr}(A)}{\sqrt{\mathrm{tr}(AA^T) + 2\det(A)}} \tag{8}$$

could be interpreted as a scalar measure of symmetry which perturbs the lower bound in (3) to an upper bound in the case that A is 2×2 Hurwitz. $\rho(A)$ thus appears to be indicative of how near the controller $u = -B^T\sqrt{(AA^T)^{-1}}x$ is to optimality. This implicit choice of the geometric mean $\sqrt{AA^T}$ as a symmetric part with associated scalar perturbation ρ in (7) should be compared to the arithmetic mean $\frac{A+A^T}{2}$ with associated non-scalar perturbation Δ in (5).

Example 1
A Damped Pendulum

It is well known that a linear pendulum with damping factor γ may be modeled by the equation

$$\dot{x} = \begin{pmatrix} 0 & 1 \\ -c & -\gamma \end{pmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u + \begin{pmatrix} 0 \\ 1 \end{pmatrix} w.$$

Here, we set $c = 1$. It is straightforward to evaluate (8): $\rho(A(\gamma)) = \gamma\big(1 + \big(\frac{\gamma}{2}\big)^2\big)^{-\frac{1}{2}}$. For example, already for the choice $\gamma = 4$ we obtain $\rho \approx 1.79$. The corresponding controller $u = -B^T\sqrt{(AA^T)^{-1}}x$ thus gives the simple upper bound (7) as $\|H^T(0.79AA^T + BB^T)^{-1}H\|^{\frac{1}{2}}$, where $H = (0,1)^T$, see Remark 2. Surprisingly, the upper bound reduces to the lower bound in the limit as $\gamma \to \infty$, because then $\rho \to 2$. This occurs despite the fact that $A(\gamma)$ is asymmetric with a constant off-diagonal gap for all γ.

Remark 5
Theorem 12 gives a simple analytical solution to Lyapunov equations $A^TP + PA = -Q$ with $Q = \rho I, \rho > 0$ (by scaling P_*) provided that A is 2×2 and Hurwitz. This should be compared to the well-known solution $P = \int_0^\infty e^{A^Tt}Qe^{At}\mathrm{dt}$ which is more complex, see e.g., Theorem 7.11 in [Rugh, 1996]. Note that counterexamples to the above occur as a rule for general $n \times n$ matrices.

3. Numerical Example

In this section, we give a numerical example to illustrate the use and benefits of Theorem 11 from an application point of view. All computations were performed in Matlab [MATLAB, 2020].

3.1 The Buffer Network

Consider the small buffer network in Figure 1. In practice, the three buffers could represent containers (e.g., rooms, biological compartments or water tanks) which exchange some contents (e.g., heat or water). Figure 1 suggests that the transfer of contents is caused by both dynamics and control. For

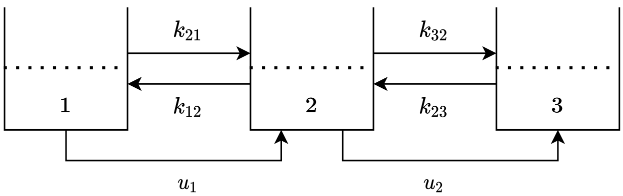

Figure 1. The buffer network system considered in Section 3.

more on such applications, see e.g., [Rantzer and Valcher, 2018],[Åström and Murray, 2010].

More generally, we denote by x_i the deviation in quantity of the i:th buffer from its steady state level marked by dotted lines in Figure 1, and we suppose its rate of change is affected by the j:th buffer as $k_{ij}x_j$. We suppose further that we may transfer contents between the buffers by means of control: u_i denotes the flow rate from buffer i to buffer $i+1$. This could correspond to constructing flow links with pumps between the water tanks. Moreover, disturbances w_i may affect each buffer; this could correspond to disturbing water inflows to the tanks.

Figure 1 together with the interpretation given above suggests the following plausible dynamics.

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -k_{11} & k_{12} & 0 \\ k_{21} & -k_{22} & k_{23} \\ 0 & k_{32} & -k_{33} \end{pmatrix} x + \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} u + w \\ &= \begin{pmatrix} -1.5 & 0.24 & 0 \\ 0.1a & -2.6 & 0.35 \\ 0 & 0.52a & -4.2 \end{pmatrix} x + \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} u + w\end{aligned}$$

Here, $a \geq 1$ is a constant which affects the dynamical transfer of contents between buffers in one direction. Increasing a thus has the effect of increasing the asymmetry in the state matrix A, which depends on a. Note in particular that A is asymmetric for all a. Hence, the H_∞ optimal controller given in Corollary 2 cannot be used without running the risk of obtaining, possibly, a very high H_∞ norm. However, comparing the superdiagonal elements with the subdiagonal elements in A, we note that they are relatively similar for $a = 1$. At this a, A could thus be suspected to be near enough symmetric. We would therefore like to employ Theorem 11 in order to obtain controllers with good H_∞ norm bounds.

3.2 Controller Comparison

We shall compare the H_∞ norm and the bounds in (3) corresponding to two different controllers K_P against a standard optimal controller obtained by solving Riccati equations in Matlab. The first of the two controllers is obtained through the natural decomposition (6). With $S_{\text{arthm}} = \frac{A+A^T}{2}$ and $P_{\text{arthm}} = -S_{\text{arthm}}^{-1}$, we have

$$u = K_{\text{arthm}}x = -B^T P_{\text{arthm}}x = 2B^T(A + A^T)^{-1}x.$$

The second controller is chosen through a different decomposition (4), namely $S_{\text{diag}} = \text{diag}(-1.5, -2.6, -4.2)$. S_{diag} is thus a diagonal matrix with a diagonal identical to that of A. Hence, with $P_{\text{diag}} = -S_{\text{diag}}^{-1}$, we have

$$u = K_{\text{diag}}x = -B^T P_{\text{diag}}x = B^T S_{\text{diag}}^{-1}x.$$

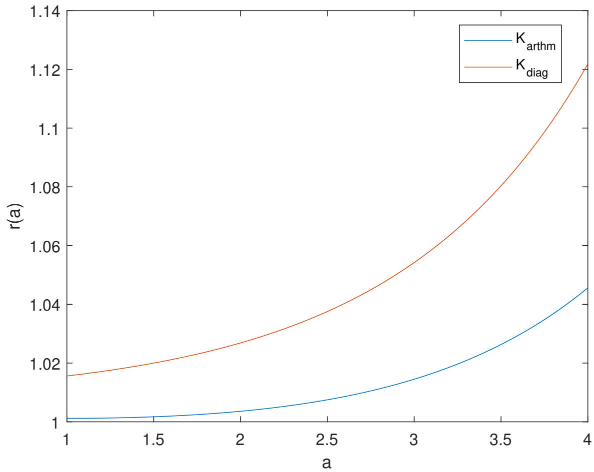

Figure 2. The ratio r between the upper and lower bound in (3) on the H_∞ norm of the closed-loop system corresponding to the controllers K_{arthm} and K_{diag} for different degrees of asymmetry a in the state matrix A.

Table 1. The H_∞ norm bounds in (3) for the two controllers K_{arthm} and K_{diag}. In each box, the left (right) value corresponds to the asymmetry degree $a = 1$ ($a = 4$) in A. In this case, the lower bounds coincide with the optimal values.

	Lower Bound	$\|G_K\|_\infty$	Upper bound
K_{arthm}	0.6054 – 0.6717	0.6056 – 0.6728	0.6061 – 0.7023
K_{diag}	0.6054 – 0.6717	0.6063 – 0.6757	0.6149 – 0.7535

We now apply Theorem 11 to obtain the bounds in (3) corresponding to the two controllers K_{arthm} and K_{diag}. In addition, we will observe the change in these values over varying degrees of asymmetry in A by increasing a. Since both the upper and lower bound depend on A and therefore a, we study their ratio $r(a)$ for various a. This is shown in Figure 2. As expected, a higher degree of asymmetry in A leads to an increased gap between the upper and lower bound in (3) for both controllers. Surprisingly, however, the gap appears to be small even for $a = 4$, when the asymmetry in A is significant. This is seen more clearly in Table 1, which supplies explicit values for the bounds when $a = 1$ and $a = 4$.

It is particularly interesting to note the success of the diagonal choice $S = S_{\text{diag}}$. Symmetry is then considered with respect to the diagonal. In essence, the choice corresponds to neglecting the dynamics between the buffers. The asymmetric part – i.e., the off-diagonal part of A – thus becomes significant. In view of (5), this explains why the upper bound corresponding to K_{arthm} performs better in Figure 2 and Table 1. What is perhaps surprising is that the upper bound for K_{diag} nonetheless remains near optimality even for high degrees of asymmetry ($a = 4$).

3.3 Controller Sparsity

An additional very important property of the controller K_{diag} besides its simplicity is that it preserves the sparsity pattern of B^T. This follows because the diagonal matrix P_{diag} acts on B^T in $K_{\mathrm{diag}} = -B^T P_{\mathrm{diag}}$. Consequently, unlike K_{arthm} or an optimal controller obtained by means of Riccati equations, K_{diag} will remain sparse for much larger networks than the one considered in this section, provided that B is sparse. From a large-scale system perspective, where it is desirable that local controllers only require local information, K_{diag} is quite superior to K_{arthm}. This shows that decompositions (4) of A other than (6) may be of interest.

4. Conclusions and Future Works

In this letter, an upper bound on the deviation from H_∞ optimality of a class of controllers has been derived. For a particular subset of these controllers, this upper bound simplifies to a well-known lower bound perturbed by the deviation from symmetry in the state matrix. This is a strong continuity statement which allows one to quantify the maximum deviation from optimality by direct use of the deviation from symmetry in the state matrix. Furthermore, the corresponding controllers will have the same structural benefits as the well-known optimal controller obtained in the special case that the state matrix is symmetric and Hurwitz. This letter has thus justified working with nearly symmetric state matrices as though they were symmetric.

With regards to future work, one direction is to establish analogous bounds on the H_∞ norm of the closed-loop system corresponding to other controllers, e.g., PI controllers. The corresponding output feedback case could also be considered.

5. Appendix

In this section, we give proofs of the results in Section 2.

Proof. *Theorem 11*
Consider the controller $u = K_P x$ with $K_P = -B^T P$ for some $P \succ 0$ which satisfies (2). We wish to show closed-loop stability and derive the upper

bound in (3). For this purpose, note that

$$\begin{aligned}
0 &\succ A^T P + PA + I - K_P^T K_P \\
&= A^T P + PA + I - K_P^T K_P - K_P^T K_P + K_P^T K_P \\
&= A^T P + PA + I + PBK_P + K_P^T B^T P + K_P^T K_P \\
&= (A + BK_P)^T P + P(A + BK_P) + I + K_P^T K_P \\
&= A_{cl}^T P + PA_{cl} + I + K_P^T K_P
\end{aligned} \tag{9}$$

where the first inequality follows by assumption (2). The first implication of (9) is that for the closed-loop state matrix $A_{cl} = A + BK_P$, a $P \succ 0$ satisfies the Lyapunov equation $A_{cl}^T P + PA_{cl} = Q$ for some $Q \prec 0$. Hence, the closed-loop system is asymptotically stable, see e.g., Theorem 7.11 in [Rugh, 1996].

Set now

$$\gamma_* = \|(AA^T + BB^T - (P^{-1} + A)(P^{-1} + A)^T)^{-1}\|^{\frac{1}{2}}.$$

We have immediately

$$\begin{aligned}
0 &\succeq -\gamma_*^2 I + (AA^T + BB^T - (P^{-1} + A)(P^{-1} + A)^T)^{-1} \\
&= -\gamma_*^2 I - P(A^T P + PA + I - K_P^T K_P)^{-1} P \\
&= -\gamma_*^2 I - P(A_{cl}^T P + PA_{cl} + I + K_P^T K_P)^{-1} P
\end{aligned}$$

where in the second line (9) was invoked. Since (9) also gives that $A_{cl}^T P + PA_{cl} + I + K_P^T K_P \prec 0$, we may employ the Schur complement lemma, see e.g., [Zhang, 2006] p. 34, to obtain

$$0 \succeq \begin{pmatrix} A_{cl}^T P + PA_{cl} + I + K_P^T K_P & P \\ P & -\gamma_*^2 I \end{pmatrix}.$$

with $P \succ 0$ by assumption. Now, taking any nonzero disturbance $w \in L_2[0, \infty)$ with its resulting state trajectory $x(t)$, we have by definition of negative semidefiniteness at any time $t \geq 0$

$$\begin{aligned}
0 &\geq \begin{pmatrix} x \\ w \end{pmatrix}^T \begin{pmatrix} A_{cl}^T P + PA_{cl} + I + K_P^T K_P & P \\ P & -\gamma_*^2 I \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix} \\
&= (A_{cl}x + w)^T Px + x^T P(A_{cl}x + w) \\
&\quad + x^T x + (K_P x)^T K_P x - \gamma_*^2 w^T w \\
&= \frac{d}{dt}(x^T Px) + x^T x + u^T u - \gamma_*^2 w^T w
\end{aligned}$$

where the dependence on t has been suppressed for lack of space. Note that in the second equality, the system dynamics (1) were invoked and $P \succ 0$ was used. Integrating, we now have

$$
\begin{aligned}
0 &\geq \lim_{t\to\infty} \Big[x^T P x\Big]_0^t + \|x\|^2 + \|u\|^2 - \gamma_*^2 \|w\|^2 \\
&= \|x\|^2 + \|u\|^2 - \gamma_*^2 \|w\|^2
\end{aligned}
$$

since the system is stable and $x(0) = 0$ by assumption. But this implies

$$\frac{\|z\|}{\|w\|} \leq \gamma_*$$

for all nonzero $w \in L_2[0, \infty)$. Consequently, the L_2 gain is less than or equal to γ_*. But the L_2 gain of a stable linear time-invariant system is known to equal the H_∞ norm of the corresponding transfer function. We thus have

$$\|G_{K_P}\|_\infty \leq \gamma_*$$

which is the desired upper bound.

As for the lower bound in (3), a simple least squares argument gives

$$\|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}} \leq \inf_K \|G_K\|_\infty,$$

see Section 2 in [Bergeling et al., 2020], where this lower bound has previously appeared. In other words, the above expression lower bounds the H_∞ norm of the closed-loop system corresponding to all stabilizing controllers, not only those on the form $u = K_P x$ considered above. This completes the proof. □

Proof. *Theorem 12*
Suppose $A \in \mathbb{R}^{2\times 2}$ with A Hurwitz and set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

for some $a, b, c, d \in \mathbb{R}$. It is well known that the square root of a matrix $M \in \mathbb{R}^{2\times 2}$ such that $M \succ 0$ is given by

$$M = \frac{1}{k}\big(M + \sqrt{\det(M)} I\big) \tag{10}$$

where $k = \sqrt{\operatorname{tr}(M) + 2\sqrt{\det(M)}}$, see e.g., [Levinger, 1980].

Take now $M = AA^T \succ 0$. By (10), we have

$$
\begin{aligned}
A^{-1}\sqrt{AA^T} &= \frac{1}{k} A^{-1}(AA^T + \det(A) I) \\
&= \frac{1}{k}\left(\begin{pmatrix} a & c \\ b & d \end{pmatrix} + \det(A) \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right) \\
&= \frac{1}{k} \begin{pmatrix} a+d & -(b-c) \\ b-c & a+d \end{pmatrix}
\end{aligned}
$$

where we used that $\sqrt{\det(AA^T)} = \det(A)$ since A is 2×2 and Hurwitz. We also used the well-known expression for the inverse of a 2×2 matrix. Noting that $\sqrt{AA^T}A^{-T} = \left(A^{-1}\sqrt{AA^T}\right)^T$, the above now gives

$$A^{-1}\sqrt{AA^T} + \sqrt{AA^T}A^{-T} = \frac{2(a+d)}{k}I = -\rho I \tag{11}$$

from which we identify

$$\rho = \frac{-2\ \mathrm{tr}(A)}{\sqrt{\mathrm{tr}(AA^T) + 2\det(A)}} \tag{12}$$

since $\mathrm{tr}(A) = a + d$. Now, multiplying (11) by $\sqrt{(AA^T)^{-1}}$ from both sides, we obtain

$$\sqrt{(AA^T)^{-1}}A^{-1} + A^{-T}\sqrt{(AA^T)^{-1}} = -\rho(AA^T)^{-1}$$

which may be multiplied by A^T from the left and A from the right to yield

$$A^T\sqrt{(AA^T)^{-1}} + \sqrt{(AA^T)^{-1}}A = -\rho I$$

and thus $P_* = \sqrt{(AA^T)^{-1}}$ solves the Lyapunov equation $A^T P + PA = -\rho I$, as was to be shown.

Finally, a different expression for ρ is easily seen to be given by $\rho = -\mathrm{tr}(P_* A)$. Now, with the singular value decomposition $A = USV^T$, we have

$$P_* A = \sqrt{(USV^TVSU^T)^{-1}}USV^T = UV^T.$$

Since we have $(UV^T)^T(UV^T) = VU^TUV^T = VV^T = I$, the product $UV^T = P_* A$ is orthogonal by definition. But orthogonal matrices have eigenvalues with unit length only. Thus, $|\mathrm{tr}(P_* A)| = |\mathrm{tr}(UV^T)| \leq 2$, recalling that the trace of a matrix equals the sum of its eigenvalues. Because both numerator and denominator in (12) are positive (A is 2×2 Hurwitz), it follows that $\rho \in [0, 2]$. This completes the proof. □

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References

Åström, K. J. and R. M. Murray (2010). *Feedback Systems: An Introduction for Scientists and Engineers*. Princeton University Press. ISBN: 9780691193984.

Bergeling, C., R. Pates, and A. Rantzer (2020). "H-infinity optimal control for systems with a bottleneck frequency". *IEEE Trans. Autom. Control* **66**:6, pp. 2732–2738.

Doyle, J., K. Glover, P. Khargonekar, and B. Francis (1988). "State-space solutions to standard H_2 and H_∞ control problems". In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 1691–1696.

Gahinet, P. and P. Apkarian (1994). "A linear matrix inequality approach to H_∞ control". *Int. J. Robust Nonlinear Control* **4**:4, pp. 421–448.

Levinger, B. W. (1980). "The square root of a $2\times$ 2 matrix". *Math. Mag.* **53**:4, pp. 222–224.

Lidström, C. and A. Rantzer (2016). "Optimal H_∞ state feedback for systems with symmetric and Hurwitz state matrix". In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 3366–3371.

MATLAB (2020). *MATLAB and Control Synthesis Toolbox Release 2020a, The Mathworks, Inc.* Natick, MA, USA.

Meyer, C. D. (2000). *Matrix Analysis and Applied Linear Algebra.* Vol. 71. Philadelphia, PA, USA: SIAM.

Rantzer, A., C. Lidström, and R. Pates (2017). "Structure preserving H-infinity optimal PI control". *IFAC-PapersOnLine* **50**:1, pp. 2573–2576.

Rantzer, A. and M. E. Valcher (2018). "A tutorial on positive systems and large scale control". In: *Proc. IEEE Conf. Decis. Control (CDC)*, pp. 3686–3697.

Rugh, W. J. (1996). *Linear System Theory.* Upper Saddle River, NJ, USA: Prentice-Hall, Inc. ISBN: 9780134412054.

Vladu, E., C. Bergeling, and A. Rantzer (2021). "Global solution to an H-infinity control problem with input nonlinearity". In: *Proc. 60th IEEE Conf. Decis. Control (CDC)*, pp. 3237–3242.

Zhang, F. (2006). *The Schur Complement and its Applications.* Vol. 4. New York, NY, USA: Springer Science & Business Media. ISBN: 9780387242712.

Zhou, K., J. C. Doyle, K. Glover, et al. (1996). *Robust and Optimal Control.* Vol. 40. Upper Saddle River, NJ, USA: Prentice Hall. ISBN: 9780134565675.

Paper II

A Unifying Statement for an H-infinity Optimal Controller with Positivity Properties

Emil Vladu

Abstract

In this paper, we unify two already published results on state feedback H-infinity optimality. Previously, optimality has been shown for a particular controller structure in the case that the open-loop state matrix is symmetric, as well as in the case that the closed-loop system is internally positive. By contrast, the main result of the present paper gives optimality based on neither of these two properties. As a result, when applied to a class of buffer networks, it succeeds not only in showing optimality when the system parameters are chosen so as to give open-loop symmetry and closed-loop positivity, respectively, but also when both of these properties are absent.

1. Introduction

In control engineering, it is often of interest to suppress the impact of various disturbances on a desired output. In response to this challenge, H_∞ control developed, a field concerned roughly with synthesizing controllers which suppress worst-case disturbances, see e.g., [Zhou et al., 1996; Dullerud and Paganini, 2013; Doyle et al., 1988]. At the same time, much effort has been devoted to large-scale systems specifically, with the aim of developing scalable analysis and synthesis methods, see e.g., [Anderson et al., 2019] and the references therein. Finally, positive systems – systems which preserve the nonnegative cone in signal space – have surfaced recently in the control community due to their favorable scalability properties in both control analysis and synthesis, see e.g., [Berman and Plemmons, 1994; Farina and Rinaldi, 2000; Rantzer and Valcher, 2018] and the references therein.

The present paper lies at the intersection of these three fields. In particular, we pursue a line of research concerned with a promising controller structure, namely $u = B^T A^{-T} x$, where A is the state matrix and B is the input matrix of a linear time-invariant (LTI) system. This controller has proven to be not only H_∞ optimal but also sparse for certain classes of systems motivated by applications, see e.g., [Lidström and Rantzer, 2016; Bergeling et al., 2020; Vladu and Rantzer, 2022]. However, the set of systems for which optimality, let alone closed-loop stability, can be guaranteed remains restricted. We illustrate this by considering the system

$$\dot{x} = \begin{pmatrix} -1 & 0 \\ \alpha & -1 \end{pmatrix} x + \begin{pmatrix} -1 \\ 1 \end{pmatrix} u + w$$

where x is the state, u the control input and w the disturbance, respectively. This system could, for example, represent two leaking water tanks with deviations x from some desired equilibrium water level, one of which leaks partially into the other with rate $\alpha \geq 0$. The control signal amounts to shifting water from one tank to the other, and w may be thought of as disturbing inflows. The goal is to suppress the worst-case impact on the output (x, u) over the set of unity disturbances in an L_2 sense, i.e., minimize the H_∞ norm of the closed-loop system from w to (x, u). Here, u is penalized along with x, as is customary in order to account for the fact that in reality, u cannot be allowed to grow arbitrarily large.

At present, optimality for this system can be shown only when $\alpha = 0$ by invoking the symmetry of A [Lidström and Rantzer, 2016] and when $\alpha = 1$ by invoking the internal positivity of the closed-loop system from w to (x, u) [Vladu and Rantzer, 2022]. By contrast, a direct application of the main result in this paper now allows us to infer optimality in the entire range $0 \leq \alpha \leq 1$. This is a significant improvement, as it not only unifies two seemingly dissimilar phenomena, but also proceeds beyond them. The conditions for optimality offered in the present paper may thus be thought of as more natural than the previous ones. More specifically, the main result guarantees that $u = K_* x$ is H_∞ optimal if both A and $A + BK_*$ are Metzler Hurwitz and if in addition a certain other matrix is Metzler. This latter matrix will be analyzed at length in the following, and it often simplifies significantly in practice.

Additionally, a novel criterion for determining closed-loop stability is provided, namely that $A + A^T \prec 0$. The simplicity and the dependence only on A makes the criterion very convenient. To the author's knowledge, no previous stability criterion for the controller $u = K_* x$ exists for asymmetric A. Finally, we not only supply new proofs based on the main theorem to some previous results, but we also use it to guarantee optimality for a new class of systems motivated by applications for which the optimal controller continues to be sparse and thus interesting from a large-scale perspective.

1.1 Mathematical Notation

Let $\mathbb{R}$ denote the set of real numbers and $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ the set of n-dimensional vectors and $n \times m$ matrices, respectively, with entries in $\mathbb{R}$. For $v \in \mathbb{R}^n$, $\mathrm{diag}(v)$ is the square diagonal matrix with v along its diagonal. For any matrix M, $M_{i,j}$ denotes the entry (i, j). $\mathbb{1}$ is the vector with all entries equal to 1 and I is the identity matrix, with context determining the size. $M > (\geq)0$ means that M is entrywise positive (nonnegative). If $A \in \mathbb{R}^{n\times n}$, $\|A\|$ denotes the spectral (2-induced) norm of A, and $\mathrm{diag}(A)$ is the diagonal matrix with the same diagonal as A. If all eigenvalues of A have negative real part, we say that A is Hurwitz; if all offdiagonal elements are nonnegative, we say that A is Metzler. For symmetric A, $A \succ (\prec)0$ means that A is positive (negative) definite. Finally, the H_∞ norm of any stable transfer function G is $\|G\|_\infty = \sup_\omega \|G(i\omega)\|$.

2. Results

Consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu + w \\ z &= \begin{pmatrix} x \\ u \end{pmatrix} \end{aligned} \tag{1}$$

where $x \in \mathbb{R}^n$ denotes the state, $u \in \mathbb{R}^m$ the control input, $w \in \mathbb{R}^n$ the disturbance input and $z \in \mathbb{R}^{n+m}$ the regulated output. Here, $A \in \mathbb{R}^{n\times n}$ is the state matrix and $B \in \mathbb{R}^{n\times m}$ is the input matrix. Denote further by G_K the corresponding closed-loop transfer function from w to z given the controller $u = Kx$, $K \in \mathbb{R}^{m\times n}$. For an invertible A, set $K_* = B^T A^{-T}$ and $A_{cl} = A + BK_*$.

We have the following main result.

THEOREM 13
Consider system (1) and suppose that A and A_{cl} are Metzler Hurwitz and that

$$K_*^T K_* + A_{cl}^T \mathrm{diag}(-A)^{-1} + \mathrm{diag}(-A)^{-1} A_{cl} \tag{2}$$

is Metzler. Then $u = K_* x$ is an H_∞ optimal controller, i.e.,

$$\min_K \|G_K\|_\infty = \|G_{K_*}\|_\infty = \|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}}.$$

Proof. See Appendix. □

In addition, we also provide the following sufficient condition for the stability of the closed-loop system.

PROPOSITION 4
Consider system (1) and suppose that $A^T + A \prec 0$. Then $u = K_* x = B^T A^{-T} x$ is a stabilizing controller, i.e., $A_{cl} = A + BK_*$ is Hurwitz.

Proof. See Appendix. □

The importance of the above two results is twofold. First, Proposition 4 provides a simple criterion dependent only on A for K_* to be stabilizing even for asymmetric A, thereby essentially settling the stability question for this controller structure. Second, Theorem 13 unites the following two seemingly disparate results already published.

COROLLARY 3—SPECIAL CASE OF THEOREM 1 IN [LIDSTRÖM AND RANTZER, 2016]
Consider system (1) with A diagonal and Hurwitz and $-BB^T$ Metzler. Then $u = K_* x = B^T A^{-1} x$ is an H_∞ optimal controller.

COROLLARY 4—THEOREM 2.1 IN [VLADU AND RANTZER, 2022]
Suppose a directed graph $(\mathcal{V}, \mathcal{E})$ is given such that if $(i,j) \in \mathcal{E}$, then $i > j$ and $(k,j) \notin \mathcal{E}$ for $k \neq i$. Consider now the associated system with dynamics

$$\dot{x}_i = -a_i x_i + \sum_{(i,j)\in\mathcal{E}} a_j x_j + \sum_{(i,j)\in\mathcal{U}} u_{ij} - \sum_{(j,i)\in\mathcal{U}} u_{ji} + w_i \tag{3}$$

for $i = 1, \dots, n$, where it is assumed that $a_i > 0$ and $\mathcal{U} \subseteq \mathcal{E}$. Then an H_∞ optimal controller is given by

$$u_{ij}(x) = \frac{x_j}{a_j}$$

for $(i,j) \in \mathcal{U}$. Furthermore, this controller gives a closed-loop positive system from w to z.

Proof. See Appendix. □

In particular, the proof of Corollary 3 no longer depends on the symmetry of A and that of Corollary 4 no longer depends on the positivity of the closed-loop system from w to z.

Additionally, Theorem 13 may be readily invoked to show optimality for systems not covered by previous results. We illustrate this by considering a class of systems motivated by applications which lack closed-loop positivity in general and have block triangular A.

COROLLARY 5
Consider system (1) with $n \geq 2$ such that

$$\dot{x}_i = -a_{i,i}x_i + a_{i,i-1}x_{i-1} + u_{i-1} - u_i + w_i$$

for $1 < i < n$ when $n > 2$,

$$\dot{x}_1 = -a_{1,1}x_1 - u_1 + w_1$$

and

$$\dot{x}_n = -a_{n,n}x_n + a_{n,n-1}x_{n-1} + u_{n-1} + w_n.$$

Suppose now that $a_{i,i} > 0$ and that $a_{i,i-1} \geq 0$ can never be positive for two consecutive i. Suppose further that $a_{i,i-1} > 0$ implies $a_{i-1,i-1} \geq 2a_{i,i-1}$ and $16a_{i,i} > a_{i-1,i-1}$. Then $u = K_*x = B^TA^{-T}x$ is an H_∞ optimal controller and each u_i will depend on at most three states.

Proof. See Appendix. □

Note in particular that since each control signal u_i requires information from no more than three nodes for any n, this optimal controller should be an appealing candidate for large-scale networks of this type. Corollary 5 will be illustrated on a buffer network system in Section 3 below.

We close this section with some remarks.

REMARK 6
It is interesting to compare Theorem 2.2 in [Vladu and Rantzer, 2022] with Theorem 13 in the present paper. In the former, A_{cl} Metzler Hurwitz and $K_*^TK_* \geq 0$ (≤ 0) are sufficient for optimality, whereas the latter essentially relaxes the constraint $K_*^TK_* \geq 0$ through assumption (2) in exchange for a Metzler constraint on A. The sets of systems covered by the two theorems are thus non-overlapping. There is good cause, however, to prefer the latter over the former: besides its capability to go further and unify, as demonstrated above, the open-loop state matrix is arguably more likely to be Metzler in the first place for systems in which a closed-loop positive system is attained or even desired. In a sense then, as in the above results, the Metzler property of A may come along naturally in applications.

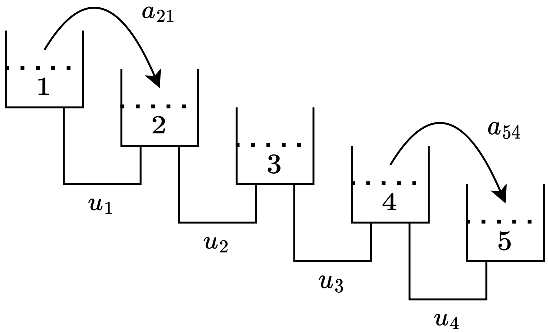

Figure 1. The network of five buffers considered in Section 3. The buffers are connected through flow links, which allow for a transfer of contents subject to control.

Remark 7
Assumption (2) in Theorem 13 merits some further discussion. First, since A_{cl} is Metzler, it follows that the second and third term in (2) can only assist towards making (2) Metzler. As for the term $K_*^T K_*$, it often has a favorable form in relation to the remaining terms: for example, in Corollary 3 it will immediately cancel out part of the second term, since $\mathrm{diag}(-A)^{-1} = -A^{-1}$, *whereas in Corollary 4, $K_*^T K_* \geq 0$. This suggests that the constraint (2) is in fact often readily checked.*

Remark 8
It is curious to note that despite the hint of positivity in Corollary 3, the closed-loop system from w to z will in general not even be externally positive, let alone internally positive. Simple systems suffice in order to demonstrate this, e.g., the system considered in Section 1 with $\alpha = 0$. Hence, despite its clear flavor of positivity, Theorem 13 appears to be connected to something beyond this traditional concept.

3. Illustrative Example

In this section, we illustrate some of the key points made in Section 2 and apply Corollary 5 to a buffer network system.

A buffer may be thought of as an abstraction of a container of some commodity: examples include tanks with varying water levels, biological cells with varying chemical concentrations or rooms with varying temperatures, see e.g., [Rantzer and Valcher, 2018]. If in addition the buffers are connected according to a string topology by flow links enabling a transfer of contents by means of control, and no two consecutive buffers share contents with any other buffer naturally, then the resulting system is of the type considered in Corollary 5. We consider an example with state matrix

$$A = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0.4 & -0.2 & 0 & 0 & 0 \\ 0 & 0 & -0.8 & 0 & 0 \\ 0 & 0 & 0 & -1.3 & 0 \\ 0 & 0 & 0 & 0.6 & -0.7 \end{pmatrix}$$

and input matrix

$$B = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

illustrated in Fig. 1. Here, each state x_i may be thought of as a deviation from some equilibrium level, and the goal is to minimize the worst-case deviation over the set of unity disturbances – which we assume can affect each buffer – in an L_2 sense.

Now, since $a_{i,i} > 0$ and $a_{i,i-1} \geq 0$ with $a_{i,i-1} > 0$ only for $i = 2$ and $i = 5$, which are not consecutive, and since $1 = a_{1,1} \geq 2a_{2,1} = 0.8$, $1.3 = a_{4,4} \geq 2a_{5,4} = 1.2$, $3.2 = 16a_{2,2} > a_{1,1} = 1$ and $11.2 = 16a_{5,5} > a_{4,4} = 1.3$, Corollary 5 yields that

$$K_* \approx \begin{pmatrix} 1 & -3 & 0 & 0 & 0 \\ 0 & 5 & -1.25 & 0 & 0 \\ 0 & 0 & 1.25 & -0.77 & -0.66 \\ 0 & 0 & 0 & 0.77 & -0.77 \end{pmatrix}$$

is an H_∞ optimal controller. For simulations of the corresponding closed-loop system with this controller structure when A is symmetric, see e.g., [Lidström and Rantzer, 2016].

Finally, note that optimality has been shown despite the fact that A is asymmetric and the closed-loop system lacks internal positivity. Note also that no row in K_* contains more than three nonzero elements, i.e., each control signal u_i will depend on no more than three states. As seen in Corollary 5, this observation will hold for any n, demonstrating that K_* continues to be an appealing candidate controller in the context of large-scale systems even when the two aforementioned properties are absent.

4. Conclusions

In this paper, we consider a certain controller structure and offer a set of positivity-based conditions for it to be H_∞ optimal. Interestingly, for a particular class of systems based on applications, these conditions are sufficient to show optimality both when the closed-loop system is internally positive and when A is symmetric but the closed-loop system lacks even external positivity. As such, with regards to future works, it may be interesting to explore this set of conditions further, especially in relation to the traditional concept of positivity. Additionally, efforts may also be directed at expanding the main result beyond its positivity-based setting so as to fully encompass the symmetric case.

Appendix

In this section, we supply proofs corresponding to the results in Section 2.

Proof. *Theorem 13*
The main idea of the proof is to construct a certain diagonal matrix $D \succ 0$ which solves a Riccati inequality pertaining to H_∞ control for the optimal γ-level. We distinguish between the cases when $(A_{cl}A^T)^{-1}$ is irreducible and reducible, respectively.

Observe first that since A is Hurwitz, $K_* = B^T A^{-T}$ and $A_{cl} = A + BK_*$ are well-defined. Define further $Q = AA^T + BB^T$ and set $\gamma_* = \|Q^{-1}\|^{\frac{1}{2}}$, noting that $Q \succ 0$ and $Q = A_{cl}A^T$. Since both A and A_{cl} are assumed to be Metzler Hurwitz, it follows that $A^{-1} \leq 0$ and $A_{cl}^{-1} \leq 0$ – see e.g., Theorem 2.5.3 in [Horn and Johnson, 1991] – and so $Q^{-1} = A^{-T}A_{cl}^{-1} \geq 0$.

Suppose now that Q^{-1} is irreducible. By the Perron-Frobenius theorem for irreducible nonnegative matrices, see e.g., Theorem 1.3 in Chapter 2.1 in [Berman and Plemmons, 1994], Q^{-1} must have an elementwise positive eigenvector $v > 0$ corresponding to the largest eigenvalue, i.e., $Q^{-1}v = \gamma_*^2 v$. Choose now

$$D = -\text{diag}(A^T v)\text{diag}(v)^{-1}.$$

In order to see that $D \succ 0$, note first that $0 > A_{cl}^{-1}v$. This follows since $A_{cl}^{-1} \leq 0$ cannot contain a zero row (or else A_{cl} would not be invertible, let alone Hurwitz, as assumed). Thus,

$$A^T v = \gamma_*^{-2} A^T(\gamma_*^2 v) = \gamma_*^{-2} A^T Q^{-1} v = \gamma_*^{-2} A_{cl}^{-1} v < 0.$$

This implies that $\text{diag}(A^T v) \prec 0$ and so $D \succ 0$.

Next, we define the matrix

$$R = (A^{-T}D + I)^T Q (A^{-T}D + I) - Q$$

and observe that

$$A^{-T}Dv = -A^{-T}\text{diag}(A^T v)\mathbb{1} = -A^{-T}A^T v = -v.$$

This means that $(A^{-T}D + I)v = 0$, and so $Rv = -\gamma_*^{-2}v$, i.e., v is an eigenvector to R.

In the final step, we show that R is Metzler. For this purpose, note that

$$D_{i,i} = -\frac{1}{v_i}\sum_{k=0}^{n} A_{k,i}v_k = -A_{i,i} + \sum_{k\neq i} -A_{k,i}\frac{v_k}{v_i} \leq -A_{i,i}$$

since A is assumed to be Metzler. Thus, $\text{diag}(-A) \geq D$ and so $D^{-1} \geq \text{diag}(-A)^{-1}$. Crucially it follows that exchanging $\text{diag}(-A)^{-1}$ for D^{-1} in the matrix $\text{diag}(-A)^{-1}A_{cl}$ can never decrease its offdiagonal entries, as A_{cl} was assumed to be Metzler. Invoking the last assumption of the theorem, namely that (2) is Metzler, it follows that

$$A^{-1}QA^{-T} + A_{cl}^T D^{-1} + D^{-1}A_{cl}$$

is also Metzler ($A^{-1}QA^{-T} = I + K_*^T K_*$) and therefore also

$$D(A^{-1}QA^{-T} + A_{cl}^T D^{-1} + D^{-1}A_{cl})D = R.$$

The last equality follows by completing the square and noting that $A_{cl} = QA^{-T}$.

Altogether, R is Metzler and has a positive eigenvector v with corresponding eigenvalue $-\gamma_*^{-2}$. As a result, $-\gamma_*^{-2}$ has to be the largest eigenvalue of R, see e.g., Corollary 1.12, Chapter 2, in [Berman and Plemmons, 1994] (shift R by ρI with $\rho > 0$ large enough). It follows that

$$R + \gamma^{-2}I \prec 0$$

for all $\gamma > \gamma_*$. Now, a congruence transformation with D^{-1} gives

$$A^{-1}QA^{-T} + A_{cl}^T D^{-1} + D^{-1}A_{cl} + \gamma^{-2}D^{-2} \prec 0$$

for all $\gamma > \gamma_*$. But a solution $P = D^{-1} \succ 0$ to this Riccati inequality means exactly that a system with state matrix A_{cl}, input matrix I and output matrix C such that $C^TC = A^{-1}QA^{-T}$ achieves an H_∞ norm at or below γ_*, see e.g., Theorem 2 (the linear case with $V(x) = \frac{x^TPx}{2}$) in [Van Der Schaft, 1992]. Now, since $A^{-1}QA^{-T} = I + K_*^T K_*$, this can be applied to the present closed-loop system from w to z with output matrix $C = \big(I, K_*^T\big)^T$, i.e.,

$$\|G_{K_*}\|_\infty \leq \gamma_* = \|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}}.$$

But this value is also known to be a lower bound over the set of all stabilizing controllers – see e.g., Theorem 2 or Section 2 with asymmetric A in [Bergeling et al., 2020] – and so the conclusion follows.

The above now remains to be shown when Q^{-1} is reducible. The main issue here is that v could have zero entries, but barring this the proof essentially follows the same steps and so we shall be more succinct in the following.

Suppose then that Q^{-1} is reducible and let $\epsilon > 0$ be given. By definition (see e.g., Definition 1.2, Chapter 2, in [Berman and Plemmons, 1994]), there is a permutation matrix $\bar{P}$ such that

$$\bar{P}Q^{-1}\bar{P}^T = \begin{pmatrix} \bar{U} & 0 \\ \bar{W} & \bar{V} \end{pmatrix}$$

with $\bar{U}, \bar{V}$ square. As a result, Q must have zero entries: to see this, invert the above block-triangular matrix and note that permutation matrices only permute rows and columns. We now introduce the perturbation $Q_\delta = Q - P_\delta$ with $P_\delta \in \mathbb{R}^{n\times n}$ such that

$$\left(P_\delta\right)_{i,j} = \begin{cases} \delta, & \text{if } Q_{i,j} = 0 \\ 0, & \text{otherwise} \end{cases}$$

for some sufficiently small $\delta > 0$ such that Q_δ remains positive definite. Note that Q_δ has no zero entries and so Q_δ^{-1} must be irreducible, or else again Q_δ would have zero entries. Hence, as before, Q_δ^{-1} must have a positive eigenvector $v_\delta > 0$ corresponding to its largest eigenvalue $\gamma_\delta^2 > 0$, i.e., $Q_\delta^{-1} v_\delta = \gamma_\delta^2 v_\delta$. Choose now

$$D_\delta = -\text{diag}(A^T v_\delta)\text{diag}(v_\delta)^{-1}$$

and note that

$$\begin{aligned} A^T Q_\delta^{-1} &= A^T(Q - P_\delta)^{-1} = A^T(I - Q^{-1}P_\delta)^{-1}Q^{-1} \\ &= A^T \sum_{k=0}^{\infty}(Q^{-1}P_\delta)^k Q^{-1} \\ &= A^T Q^{-1} + A^T Q^{-1} P_\delta \sum_{k=0}^{\infty}(Q^{-1}P_\delta)^k Q^{-1} \\ &= A_{cl}^{-1} + A_{cl}^{-1} P_\delta \sum_{k=0}^{\infty}(Q^{-1}P_\delta)^k Q^{-1} \leq A_{cl}^{-1} \leq 0 \end{aligned}$$

as $-A_{cl}^{-1}, Q^{-1}, P_\delta \geq 0$ (see above). In the second equality, δ is assumed to be sufficiently small for Theorem 3.9 in [Dullerud and Paganini, 2013] to be invoked. Altogether, the above implies that $A^T v_\delta < 0$ as in the irreducible case above, and so $D_\delta \succ 0$.

Define now

$$R_\delta = (A^{-T}D_\delta + I)^T Q (A^{-T}D_\delta + I) - Q_\delta$$

and note exactly as before that $R_\delta v_\delta = -\gamma_\delta^{-2} v_\delta$. Further, proceeding again as in the irreducible case above, we have that $\mathrm{diag}(-A) \geq D_\delta$ and consequently that

$$(A^{-T} D_\delta + I)^T Q (A^{-T} D_\delta + I) - Q = R_\delta - P_\delta$$

is Metzler. Adding $P_\delta \geq 0$ will not change this fact, and so R_δ is Metzler as well. Again it follows that $-\gamma_\delta^{-2}$ is the largest eigenvalue of R_δ. Crucially now, we may choose $\delta > 0$ small enough so that by the continuity of eigenvalues, the largest eigenvalue of $R_\delta - P_\delta$ can be made arbitrarily close to that of R_δ, namely $-\gamma_\delta^{-2}$. But since $\gamma_\delta \longrightarrow \gamma_*$ as $\delta \longrightarrow 0$, there must be a small enough $\delta > 0$ such that

$$R_\delta - P_\delta + \gamma^{-2} I \prec 0$$

for all $\gamma > \epsilon + \gamma_*$. As in the irreducible case above, a congruence transformation with D_δ^{-1} gives

$$A^{-1} Q A^{-T} + A_{cl}^T D_\delta^{-1} + D_\delta^{-1} A_{cl} + \gamma^{-2} D_\delta^{-2} \prec 0$$

for all $\gamma > \epsilon + \gamma_*$. As before, the satisfaction of this Riccati inequality by $P = D_\delta^{-1} \succ 0$ shows that $\|G_{K_*}\|_\infty \leq \gamma_* + \epsilon$. Now, since this procedure of constructing a solution $P \succ 0$ is applicable to all $\epsilon > 0$, it follows that $\|G_{K_*}\|_\infty \leq \gamma_*$, and so the above conclusion follows also in the reducible case. □

Proof. *Proposition 4*
Define $Q = AA^T + BB^T$ and note first that by assumption, $A^T + A \prec 0$. This is a Lyapunov inequality with $P = I \succ 0$ and it follows that A is Hurwitz and thus invertible, see e.g., Theorem 3.6 in [Khalil, 1992]. As a result, $A_{cl} = A + BB^T A^{-T} = QA^{-T}$ is well-defined and $Q \succ 0$, and so $A^{-T} = Q^{-1} A_{cl}$. It follows that $A^{-1} + A^{-T} = A_{cl}^T Q^{-1} + Q^{-1} A_{cl}$. Now, invoking the assumption again and noting that congruence transformations preserve definiteness, we have $0 \succ A^{-1}(A^T + A)A^{-T} = A^{-1} + A^{-T}$. Hence, $A_{cl}^T Q^{-1} + Q^{-1} A_{cl} \prec 0$, another Lyapunov inequality with $P = Q^{-1} \succ 0$, showing that A_{cl} is in fact Hurwitz. □

Proof. *Corollary 3*
Note first that since A is diagonal Hurwitz and $-BB^T$ is Metzler by assumption, A and A_{cl} are both Metzler (multiplication by a positive diagonal matrix preserves the Metzler property). The stability of A_{cl} follows by noting that $A^T + A \prec 0$ and invoking Proposition 4.

Finally, A diagonal gives $\mathrm{diag}(-A) = -A$. Thus, after canceling terms, expression (2) reduces to

$$-2I + (-A^{-1})(-BB^T)(-A^{-1})$$

which is Metzler due to the assumptions made. The desired conclusion now follows from Theorem 1. □

Proof. *Corollary 4*
Observe first that A and A_{cl} are Metzler Hurwitz and that $K_*^T K_* \geq 0$, see e.g., the proof of Theorem 2.1 in [Vladu and Rantzer, 2022]. The assumptions of Theorem 13 are thus clearly satisfied and the corollary follows. Note that the positivity of the closed-loop system from w to z, which was fundamental in the corresponding proof in [Vladu and Rantzer, 2022], has not been invoked here. □

Proof. *Corollary 5*
Clearly, A is Metzler Hurwitz and block lower triangular with blocks of size $p = 1$ or $p = 2$. At the same time, $B \in \mathbb{R}^{n\times(n-1)}$ will have the form $B_{i,i} = -1$, $B_{i+1,i} = 1$ for $1 \leq i \leq n-1$ and zero otherwise. It follows that BB^T will be tridiagonal with $BB^T_{n,n} = 1$, $BB^T_{i,i+1} = -1$, $BB^T_{i+1,i} = -1$ and $BB^T_{i,i} = 2$ for $1 \leq i \leq n-1$ with the exception that $BB^T_{1,1} = 1$. To show this, split B into two terms, one containing only the last column of B and the other containing the rest, then use induction. For an illustration of A and B matrices on this form when $n = 5$, see Section 3.

Set now $D = \text{diag}(A^{-1}) = \text{diag}(A)^{-1}$ and $L = A^{-1} - D$. Since the inverse of a 2×2 lower triangular matrix is

$$\begin{pmatrix} v_{11} & 0 \\ v_{21} & v_{22} \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{v_{11}} & 0 \\ -\frac{v_{21}}{v_{11}v_{22}} & \frac{1}{v_{22}} \end{pmatrix}$$

it follows that A will have the same offdiagonal zero pattern as A^{-1} and therefore L. Further, writing out

$$BB^T L^T = \big(BB^T(L^T)_1, \ldots, BB^T(L^T)_n\big), \tag{4}$$

it is clear that $BB^T L^T$ will contain negative values only in the entries in which L^T is nonzero. Altogether, $(BB^T L^T)_{j,i} < 0$ if and only if $j = i-1$ and $a_{i,j} \neq 0$, with value $(BB^T L^T)_{i-1,i} = -\frac{2a_{i,i-1}}{a_{i-1,i-1}a_{i,i}}$ except if $i = 2$ in which case a factor 2 disappears. But A and $-BB^T$ are Metzler, and $A_{cl} = A + BB^T A^{-T} = A + BB^T D + BB^T L^T$, and so in order for A_{cl} to be Metzler, it is necessary and sufficient to demand that $\frac{1}{a_{i,i}} - \frac{2a_{i,i-1}}{a_{i-1,i-1}a_{i,i}} \geq 0$ for all i such that $a_{i,i-1} \neq 0$. Here, the first term $\frac{1}{a_{i,i}}$ comes from the term $BB^T D$. But this is exactly what the assumption $a_{i-1,i-1} \geq 2a_{i,i-1}$ guarantees.

As for A_{cl} being Hurwitz, note that each non-diagonal 2×2 block in $A + A^T$ has negative trace and determinant $4a_{i-1,i-1}a_{i,i} - a_{i,i-1}^2$. But invoking again the assumption $a_{i-1,i-1} \geq 2a_{i,i-1}$, the determinant can be lower bounded by $4a_{i-1,i-1}a_{i,i} - \frac{a_{i-1,i-1}{}^2}{4} = \frac{a_{i-1,i-1}}{4}(16a_{i,i} - a_{i-1,i-1})$ which in turn must be positive due to the assumption $16a_{i,i} > a_{i-1,i-1}$. Thus, $A + A^T \prec 0$ holds and by Proposition 4, A_{cl} must be Hurwitz.

For the final step, note that after substituting $A^{-1} = D + L$ in (2) and cancelling some terms, expression (2) reduces to

$$LBB^T L^T - DBB^T D - A^T D - DA. \tag{5}$$

We now argue that the first term in (5) is in fact diagonal. For this purpose, recall again (4). It is clear that each column in $BB^T L^T$ contains at most three consecutive nonzero numbers centered around the negative numbers in the entries where L^T is nonzero, see above. But due to the assumption that $a_{i,i-1}$ cannot be nonzero for two consecutive i, each nonzero column in $BB^T L^T$ must be surrounded by columns with zero entries. Consequently, each row in $BB^T L^T$ with a negative value contains at most one nonzero entry, namely the one in which L^T is also nonzero, and $LBB^T L^T$ is thus diagonal. Now, since A and $-BB^T$ are Metzler, (5) and therefore (2) must be Metzler, and so Theorem 13 may be invoked to show that $u = K_* x$ is optimal. Of course, since no two consecutive rows in A and therefore A^{-T} can ever contain more than three columns with nonzero entries, each row in $K_* = B^T A^{-T}$ can have at most three nonzero entries, and so each control signal u_i can depend on at most three states. □

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References

Anderson, J., J. C. Doyle, S. H. Low, and N. Matni (2019). "System level synthesis". *Annual Reviews in Control* **47**, pp. 364–393.

Bergeling, C., R. Pates, and A. Rantzer (2020). "H-infinity optimal control for systems with a bottleneck frequency". *IEEE Trans. Autom. Control* **66**:6, pp. 2732–2738.

Berman, A. and R. J. Plemmons (1994). *Nonnegative Matrices in the Mathematical Sciences*. SIAM. ISBN: 9780120922505.

Doyle, J., K. Glover, P. Khargonekar, and B. Francis (1988). "State-space solutions to standard H_2 and H_∞ control problems". In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 1691–1696.

Dullerud, G. E. and F. Paganini (2013). *A Course in Robust Control Theory: a Convex Approach.* Vol. 36. Springer Science & Business Media. ISBN: 9780387989457.

Farina, L. and S. Rinaldi (2000). *Positive Linear Systems: Theory and Applications.* Vol. 50. John Wiley & Sons. ISBN: 9780471384564.

Horn, R. and C. Johnson (1991). *Topics in Matrix Analysis.* Cambridge University Press, Cambridge. ISBN: 9780521467131.

Khalil, H. K. (1992). *Nonlinear Systems.* NY, USA: Macmillan Publishing Company. ISBN: 9780130673893.

Lidström, C. and A. Rantzer (2016). "Optimal H_∞ state feedback for systems with symmetric and Hurwitz state matrix". In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 3366–3371.

Rantzer, A. and M. E. Valcher (2018). "A tutorial on positive systems and large scale control". In: *Proc. IEEE Conf. Decis. Control (CDC)*, pp. 3686–3697.

Van Der Schaft, A. J. (1992). "L_2-gain analysis of nonlinear systems and nonlinear state feedback H_∞ control". *IEEE Transactions on Automatic Control* **37**:6, pp. 770–784.

Vladu, E. and A. Rantzer (2022). "On decentralized H-infinity optimal positive systems". *IEEE Control Syst. Lett.* **7**, pp. 391–394.

Zhou, K., J. C. Doyle, K. Glover, et al. (1996). *Robust and Optimal Control.* Vol. 40. Upper Saddle River, NJ, USA: Prentice Hall. ISBN: 9780134565675.

Paper III

A Cone-preserving Solution to a Nonsymmetric Riccati Equation

Emil Vladu **Anders Rantzer**

Abstract

In this paper, we provide the following simple equivalent condition for a nonsymmetric Algebraic Riccati Equation to admit a stabilizing cone-preserving solution: an associated coefficient matrix must be stable. The result holds under the assumption that said matrix be cross-positive on a proper cone, and it both extends and completes a corresponding sufficient condition for nonnegative matrices in the literature. Further, key to showing the above is the following result which we also provide: in order for a monotonically increasing sequence of cone-preserving matrices to converge, it is sufficient to be bounded above in a single vectorial direction.

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1. Introduction

Algebraic Riccati equations have been studied extensively in the literature over the years, e.g., [Lancaster and Rodman, 1995] and the references therein. They often appear in the form

$$XBX + DX + XA + C = 0,$$

where $A, B, C, D \in \mathbb{R}^{n\times n}$, and are not only of theoretical but also of practical interest. Often, as in control theory [Dullerud and Paganini, 2013][Willems, 1971], D is taken as A^T with B, C symmetric, and a symmetric solution X is sought for. A typical result under the assumption of stabilizability and detectability may provide a unique stabilizing solution in certain settings, e.g., [Kučera, 1973]. However, more recently there has been an increased interest in its nonsymmetric counterpart, see e.g., [Freiling, 2002] and the references therein.

The specific case in which the negative of the matrix

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

forms an M-matrix, i.e., an anti-stable matrix with nonpositive offdiagonal elements, with applications in transport theory and Markov models, has attracted some interest, see e.g., [Guo, 2001][Guo and Higham, 2007] and the references therein. In particular, part of [Guo and Higham, 2007, Theorem 1.1] provides the following sufficient condition for the existence of a stabilizing, entrywise nonnegative solution: $-L$ should be an M-matrix. Analogue statements for irreducible singular M-matrices and more recently also regular singular M-matrices [Guo and Lu, 2016] are known to hold.

The objective of the present paper is to both generalize the above result to proper cones as well as to complete it into an equivalence (Theorem 14). Earlier papers on the topic appear to be focused exclusively on the M-matrix case and its applications, with no mention about proper cones to the best of the authors' knowledge. By contrast, the main purpose of this paper is the desire to better understand the result by identifying the structure which generates it: the conic structure of the nonnegative orthant in $\mathbb{R}^n$. In addition, the main result turns out to be useful also in a different context, namely control theory [Vladu, 2024].

In order to prove the main result, a fixed-point iteration approach similar to the one in [Guo, 2001] is used in one direction. However, rather than exploiting the Kronecker product, we shall pass through the analytical integral solution of the Sylvester equation. Further, and more importantly, convergence no longer becomes a simple matter of applying the monotone convergence theorem elementwise. As a result, we also provide a somewhat unexpected convergence result (Theorem 15) for a sequence of monotonically increasing cone-preserving matrices with a vectorial upper bound. This is novel to the best of the authors' knowledge.

The outline of the paper is as follows: Section 2 reviews the crucial notion of cross-positivity on a proper cone as well as some elementary facts in cone theory. In Section 3, we present the results of the paper along with their proofs. Section 4 subsequently concludes the paper with some suggestions for future works.

2. Preliminaries

In this section, we clarify the notation used in the paper and provide the necessary background required for the results.

2.1 Definitions and Notation

In this subsection, we explain some definitions and notation used throughout the paper. $\mathbb{R}$, $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ refer to the set of real numbers, n-dimensional vectors and $n \times m$ matrices with entries in $\mathbb{R}$, respectively. $\succeq_K$ refers to the partial order induced by a proper cone K, see Subsection 2.2. If $A \in \mathbb{R}^{n\times n}$, then A is said to be stable if all its eigenvalues have negative real part. $\|A\|$ refers to the corresponding norm induced by an inner product, if such a function has been supplied, and otherwise to the spectral norm.

2.2 Cone Theory

In this subsection, we provide some necessary background on cone theory from the literature. For more results on this topic, see e.g., [Berman and Plemmons, 1994][Schneider and Tam, 2006] and the references therein.

A set $K \subseteq \mathbb{R}^n$ is said to be a cone if $x \in K$ and $\alpha \geq 0$ imply $\alpha x \in K$. A convex, closed and pointed ($K \cap -K = \{0\}$) cone with non-empty interior is said to be proper. A proper cone K induces a partial order $\succeq_K$ such that $x \succeq_K y$ if and only if $x - y \in K$. If $x - y$ lies in the interior of K, we say that $x \succ_K y$. The standard example of a proper cone is the nonnegative orthant in $\mathbb{R}^n$, and $\succeq_K$ then reduces to the usual inequality between real numbers applied entrywise.

Given a proper cone $K \subseteq \mathbb{R}^n$, the associated dual cone is defined as

$$K_* = \{y \in \mathbb{R}^n \mid y^T x \geq 0 \text{ for all } x \in K\}.$$

The interior of the dual cone consists of all those y such that $y^T x > 0$ for all nonzero $x \in K$. If a cone is proper, the associated dual cone is also proper [Boyd and Vandenberghe, 2004, Chap. 2.6.1]. Further, it is routine to verify from the definitions that $\big(K \times K\big)_* = K_* \times K_*$ and that $K \times K$ is a proper cone in $\mathbb{R}^{2n}$.

A matrix $A \in \mathbb{R}^{n \times n}$ is said to be cross-positive on K if $x \in K$ and $y \in K_*$ with $y^T x = 0$ imply $y^T A x \geq 0$. In particular, the set of cross-positive matrices on the nonnegative orthant is simply the set of matrices with nonnegative offdiagonal elements. For more on cross-positive matrices, see e.g., [Schneider and Vidyasagar, 1970].

A matrix $A \in \mathbb{R}^{n \times n}$ such that $AK \subseteq K$ for a proper cone $K \subseteq \mathbb{R}^n$ is said to be K-nonnegative or leave K invariant. The set of such matrices is denoted $\pi(K)$ and is itself known to be a proper cone in $\mathbb{R}^{n \times n}$ [Berman and Plemmons, 1994, Chap. 1.1] in the vector sense after a standard identification with $\mathbb{R}^{n^2}$. Hence, if a proper cone $K \subseteq \mathbb{R}^n$ is specified, $X \succeq_K Y$ where $X, Y \in \mathbb{R}^{n \times n}$ means that $X - Y \in \pi(K)$, i.e., $X - Y$ is K-nonnegative.

Next, we gather some elementary facts about K-nonnegativity and cross-positivity of which we shall make frequent use.

Proposition 5
Given $x, y \in \mathbb{R}^n$, $A, B \in \mathbb{R}^{n \times n}$ and a proper cone $K \subseteq \mathbb{R}^n$, the following holds:

(i) If A and B are K-nonnegative, then so is $A + B$ and AB.

(ii) If A is K-nonnegative and $x \succeq_K y$, then $Ax \succeq_K Ay$.

(iii) If $A \succeq_K B$ and $x \succeq_K 0$, then $Ax \succeq_K Bx$.

(iv) If A is K-nonnegative, then A is cross-positive on K.

(v) If A and B are cross-positive on K, then so is $A + B$.

Proof. Regarding (i), if $x \in K$, then $(A + B)x = Ax + Bx \succeq_K 0$ and $ABx = Ay \succeq_K 0$ with $y = Bx \succeq_K 0$ since K is a proper cone. As for (ii), note that $Ax - Ay = A(x - y) \succeq_K 0$ so that $Ax \succeq_K Ay$, since $x - y \in K$ by assumption. Similarly, in (iii) we note that $A - B$ is K-nonnegative by assumption, and so $(A - B)x \succeq_K 0$, i.e., $Ax \succeq_K Bx$. Further, (iv) follows by definition of cross-positivity, as $z = Ax \succeq_K 0$ if $x \succeq_K 0$ by assumption, so that $y^T A x = y^T z \geq 0$ if $y \in K_*$ by definition of the dual cone. Finally, if $x \in K$, $y \in K_*$ with $y^T x = 0$, we have $y^T (A + B)x = y^T A x + y^T B x \geq 0$ by assumption and so $A + B$ is cross-positive, i.e., (v) holds. □

We now recall the following monotone convergence result.

Lemma 1
[Berman and Plemmons, 1974, Lemma 1] Let $K \subseteq \mathbb{R}^n$ be a proper cone and let $\{s_i\}_{i=1}^{\infty}$ be such that $s_i \preceq_K s_{i+1}$. Let $t \in \mathbb{R}^n$ be such that $s_i \preceq_K t$ for every positive integer i. Then the sequence $\{s_i\}_{i=1}^{\infty}$ converges.

We close this section with two important results on cross-positivity.

Lemma 2
[Schneider and Vidyasagar, 1970, Theorem 3] Let $K \subseteq \mathbb{R}^n$ be a proper cone and $A \in \mathbb{R}^{n\times n}$. Then A is cross-positive on K if and only if e^{At} is K-nonnegative for all $t \geq 0$.

Lemma 3
[Schneider and Tam, 2006, Facts 7.1, 7.3, 7.5] Suppose $A \in \mathbb{R}^{n\times n}$ is cross-positive on a proper cone $K \subseteq \mathbb{R}^n$. Then the following are equivalent:

(i) A is stable.

(ii) There exists $x \succ_K 0$ such that $Ax \prec_K 0$.

(iii) $-A^{-1}$ is K-nonnegative.

3. Results

In this section, we provide the results of the paper. The main result is the following.

Theorem 14
Let the proper cone $K \subseteq \mathbb{R}^n$ and the matrices $A, B, C, D \in \mathbb{R}^{n\times n}$ be given. Suppose now that

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is cross-positive on $K \times K$. Then L is stable if and only if

$$XBX + DX + XA + C = 0 \tag{1}$$

has a solution $X_* \succeq_K 0$ such that $A + BX_*$ and $D + X_*B$ are stable and cross-positive on K.

Proof. See below. □

Remark 9
In the special case that K is taken as the nonnegative orthant, the direction corresponding to sufficiency in Theorem 14 is equivalent to the part of the statement in [Guo, 2001, Theorem 3.1] corresponding to (nonsingular) M-matrices, see Section 1. This follows because cross-positive matrices on the nonnegative orthant have nonnegative offdiagonal elements and vice versa.

The next result is not only instrumental in proving Theorem 14 but is also interesting in its own right.

Theorem 15
Let the proper cone $K \subseteq \mathbb{R}^n$ and the sequence $\{X_i\}_{i=1}^{\infty}$ in $\mathbb{R}^{n\times n}$ be given. Suppose now that $0 \preceq_K X_i \preceq_K X_{i+1}$ and that there exist $s, r \in \mathbb{R}^n$ with $r \succ_K 0$ such that $X_i r \preceq_K s$ for all positive integers i. Then $\{X_i\}_{i=1}^{\infty}$ converges.

Proof. Let $w \in \mathbb{R}^n$ be given and note that because $r \succ_K 0$, there must exist some $\varepsilon > 0$ such that both $r - \varepsilon w \succeq_K 0$ and $r + \varepsilon w \succeq_K 0$. Define now $a_i = X_i r$ and $b_i = X_i(r - \varepsilon w)$. First, it is clear from the assumptions and the fact that $r - \varepsilon w \succeq_K 0$ that $(X_{i+1} - X_i)r \succeq_K 0$ and $(X_{i+1} - X_i)(r - \varepsilon w) \succeq_K 0$, i.e., $a_i \preceq_K a_{i+1}$ and $b_i \preceq_K b_{i+1}$, respectively. Second, since by assumption $X_i \succeq_K 0$ and $X_i r \preceq_K s$, we have $a_i \preceq_K s$ and

$$0 \preceq_K X_i(r + \varepsilon w) = X_i(2r - r + \varepsilon w) = 2X_i r - X_i(r - \varepsilon w) = 2X_i r - b_i$$

so that $b_i \preceq_K 2X_i r \preceq_K 2s$. It follows by Lemma 1 that both $\{a_i\}_{i=1}^{\infty}$ and $\{b_i\}_{i=1}^{\infty}$ converge. But then the sequence with elements

$$X_i w = \frac{1}{\varepsilon} X_i(r - r + \varepsilon w) = \frac{1}{\varepsilon}(a_i - b_i)$$

must also converge. Now, since this holds for all $w \in \mathbb{R}^n$, one can choose w so as to pick out each column of X_i to show that they all converge. But by the equivalence of norms, this implies that $\{X_i\}_{i=1}^{\infty}$ converges entrywise and the conclusion follows. □

In order to prove Theorem 14, we shall require the following additional lemmata.

Lemma 4
Let the proper cone $K \subseteq \mathbb{R}^n$ and the matrices $A, C, D \in \mathbb{R}^{n\times n}$ be given. Suppose now that A and D are stable and cross-positive on K and that $C \succeq_K 0$. Then

$$DX + XA + C = 0 \tag{2}$$

has a unique solution $X_* \succeq_K 0$.

Proof. Existence and uniqueness of a solution X_* follows from Theorem 4.4.6 in [Horn and Johnson, 1991], as A and D are stable by assumption. Further,

$$X_* = \int_0^{\infty} e^{Dt} C e^{At} \mathrm{dt}, \tag{3}$$

as

$$-C = \left[e^{Dt} C e^{At}\right]_0^{\infty} = \int_0^{\infty} \frac{d}{dt}\left(e^{Dt} C e^{At}\right) \mathrm{dt}$$
$$= \int_0^{\infty} \left(De^{Dt} C e^{At} + e^{Dt} C e^{At} A\right) \mathrm{dt} = DX_* + X_* A,$$

similar to the well-known Lyapunov equation solution. It follows now from Lemma 2 and Proposition 5 (i) that the integrand in (3) is K-nonnegative for all $t \geq 0$. As a result, we have $X_* \succeq_K 0$, as $\pi(K)$ is a proper cone and is therefore closed and preserves nonnegative linear combinations, see Section 2. □

Lemma 5
Let the proper cone $K \subseteq \mathbb{R}^n$ and the matrices $A, B, C, D \in \mathbb{R}^{n\times n}$ be given. Then

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is cross-positive on $K \times K$ if and only if A and D are cross-positive on K and $B, C \succeq_K 0$.

Proof. In the first direction, suppose that L is cross-positive on $K \times K$ and take any $x \in K$ and $y \in K_*$ such that $y^T x = 0$. It follows that $\bar{y}^T \bar{x} = 0$, where $\bar{x} = (x^T, 0^T)^T \in K \times K$ and $\bar{y} = (y^T, 0^T)^T \in \big(K \times K\big)_*$, and so by assumption $y^T Ax = \bar{y}^T L\bar{x} \geq 0$, implying that A is cross-positive on K. Taking instead $\bar{x} = (0, x^T)^T \in K \times K$ and noting that $\bar{y}^T \bar{x} = 0$ – this time for any $x \in K$ and $y \in K_*$ – again by cross-positivity $y^T Bx = \bar{y}^T L\bar{x} \geq 0$. Since this holds for any fixed $x \in K$ as y ranges over all elements in K_*, we must have $Bx \in K_{**}$, i.e., $B \succeq_K 0$ as for proper cones $K_{**} = K$, see e.g., [Boyd and Vandenberghe, 2004, Chap. 2.6.2]. Similar reasoning gives that D is cross-positive on K and that $C \succeq_K 0$.

As for the other direction, suppose that $y = (y_1^T, y_2^T)^T \in \big(K \times K\big)_*$ and $z = (z_1^T, z_2^T)^T \in K \times K$ are given such that $y^T z = 0$, i.e.,

$$y^T z = y_1^T z_1 + y_2^T z_2 = 0. \tag{4}$$

Note now that since $\big(K \times K\big)_* = K_* \times K_*$, we must have $y_1^T z_1 \geq 0$ and $y_2^T z_2 \geq 0$. As such, the only possibility for (4) to hold is that both $y_1^T z_1 = 0$ and $y_2^T z_2 = 0$. Hence, since A and D are cross-positive on K by assumption, it follows that $y_1^T Az_1 \geq 0$ and $y_2^T Dz_2 \geq 0$. Consequently,

$$y^T Lz = y_1^T Az_1 + y_1^T Bz_2 + y_2^T Cz_1 + y_2^T Dz_2 \geq 0$$

since by assumption $B, C \succeq_K 0$. But this means exactly that L is cross-positive on $K \times K$. □

Lemma 6
Let the proper cone $K \subseteq \mathbb{R}^n$ and the matrices $A, B, C, D \in \mathbb{R}^{n\times n}$ be given. Suppose now that

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is stable and cross-positive on $K \times K$. Then A and D are stable and there exist $u_1, u_2, v_1, v_2 \in \mathbb{R}^n$ with $v_1, v_2 \succ_K 0$ such that

$$Av_1 + Bv_2 = u_1 \prec_K 0, \quad Cv_1 + Dv_2 = u_2 \prec_K 0. \tag{5}$$

Proof. Invoke immediately Lemma 3 to show that there exists a $v \succ_{K\times K} 0$ such that $Lv = u \prec_{K\times K} 0$. With the even partitioning $u = (u_1^T, u_2^T)^T$, $v = (v_1^T, v_2^T)^T$, we thus obtain (5) as desired. In order to see that $v_1, v_2 \succ_K 0$, note that since v lies in the interior of $K \times K$, there is an $\varepsilon > 0$ such that for all $y \in \mathbb{R}^{2n}$ with $\|y - v\| < \varepsilon$ we have $y \in K \times K$. Consider now any $y_1 \in \mathbb{R}^n$ such that $\|y_1 - v_1\| < \varepsilon$. Since $\|y_1 - v_1\| = \|y - v\|$ with $y = (y_1^T, v_2^T)^T$, it follows that $y \in K \times K$ and so $y_1 \in K$. Hence, v_1 lies in the interior of K, i.e., $v_1 \succ_K 0$. Similar reasoning gives $v_2 \succ_K 0$.

Finally, Lemma 5 implies that A and D are cross-positive on K and that $B, C \succeq_K 0$. Consequently, $Bv_2 \succeq_K 0$ and $Cv_1 \succeq_K 0$, and as a result, the above inequalities (5) imply that $Av_1 \prec_K 0$ and $Dv_2 \prec_K 0$, as $p \succ_K 0$ and $q \succeq_K 0$ imply $p + q \succ_K 0$. Another application of Lemma 3 therefore implies that A and D are both stable. □

We are now ready for the proof of Theorem 14.

***Proof.* Theorem 14**
Suppose in the first direction that L is stable. The proof outline is then as follows: we construct a monotonically increasing sequence of matrices which is shown to converge to the desired solution of (1). For this purpose, consider the recursion

$$DX_{i+1} + X_{i+1}A = -X_iBX_i - C \tag{6}$$

and set $X_0 = 0$. We proceed by induction to show that the sequence $\{X_i\}_{i=0}^{\infty}$ generated by (6) is both well-defined and monotonically increasing. In the case that $i = 1$, an application of Lemma 4 directly gives a unique solution $X_1 \succeq_K 0 = X_0$. Suppose now that X_i is well-defined through (6) and that $X_i \succeq_K X_{i-1}$ up until $i = k$ for some positive integer k. By Lemma 5, A and D are both cross-positive on K and $B, C \succeq_K 0$. Thus, $X_kBX_k + C \succeq_K 0$ by the induction assumption and so by Lemma 4, X_{k+1} follows uniquely from (6), as A and D are stable by Lemma 6. Further, subtracting $DX_k + X_kA$ from both sides in (6) yields

$$\begin{aligned} D(X_{k+1} - X_k) + (X_{k+1} - X_k)A &= -X_kBX_k - DX_k - X_kA - C \\ &\preceq_K -X_{k-1}BX_{k-1} - DX_k - X_kA - C = 0. \end{aligned}$$

Here, the induction assumption (6) was invoked in the second equality, and $X_k \succeq_K X_{k-1}$ was invoked to give $X_kBX_k \succeq_K X_{k-1}BX_{k-1}$, hence the inequality. The latter statement follows from the assumption $B \succeq_K 0$, repeated application of Proposition 5 (i) and transitivity. But again by Lemma 4, this means that $X_{k+1} - X_k \succeq_K 0$, i.e., $X_{k+1} \succeq_K X_k$. Thus, the sequence $\{X_i\}_{i=0}^{\infty}$ generated by (6) is well-defined and monotonically increasing by induction.

Next, we show that $\{X_i\}_{i=0}^{\infty}$ is bounded above in the sense that there exist $r, s \in \mathbb{R}^n$ with $r \succ_K 0$ such that $X_i r \preceq_K s$. For this purpose, apply Lemma 6 to obtain (5), choose $r = v_1$ and $s = v_2 - D^{-1}u_2$, where $-u_1, -u_2, v_1, v_2 \succ_K 0$, and proceed by induction. The $i = 0$ case follows immediately: $s = v_2 - D^{-1}(Cv_1 + Dv_2) = -D^{-1}Cv_1 \succeq_K 0 = X_0 r$ by Proposition 5 (i), since $-D^{-1} \succeq_K 0$ by Lemma 3 as D is cross-positive on K by assumption and stable from Lemma 6. Suppose now $X_i v_1 \preceq_K s = v_2 - D^{-1}u_2$ where $i = k$ for some $k \geq 0$. We have

$$\begin{aligned}
-DX_{k+1}v_1 &= X_{k+1}Av_1 + X_kBX_kv_1 + Cv_1 \\
&\preceq_K X_{k+1}Av_1 + X_kBv_2 - Dv_2 + u_2 \\
&= (X_{k+1} - X_k)Av_1 + X_ku_1 - Dv_2 + u_2 \preceq_K -Dv_2 + u_2.
\end{aligned}$$

Here, the first equality comes from (6). In the first inequality, the second expression in (5) and $X_k v_1 \preceq_K v_2$ were invoked. In the second equality, the first expression in (5) was used, and in the second inequality monotonicity was invoked along with the fact that $Av_1 = u_1 - Bv_2 \prec_K 0$ and $X_k u_1 \preceq_K 0$. Multiplication by $-D^{-1} \succeq_K 0$ thus gives $X_{k+1}v_1 \preceq_K v_2 - D^{-1}u_2$, and so the desired conclusion follows by induction.

Altogether, we have shown that $0 = X_0 \preceq_K X_i \preceq_K X_{i+1}$ and that there exist $r, s \in \mathbb{R}^n$ with $r \succ_K 0$ such that $X_i r \preceq_K s$ for all $i \geq 0$. An application of Theorem 15 thus shows that $\{X_i\}_{i=1}^{\infty}$ is convergent, i.e., $X_i \to X_*$ as $i \to \infty$ for some $X_* \in \mathbb{R}^{n\times n}$. Consequently, since the sequence satisfies (6), we have

$$X_*BX_* + DX_* + X_*A + C = 0.$$

Further, since $\pi(K)$ is a proper cone and hence closed, see Section 2, $\{X_i\}_{i=1}^{\infty}$ converges inside $\pi(K)$, i.e., $X_* \succeq_K 0$.

It remains to show that $A + BX_*$ and $D + X_*B$ are stable and cross-positive on K. For this purpose, note that $X_*v_1 \preceq_K v_2$ because K is closed. Together with the assumption $B \succeq_K 0$ and Proposition 5 (ii), we thus have

$$(A + BX_*)v_1 \preceq_K Av_1 + Bv_2 = u_1 \prec 0,$$

where in the final equality the first expression in (5) was invoked. But $A+BX_*$ is cross-positive on K according to Proposition 5 (i), (iv) and (v), since A is also cross-positive on K by assumption. It thus follows from Lemma 3 that $A + BX_*$ is stable.

As for $D + X_*B$, note that the matrix

$$\begin{pmatrix} D^T & B^T \\ C^T & A^T \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

has the same spectrum as L and is therefore stable, as it is a similarity transform of L^T. Further, it is cross-positive on $K_* \times K_*$ by Lemma 5, as D^T and A^T are cross-positive on K_* due to $K_{**} = K$, and $B^T, C^T \succeq_{K_*} 0$ by definition of K_*. By the part of Theorem 14 proved so far, the equation

$$ZB^TZ + A^TZ + ZD^T + C^T = 0$$

must therefore have a solution $Z_* \succeq_{K_*} 0$ such that $D^T + B^TZ_*$ is stable and cross-positive on K_*. Additionally, such a solution can be taken as the limit of a sequence $\{Z_i\}_{i=0}^{\infty}$ with $Z_0 = 0$ which satisfies the corresponding recursion (6), i.e.,

$$A^TZ_{i+1} + Z_{i+1}D^T = -Z_iB^TZ_i - C^T.$$

But according to first part of the proof, this recursion is also satisfied by $\{X_i^T\}_{i=0}^{\infty}$ with $X_0 = 0$ once (6) is transposed. Because the sequence generated by the recursion was shown to be unique, we may conclude that $Z_i = X_i^T$ and so by the continuity of the transpose operator $Z_* = X_*^T$. But then the stability and cross-positivity of $D^T + B^TZ_* = D^T + B^TX_*^T$ on K_* implies the stability and cross-positivity of $D + X_*B = (D^T + B^TX_*^T)^T$ on K. This concludes the sufficiency part of the proof.

As for necessity, note first that we have the well-known similarity transformation

$$\begin{pmatrix} I & 0 \\ -X_* & I \end{pmatrix} \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \begin{pmatrix} I & 0 \\ X_* & I \end{pmatrix} = \begin{pmatrix} A + BX_* & B \\ -X_*BX_* - DX_* - X_*A - C & -(D + X_*B) \end{pmatrix}.$$

Consequently, the assumptions yield

$$\begin{aligned} -L^{-1} &= -\left(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} A & B \\ -C & -D \end{pmatrix} \right)^{-1} \\ &= -\Bigg(\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} I & 0 \\ X_* & I \end{pmatrix} \\ &\quad \begin{pmatrix} A + BX_* & B \\ 0 & -(D + X_*B) \end{pmatrix} \begin{pmatrix} I & 0 \\ -X_* & I \end{pmatrix} \Bigg)^{-1} \\ &= \begin{pmatrix} I & 0 \\ X_* & I \end{pmatrix} \begin{pmatrix} PX_* - (A + BX_*)^{-1} & P \\ -(D + X_*B)^{-1}X_* & -(D + X_*B)^{-1} \end{pmatrix} \succeq_{K \times K} 0 \end{aligned}$$

where $P = (A+BX_*)^{-1}B(D+X_*B)^{-1}$. Here, the fact that X_* solves $XBX+DX+XA+C = 0$ was invoked in the second equality. Note also that $A+BX_*$

and $D + X_*B$ being cross-positive and stable by assumption implies that $-(A+BX_*)^{-1} \succeq_K 0$ and $-(D+X_*B)^{-1} \succeq_K 0$ by Lemma 3, and so $P \succeq_K 0$ since $B \succeq_K 0$. Further, it is clear that if all four matrices in a 2×2 block matrix are K-nonnegative, then the block matrix is $K \times K$-nonnegative. As such, since also $X_* \succeq_K 0$ by assumption, the final inequality follows. But since it is assumed that L is cross-positive on $K \times K$, another application of Lemma 3 implies that L is stable. This concludes the proof. □

4. Conclusions

In this paper, the following equivalent condition is supplied for a nonsymmetric algebraic Riccati equation to admit a stabilizing cone-preserving solution: an associated coefficient matrix should be stable. This extends and completes an already published sufficient condition on the nonnegative orthant into an equivalence for general proper cones. Many additional properties, such as the minimality of the cone-preserving solution, follow from the stability of the aforementioned coefficient matrix in the well-studied nonnegative case. While this lies beyond the scope of the present paper, establishing how well these features generalize to proper cones would be interesting for future works.

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References

Berman, A. and R. J. Plemmons (1974). “Cones and iterative methods for best least squares solutions of linear systems”. *SIAM J. Numer. Anal.* **11**:1, pp. 145–154.

Berman, A. and R. J. Plemmons (1994). *Nonnegative Matrices in the Mathematical Sciences*. SIAM. ISBN: 9780120922505.

Boyd, S. P. and L. Vandenberghe (2004). *Convex Optimization*. Cambridge University Press. ISBN: 9780521833783.

Dullerud, G. E. and F. Paganini (2013). *A Course in Robust Control Theory: a Convex Approach.* Vol. 36. Springer Science & Business Media. ISBN: 9780387989457.

Freiling, G. (2002). "A survey of nonsymmetric Riccati equations". *Linear Algebra Appl.* **351**, pp. 243–270.

Guo, C.-H. (2001). "Nonsymmetric algebraic Riccati equations and Wiener–Hopf factorization for M-matrices". *SIAM J. Matrix Anal. Appl.* **23**:1, pp. 225–242.

Guo, C.-H. and N. J. Higham (2007). "Iterative solution of a nonsymmetric algebraic Riccati equation". *SIAM J. Matrix Anal. Appl.* **29**:2, pp. 396–412.

Guo, C.-H. and D. Lu (2016). "On algebraic Riccati equations associated with regular singular M-matrices". *Linear Algebra Appl.* **493**, pp. 108–119.

Horn, R. and C. Johnson (1991). *Topics in Matrix Analysis.* Cambridge University Press, Cambridge. ISBN: 9780521467131.

Kučera, V. (1973). "A review of the matrix Riccati equation". *Kybernetika* **9**:1, pp. 42–61.

Lancaster, P. and L. Rodman (1995). *Algebraic Riccati Equations.* Clarendon Press. ISBN: 9780198537953.

Schneider, H. and B.-S. Tam (2006). "Matrices leaving a cone invariant". *Handb. Linear Algebra, ed. L. Hogben, Chapman & Hall.*

Schneider, H. and M. Vidyasagar (1970). "Cross-positive matrices". *SIAM J. Numer. Anal.* **7**:4, pp. 508–519.

Vladu, E. (2024). *Stability and performance analysis on self-dual cones,* arXiv: 2411.12100 [math.OC].

Willems, J. (1971). "Least squares stationary optimal control and the algebraic Riccati equation". *IEEE Trans. Autom. Control* **16**:6, pp. 621–634.

Paper IV

Stability and Performance Analysis on Self-dual Cones

Emil Vladu

Abstract

In this paper, we consider nonsymmetric solutions to certain Lyapunov and Riccati equations and inequalities with coefficient matrices corresponding to cone-preserving dynamical systems. Most results presented here appear to be novel even in the special case of positive systems. First, we provide a simple eigenvalue criterion for a Sylvester equation to admit a cone-preserving solution. For a single system preserving a self-dual cone, this reduces to stability. Further, we provide a set of conditions equivalent to testing a given H-infinity norm bound, as in the bounded real lemma. These feature the stability of a coefficient matrix similar to the Hamiltonian, a solution to two conic inequalities, and a stabilizing cone-preserving solution to a nonsymmetric Riccati equation. Finally, we show that the H-infinity norm is attained at zero frequency.

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1. Introduction

A monotone linear dynamical system is a system such that an input $u(t)$ and a state initial condition $x(0)$ confined to a cone together imply that the state $x(t)$ and output $z(t)$ are also confined to a cone for all $t \geq 0$. Although scarce and still at its infancy, research on the topic is warranted given the success of the special case in which the cones are taken as the nonnegative orthant. Such systems are known as positive systems and are particularly suited for analysis and synthesis of large-scale systems, e.g., [Farina and Rinaldi, 2000][Rantzer and Valcher, 2018] and the references therein. One important reason is that L_1/L_∞-gain verification and controller synthesis reduces to linear programming [Briat, 2013][Rantzer, 2015b]. Similarly, positive diagonal solutions to Lyapunov equations and linear matrix inequalities (LMIs) are necessary and sufficient for achieving stability [Berman and Plemmons, 1994] and a given H_∞ norm bound [Tanaka and Langbort, 2011], respectively. The latter then paves the way for structured synthesis in which the sparsity pattern on the controller may be specified in exchange for requiring closed-loop positivity [Tanaka and Langbort, 2011]. An additional reformulation of the Lyapunov theorem and the bounded real lemma given in [Ebihara et al., 2014] is as follows: *any* nonsymmetric solution with positive definite symmetric part is necessary and sufficient in order to certify stability and a particular H_∞ norm, respectively. By contrast, in standard Lyapunov and H_∞ theory, symmetric solutions to Lyapunov equations [Rugh, 1996] and LMIs [Boyd et al., 1994][Gahinet and Apkarian, 1994] or Riccati equations [Doyle et al., 1988] are required to certify stability and performance, respectively.

For general proper cones, the key notions appear to be cone-preservance and cross-positivity, corresponding to nonnegative matrices and Metzler matrices, respectively, see Section 2. However, the above results do not generalize in a straightforward way to general cone-preserving/monotone systems. For example, extending the connection between the L_1 gain and linear programming in [Briat, 2013] requires a shift to cone linear absolute norms in [Shen and Lam, 2017]. Further, [Tanaka, 2012, Chapter 4] shows that the H_∞ norm of a system which preserves a proper cone does not in general equal its static gain, as it does for positive systems [Rantzer, 2015b]. Instead, it is the spectral radius of a transfer function corresponding to such a system which achieves its maximum value at zero frequency [Tanaka et al., 2013]. Finally, the celebrated diagonal solution to Lyapunov equations and LMIs fails to hold more generally. However, if the solution is seen as the result of the quadratic representation of a Jordan algebra applied to a vector obtained through a conic program, then the symmetric cones, a subset of the self-dual ones, appear to be the natural setting for this property. This is indicated by recent works such as [Shen and Lam, 2016] in the bounded real lemma case and [Lu et al., 2024] in the KYP lemma case, corresponding to [Tanaka and Langbort, 2011] and [Rantzer, 2015a], respectively, for positive systems. Note in particular that the H_∞ norm is shown to equal the static gain in this setting [Shen and Lam, 2016]. Another very recent result on symmetric cones is [Dalin et al., 2024], in which a Lie-algebraic approach is taken to construct a quadratic Lyapunov function for stable cone-preserving systems which becomes diagonal w.r.t. the nonnegative orthant.

By contrast, in this paper we explore what can be said when the proper cones are self-dual only. The flavor of our results is quite different from that of the above results, most notably in that we depart from matrix symmetry, let alone diagonality. In particular, we show that for two cross-positive matrices on a proper cone, the associated Sylvester equation admits a (possibly nonsymmetric) cone-preserving solution if and only if the sum of the greatest real parts of their eigenvalues is negative. It follows that when the cone is self-dual, a cone-preserving solution to the Lyapunov equation is necessary and sufficient for stability. Additionally, we leverage a recent result in [Vladu and Rantzer, 2025] to provide a set of conditions which are equivalent to a γ-bound on the H_∞ norm for monotone systems w.r.t. self-dual cones: a) a given Riccati inequality has a solution with positive definite symmetric part, b) a stabilizing cone-preserving solution exists to a nonsymmetric Riccati equation, c) a particular coefficient matrix should be stable and d) a set of conic inequalities are satisfied by elements in the interior of the cone. Here, a) appears to generalize [Ebihara et al., 2014] above, whereas b), c) and d) are novel to the best of the authors' knowledge, even for positive systems. For comparisons to similar results in the literature, see Remark 13. Finally, we show that the symmetry of the state cone in [Shen and Lam, 2016] may in fact be relaxed to self-duality in order for the static gain to determine the H_∞ norm.

The outline of the paper is as follows: Section 2 reviews the basics of cone theory and cross-positivity in particular, and Section 3 presents the results of the paper. Section 4 illustrates the results with some examples and Section 5 provides the proofs. Finally, Section 6 concludes the paper.

2. Preliminaries

In this section, we explain our notation and supply the required background for the contents in the remaining sections.

Let $\mathbb{R}$ denote the set of real numbers. $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the set of n-dimensional vectors and $n \times m$ matrices, respectively, with entries in $\mathbb{R}$. For $M \in \mathbb{R}^{n\times m}$, $\|M\|$ denotes the spectral (2-induced) norm of M, and I is the identity matrix, with context determining its dimension. For $A \in \mathbb{R}^{n\times n}$, we say that A is Hurwitz if all its eigenvalues have negative real part, and Metzler if all its offdiagonal elements are nonnegative. $\sigma(A)$ is the spectrum of A, i.e., the set of all its eigenvalues. If A is symmetric, then $A \succ (\succeq) 0$ means that A is positive (semi)definite.

We recall at this point the following important observation, made for instance in [Ebihara et al., 2014].

LEMMA 7
Let $A \in \mathbb{R}^{n\times n}$ and suppose that there exists a $P \in \mathbb{R}^{n\times n}$ with $P + P^T \succ 0$ such that

$$A^T P^T + PA \prec 0.$$

If λ is a real eigenvalue of A, then $\lambda < 0$.

A set $K \subseteq \mathbb{R}^n$ is called a cone if $x \in \mathcal{K}$ and $\alpha \geq 0$ imply $\alpha x \in K$. If in addition K is convex, closed, pointed ($\mathcal{K} \cap -\mathcal{K} = \{0\}$) and has non-empty interior, then it is called a proper cone. A proper cone induces a partial order $\succeq_K$ on $\mathbb{R}^n$, i.e., $x \succeq_K y$ if and only if $x - y \in K$; strict inequality $x \succ_K y$ means that $x - y$ lies in the interior of K. The dual cone associated with a proper cone is

$$K_* = \{y \mid y^T x \geq 0 \text{ for all } x \in K\}.$$

If $K = K_*$, then we say that K is self-dual.

A matrix $A \in \mathbb{R}^{n\times n}$ is said to be cross-positive on K if $x \in K$, $y \in K_*$ and $y^T x = 0$ imply $y^T A x \geq 0$. In particular, cross-positive matrices on the nonnegative orthant correspond to Metzler matrices. Another important notion associated with proper cones is that of K-nonnegativity: $A \in \mathbb{R}^{n\times n}$ is said to be K-nonnegative or preserve K if $AK \subseteq K$. Since the set of K-nonnegative matrices is itself a proper cone in $\mathbb{R}^{n\times n}$ [Berman and Plemmons, 1994, Chap. 1.1], it induces a partial order which we denote also by $\succeq_K$. Thus, $X \succeq_K Y$ for matrices $X, Y \in \mathbb{R}^{n\times n}$, as opposed to vectors, means that $X - Y$ is K-nonnegative. Similarly, $X \succ_K Y$ means that $X - Y$ maps all nonzero elements in K into its interior, cf. [Schneider and Vidyasagar, 1970]. In the case that K is the nonnegative orthant, $X \succeq_K 0$ and $X \succ_K 0$ mean that X is entrywise nonnegative and positive, respectively.

Cross-positive matrices satisfy the following property.

LEMMA 8
[Schneider and Vidyasagar, 1970, Theorem 5] Let $K \subseteq \mathbb{R}^n$ be a proper cone and $A \in \mathbb{R}^{n\times n}$. Suppose now that A is cross-positive on K. Then $\mu = \max\{\mathrm{Re}(\lambda) \mid \lambda \in \sigma(A)\}$ is an eigenvalue of A. Further, K contains an eigenvector corresponding to λ.

The cross-positivity of a block matrix is connected to that of its constituents in the following way.

LEMMA 9
[Vladu and Rantzer, 2025, Lemma 5] Let the proper cone $K \subseteq \mathbb{R}^n$ and the matrices $A, B, C, D \in \mathbb{R}^{n\times n}$ be given. Then

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is cross-positive on $K \times K$ if and only if A and D are cross-positive on K and $B, C \succeq_K 0$.

An important connection between dynamical systems and the notion of cross-positivity is given by the following lemma.

LEMMA 10
[Schneider and Vidyasagar, 1970, Theorem 3] Let $K \subseteq \mathbb{R}^n$ be a proper cone and $A \in \mathbb{R}^{n\times n}$. Then A is cross-positive on K if and only if e^{At} is K-nonnegative for all $t \geq 0$.

It follows that for a system $\dot{x} = Ax$ with A cross-positive on K, the resulting trajectory $x(t)$ given $x(0) \in K$ will remain inside K for all $t \geq 0$.

We have the following stability test for cross-positive matrices, see e.g., [Shen and Lam, 2016].

LEMMA 11
[Schneider and Tam, 2006, Facts 7.1, 7.3, 7.5] Suppose $A \in \mathbb{R}^{n\times n}$ is cross-positive on a proper cone $K \subseteq \mathbb{R}^n$. Then the following are equivalent:

(i) A is stable.

(ii) There exists $x \succ_K 0$ such that $Ax \prec_K 0$.

(iii) A is invertible and $-A^{-1}$ is K-nonnegative.

A recent result characterizes the existence of a stabilizing K-nonnegative solution to a nonsymmetric Algebraic Riccati Equation.

LEMMA 12
[Vladu and Rantzer, 2025, Theorem 1] Let the proper cone $K \subseteq \mathbb{R}^n$ and the matrices $A, B, C, D \in \mathbb{R}^{n\times n}$ be given. Suppose now that

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

is cross-positive on $K \times K$. Then L is stable if and only if

$$XBX + DX + XA + C = 0$$

has a solution $X_* \succeq_K 0$ such that $A + BX_*$ and $D + X_*B$ are Hurwitz and cross-positive on K.

We close this section by considering linear time-invariant (LTI) systems with state matrix $A \in \mathbb{R}^{n\times n}$, input matrix $B \in \mathbb{R}^{n\times m}$ and output matrix $C \in \mathbb{R}^{p\times n}$. We consider only the case with no direct term. Recall now that for such stable systems with transfer function $G(s) = C(sI - A)^{-1}B$, the H_∞ norm is defined as $\|G\|_\infty = \sup_\omega \|G(i\omega)\|$. A variant of the well-known bounded real lemma is as follows.

Lemma 13
[Zhou and Doyle, 1998, Corollary 12.3] Let $\gamma > 0$ and suppose that A is Hurwitz. Define now

$$H = \begin{pmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{pmatrix}.$$

The following conditions are equivalent:

(i) $\|G\|_\infty < \gamma$

(ii) H has no eigenvalues on the imaginary axis.

(iii) There exists a $P \succeq 0$ such that

$$\frac{1}{\gamma^2}PBB^TP + A^TP + PA + C^TC = 0 \tag{1}$$

and $A + \frac{1}{\gamma^2}BB^TP$ has no imaginary axis eigenvalues.

(iv) There exists a $P \succ 0$ such that

$$\frac{1}{\gamma^2}PBB^TP + A^TP + PA + C^TC \prec 0. \tag{2}$$

3. Results

In this section, we present the results of the paper. Those related to stability are found in Subsection 3.1, whereas those related to performance are found in Subsection 3.2.

3.1 Stability Analysis

Define $\mu(A)$ to be the greatest real part of the spectrum of a square matrix A. We then have the following result.

Theorem 16
Suppose $A, D \in \mathbb{R}^{n\times n}$ are cross-positive on a proper cone K. Then there exists $P \succ_K 0$ such that

$$DP + PA \prec_K 0$$

if and only if

$$\mu(A) + \mu(D) < 0.$$

Proof. See Section 5. □

For self-dual cones in particular, this reduces to a stability test.

COROLLARY 6
Suppose $A \in \mathbb{R}^{n\times n}$ is cross-positive on a self-dual proper cone K. Then there exists $P \succ_K 0$ such that

$$A^T P + PA \prec_K 0$$

if and only if A is Hurwitz.

Proof. See Section 5. □

We close this subsection with some remarks.

REMARK 10
Corollary 6 should be compared to Lyapunov's Theorem, e.g., [Rugh, 1996]. Note, however, that compared to this standard result, the solution P in Corollary 6 does not have to be symmetric. For example, when K is taken as the nonnegative orthant so that cross-positivity reduces to the Metzler property, $P \succ_K 0$ is equivalent to $P > 0$, i.e., entrywise positivity.

REMARK 11
In this remark, we give a Lyapunov function-like interpretation of Theorem 16. Given the two systems $\dot{x} = Ax$ and $\dot{y} = D^T y$ with A and D cross-positive on K, clearly there exists a solution $P \succ_K 0$ if and only if there exists a quadratic function $V(x, y) = y^T P x$ such that $V(x, y) > 0$ and $\dot{V}(x, y) = y^T(DP + PA)x < 0$ for all nonzero $x \in K$ and $y \in K_*$. Thus, for nonzero trajectories in K and K_*, respectively, $V(x, y)$ is always positive and decreasing. Of course, it is intuitive that this can happen despite one of the systems being unstable, so long as the trajectories of the other system converge faster to the origin: this is consistent with the condition $\mu(A) + \mu(D) < 0$.

3.2 Performance Analysis

In this subsection, we consider the LTI system

$$\begin{aligned} \dot{x} &= Ax + Bu \\ z &= Cx \end{aligned} \tag{3}$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control input and $z(t) \in \mathbb{R}^p$ is the regulated output. Denote by G the open-loop transfer function from u to z.

In keeping with [Angeli and Sontag, 2003] and [Shen and Lam, 2017], we make the following definition.

Definition 1
System (3) is said to be monotone with respect to the proper cones $(K_u, K_x, K_z) \subseteq (\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p)$ *if* $u(t) \in K_u$ *for all* $t \geq 0$ *and* $x(0) \in K_x$ *together imply that* $x(t) \in K_x$ *and* $z(t) \in K_z$ *for all* $t \geq 0$.

The main result of this subsection is the following.

Theorem 17
Consider system (3) with A *Hurwitz and let* $\gamma > 0$ *and the three self-dual proper cones* $(K_u, K_x, K_z) \subseteq (\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p)$ *be given. Suppose now that (3) is monotone with respect to* (K_u, K_x, K_z). *Then the following conditions are equivalent:*

(i) $\|G\|_\infty < \gamma$.

(ii) L *is Hurwitz, where*

$$L = \begin{pmatrix} A & \frac{1}{\gamma^2}BB^T \\ C^TC & A^T \end{pmatrix}.$$

(iii) There exists $P \succeq_{K_x} 0$ *such that*

$$\frac{1}{\gamma^2}PBB^TP + A^TP + PA + C^TC = 0 \tag{4}$$

and $A + \frac{1}{\gamma^2}BB^TP$ *is Hurwitz.*

(iv) There exists $P \in \mathbb{R}^{n\times n}$ *with* $P + P^T \succ 0$ *such that*

$$\frac{1}{\gamma^2}PBB^TP^T + A^TP^T + PA + C^TC \prec 0. \tag{5}$$

(v) There exists $p, q \succ_{K_x} 0$ *such that*

$$\begin{aligned} Ap + \frac{1}{\gamma^2}BB^Tq &\prec_{K_x} 0 \\ C^TCp + A^Tq &\prec_{K_x} 0. \end{aligned}$$

Proof. See Section 5. □

The next result shows that monotone systems w.r.t. self-dual cones also achieve their H_∞ norm at zero frequency.

Theorem 18
Consider system (3) with A *Hurwitz and suppose that system (3) is monotone with respect to the self-dual proper cones* $(K_u, K_x, K_z) \subseteq (\mathbb{R}^m, \mathbb{R}^n, \mathbb{R}^p)$. *Then*

$$\|G\|_\infty = \|G(0)\|.$$

Proof. See Section 5. □

We close this subsection with some remarks.

REMARK 12
It is straightforward to verify using Lemma 10 that Definition 1 is equivalent to the fact that A is cross-positive on K_x, $BK_u \subseteq K_x$ and $CK_x \subseteq K_z$, see e.g., [Shen and Lam, 2017]. Further, when the three cones are taken as the nonnegative orthant, we regain the definition of an (internally) positive system, e.g., [Rantzer and Valcher, 2018].

REMARK 13
Conditions (ii) and (iii) in Theorem 17 appear to be analogous to the imaginary axis eigenvalue condition on the Hamiltonian and the Riccati equation solution condition in the standard bounded real lemma, respectively, cited here for convenience in Lemma 13. Condition (v) is perhaps best compared to condition (ii) in [Shen and Lam, 2016, Theorem 2] in which twice the amount of variables and inequalities are used to characterize the γ-suboptimality of the static gain for symmetric cones.

4. Illustrative Examples

In this section, we provide examples of some of the results in Section 3. For the purpose of illustration, we shall consider only the nonnegative orthant; the reader is referred to the references provided in Section 1 for the many examples and applications on alternative cones.

4.1 A Closed-Loop Positive H_∞ Optimal Controller

In this subsection, we consider the dynamics

$$\dot{x} = \begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 2 & -4 \end{pmatrix} x + \begin{pmatrix} -1 & 0 \\ 1 & -1 \\ 0 & 1 \end{pmatrix} u + w.$$

The above system can be thought of as a simple model of a small irrigation network consisting of three pools connected in series, e.g., [Cantoni et al., 2007]. The state matrix A suggests that each pool decays towards some equilibrium level, and that the decaying content does not vanish but is instead transferred over to the next pool. Further, the input matrix B suggests that we may actuate a transfer of contents between two adjacent pools. Finally, each pool is subject to disturbing inflows w.

Now, it was shown in [Vladu and Rantzer, 2022, Theorem 2] that the controller

$$K_* = B^T A^{-T} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$

results in the following three desirable properties for the closed-loop system from w to (x, u):

a) K_* is H_∞ optimal.

b) K_* is closed-loop positive.

c) K_* is diagonal.

In a sense, K_* arguably appears to be a very natural candidate controller. However, this controller cannot in fact result from the H_∞ synthesis Riccati inequality [Boyd et al., 1994, Section 7.5.1]

$$A^T P + PA + P(\frac{1}{\gamma^2} I - BB^T)P + C^T C \prec 0, \tag{6}$$

as there is in fact no symmetric P such that $K_* = -B^T P$. In order to see this, suppose on the contrary that such a

$$P = P^T = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{12} & p_{22} & p_{23} \\ p_{13} & p_{23} & p_{33} \end{pmatrix}$$

existed. We would then have

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \end{pmatrix} = \begin{pmatrix} p_{11} - p_{12} & p_{12} - p_{22} & p_{13} - p_{23} \\ p_{12} - p_{13} & p_{22} - p_{23} & p_{23} - p_{33} \end{pmatrix}.$$

But this would imply that $p_{22} = p_{12} = p_{13} = p_{23}$ so that $0 = p_{22} - p_{23} = \frac{1}{2}$, a contradiction.

On the other hand, matters become different once we consider the extended version

$$A^T P^T + PA + P(\frac{1}{\gamma^2} I - BB^T)P^T + C^T C \prec 0.$$

It is straightforward to show that the nonsymmetric matrix $P_* = -A^{-1}$ is a solution for all $\gamma > \|(AA^T + BB^T)^{-1}\|^{\frac{1}{2}}$, where the latter value is known to be a lower bound over all stabilizing controllers. Thus, invoking condition (iv) in Theorem 17 for the corresponding closed-loop system, $K_* = -B^T P_*^T = B^T A^{-T}$ is seen to be optimal.

A synthesis procedure like (6) which fails to account for a natural controller such as K_* is arguably incomplete. Of course, any stabilizing γ-suboptimal controller can still be reached by performing a variable change and searching over two matrix variables instead of one in the corresponding LMI [Boyd et al., 1994]. However, this example shows that there is substance in the middle ground offered by the above nonsymmetric extension, and searching over one variable instead of two may prove valuable in the context of large-scale systems.

4.2 A Positive Solution to the Sylvester Equation

In this subsection, we illustrate Theorem 16 on the nonnegative orthant by considering the two matrices

$$A = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}, \quad D = \begin{pmatrix} -1 & 0 \\ 4 & 1 \end{pmatrix}$$

which are clearly Metzler and thus cross-positive on the nonnegative orthant. Since further their eigenvalues lie on the diagonal, it is clear that $\mu(A) + \mu(D) = -2 + 1 = -1 < 0$ so that Theorem 16 gives the existence of a $P > 0$ such that $DP + PA < 0$. One such P is given by

$$P = \frac{1}{9}\begin{pmatrix} 3 & 4 \\ 21 & 46 \end{pmatrix} > 0$$

since then

$$DP + PA = \begin{pmatrix} -1 & -1 \\ -1 & -1 \end{pmatrix} < 0.$$

5. Proofs

In this section, we supply proofs to the results in Section 3. For this purpose, we shall require the following lemma.

LEMMA 14
Suppose $\gamma > 0$ and the matrices $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$ and $C \in \mathbb{R}^{p\times n}$ are given with A invertible. If λ is an eigenvalue of

$$H = \begin{pmatrix} A & \frac{1}{\gamma^2}BB^T \\ -C^TC & -A^T \end{pmatrix},$$

then $\lambda \neq 0$ for all $\gamma > \|CA^{-1}B\|$. Further, if $\gamma = \|CA^{-1}B\|$, then H has a zero eigenvalue.

Proof. We have

$$\begin{aligned} \det(H) &= \det(A)\det(-A^T - (-C^TC)A^{-1}(\gamma^{-2}BB^T)) \\ &= \det(A)\det(-A^T) \\ &\quad \cdot \det(I - \gamma^{-2}A^{-T}C^TCA^{-1}BB^T) \\ &= \det(A)\det(-A^T) \\ &\quad \cdot \det(I - \gamma^{-2}(B^TA^{-T}C^T)(CA^{-1}B)) \\ &= \det(A)\det(-A^T)\det(I - \gamma^{-2}Q^TQ) \end{aligned}$$

where $Q = CA^{-1}B$. Here, the first equality follows from the Schur complement determinant rule [Horn and Johnson, 2012, Ch. 0.8.5], the second equality from the standard product determinant rule and the third equality from a repeated application of the Schur complement determinant rule so that $\det(I+RS) = \det(I+SR)$ with $R \in \mathbb{R}^{r\times s}$ and $S \in \mathbb{R}^{s\times r}$. But $\|CA^{-1}B\|$ is exactly the square root of the largest eigenvalue of Q^TQ, a matrix with nonnegative eigenvalues. It follows that the eigenvalues λ of $\gamma^{-2}Q^TQ$ must satisfy $0 \le \lambda < 1$ for all $\gamma > \|CA^{-1}B\|$. Thus, $\det(I-\gamma^{-2}Q^TQ) \neq 0$ for such γ, exploiting the fact that the determinant is equal to the product of the eigenvalues. Further, since A is invertible and hence $\det(A) = \det(A^T) \neq 0$, it follows that $\det(H) \neq 0$. Thus, no eigenvalue of H can be zero for $\gamma > \|CA^{-1}B\|$, or else the eigenvalue product would be zero. If on the other hand $\gamma = \|CA^{-1}B\|$, then $\gamma^{-2}Q^TQ$ will have an eigenvalue $\lambda = 1$. As a result, similar reasoning gives $\det(H) = 0$, i.e., H must have a zero eigenvalue. □

Proof. **Theorem 16**
$\Leftarrow$: Recall first that for any $M \in \mathbb{R}^{n\times n}$ with Jordan decomposition $M = SJS^{-1}$, we have $e^{Mt} = Se^{Jt}S^{-1} = S\bar{D}(t)\bar{E}(t)S^{-1}$. Here, $\bar{D}(t)$ is diagonal and consists of entries $e^{\lambda t}$ with $\lambda \in \sigma(M)$, and $\bar{E}(t)$ has polynomial entries in t. As such, for any $Q \prec_K 0$,

$$P = \int_0^\infty e^{Dt}(-Q)e^{At}\mathrm{d}t$$

must converge. This follows due to the assumption $\mu(A) + \mu(D) < 0$, as each entry in the integrand is a sum of terms with factors $e^{(\lambda_{D\,i}+\lambda_{A\,j})t}$. In particular, $e^{Dt}Qe^{At} \to 0$ as $t \to \infty$ so that

$$\begin{aligned} Q &= \left[e^{Dt}(-Q)e^{At}\right]_0^\infty = \int_0^\infty \frac{d}{dt}\big(e^{Dt}(-Q)e^{At}\big)\mathrm{d}t \\ &= \int_0^\infty \big(De^{Dt}(-Q)e^{At} + e^{Dt}(-Q)e^{At}A\big)\mathrm{d}t = DP + PA. \end{aligned}$$

Finally, Lemma 10 implies that $P \succeq_K 0$, as $-Q \succ_K 0$ and the set of K-nonnegative matrices is closed as it is a proper cone, see Section 2. We may now lift P into the interior by adding a sufficiently small perturbation.
$\Rightarrow$: Because A is cross-positive by assumption, by Lemma 8 there is a $v \succeq_K 0$ with $v \neq 0$ such that $Av = \mu(A)v$. Thus,

$$0 \succ_K Qv = (DP + PA)v = DPv + \mu(A)Pv = (D + \mu(A)I)Pv.$$

Since $Pv \succ_K 0$ and D is cross-positive, Lemma 11 shows that $D + \mu(A)I$ is stable and the conclusion follows. □

Proof. **Corollary 6** This follows by invoking Theorem 16 after noting that A is Hurwitz if and only if $\mu(A) = \mu(A^T) < 0$, and that A^T is also cross-positive on K if A is, provided that K is self-dual. The latter follows by noting that $K_{**} = K$, see e.g., [Boyd and Vandenberghe, 2004, p. 53], so that $x \in K_{**} = K$, $y \in K_*$ and $x^T y = 0$ imply $x^T A^T y = (x^T A^T y)^T = y^T A x \geq 0$. □

Proof. **Theorem 17**
We prove the following chain of implications: $(i) \Rightarrow (iv) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. We subsequently show that $(ii) \Leftrightarrow (v)$.

$\underline{(i) \Rightarrow (iv)}$: This follows immediately from Lemma 13.

$\underline{(iv) \Rightarrow (ii)}$: We have

$$
\begin{aligned}
&\begin{pmatrix} I & 0 \\ 0 & P \end{pmatrix} \begin{pmatrix} A & \frac{1}{\gamma^2} BB^T \\ C^T C & A^T \end{pmatrix}^T \begin{pmatrix} P^T & 0 \\ 0 & I \end{pmatrix} + \\
&\begin{pmatrix} P & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & \frac{1}{\gamma^2} BB^T \\ C^T C & A^T \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & P^T \end{pmatrix} = \\
&\begin{pmatrix} A^T P^T + PA & \frac{1}{\gamma^2} PBB^T P^T + C^T C \\ \frac{1}{\gamma^2} PBB^T P^T + C^T C & A^T P^T + PA \end{pmatrix} = \\
&\begin{pmatrix} R - F & F \\ F & R - F \end{pmatrix} = \begin{pmatrix} R & 0 \\ 0 & R \end{pmatrix} + \begin{pmatrix} -F & F \\ F & -F \end{pmatrix} \prec 0
\end{aligned}
$$

where $R = \frac{1}{\gamma^2} PBB^T P^T + A^T P^T + PA + C^T C$ and $F = \frac{1}{\gamma^2} PBB^T P^T + C^T C$. In order to see why the above expression is negative definite, note that if $x = (y^T, z^T)^T \in \mathbb{R}^{2n}$, we have

$$
x^T \begin{pmatrix} -F & F \\ F & -F \end{pmatrix} x = -(y - z)^T F (y - z) \leq 0
$$

since clearly $F \succeq 0$. Thus, negative definiteness follows as $R \prec 0$ by assumption. A congruence transformation now gives

$$
L^T \begin{pmatrix} P^T & 0 \\ 0 & P^{-T} \end{pmatrix} + \begin{pmatrix} P & 0 \\ 0 & P^{-1} \end{pmatrix} L \prec 0
$$

and since $P + P^T \succ 0$ by assumption, another congruence transformation gives $P^{-T}(P^T + P)P^{-1} = P^{-1} + P^{-T} \succ 0$ so that for any real eigenvalue λ of L, $\lambda < 0$ by Lemma 7.

In order to show that this implies stability, invoke the assumption of monotonicity through Remark 12 to see that A and A^T are both cross-positive on K_x, see the proof of Corollary 6. Further, $BK_u \subseteq K_x$ and $CK_x \subseteq K_z$ clearly imply $B^T K_{x_*} \subseteq K_{u_*}$ and $C^T K_{z_*} \subseteq K_{x_*}$, respectively. It follows, since K_u, K_x and K_z are self-dual by assumption, that $B^T K_x \subseteq K_u$ and $C^T K_z \subseteq K_x$. Thus, $BB^T \succeq_{K_x} 0$ and $C^T C \succeq_{K_x} 0$. Consequently, Lemma 9 gives that L is cross-positive on $K \times K$, and it follows from Lemma 8 that L has a real eigenvalue which upper bounds the real part of any other eigenvalue. But since all real eigenvalues of L must be negative according to the above, L must be Hurwitz.

(ii) ⇒ (iii): This follows immediately from Lemma 12.

(iii) ⇒ (i): Apply the following well-known similarity transformation to H:

$$\begin{pmatrix} I & 0 \\ -P & I \end{pmatrix} \begin{pmatrix} A & \frac{1}{\gamma^2} BB^T \\ -C^T C & -A^T \end{pmatrix} \begin{pmatrix} I & 0 \\ P & I \end{pmatrix} =$$
$$\begin{pmatrix} A + \frac{1}{\gamma^2} BB^T P & \frac{1}{\gamma^2} BB^T \\ 0 & -(A + \frac{1}{\gamma^2} BB^T P^T)^T \end{pmatrix}$$

Here, the fact that P solves (4) was exploited to obtain the zero block in the last expression. Now, this final matrix is block triangular, and so its eigenvalues coincide with those of the matrices on the diagonal, i.e., $\sigma(A + \frac{1}{\gamma^2} BB^T P) \subseteq \sigma(H)$. Since $A + \frac{1}{\gamma^2} BB^T P$ is Hurwitz by assumption and contains n eigenvalues, the remaining n eigenvalues must have positive real part due to the symmetric eigenvalue distribution of Hamiltonians about the imaginary axis [Zhou and Doyle, 1998, p. 233]. It follows that H can have no eigenvalues on the imaginary axis, and condition (i) thus follows from Lemma 13.

(ii) ⇔ (v): This follows directly from Lemma 11. □

***Proof.* Theorem 18**

Suppose on the contrary that $\|G\|_\infty > \|G(0)\|$, i.e., there exists a $\gamma_* > 0$ such that $\|G(0)\| < \gamma_* < \|G\|_\infty$. By Theorem 17, this means that $L(\gamma_*)$ cannot be Hurwitz, where

$$L(\gamma) = \begin{pmatrix} A & \frac{1}{\gamma^2} BB^T \\ C^T C & A^T \end{pmatrix}.$$

At the same time, by the continuity of eigenvalues, L must be Hurwitz for some sufficiently large γ, say γ_+, as A is Hurwitz by assumption. But since L is cross-positive on $K \times K$ (see the proof of Theorem 17), Lemma 8 implies that $L(\gamma)$ has a real eigenvalue $\lambda(\gamma)$ with maximal real part over its spectrum for all $\gamma_* \leq \gamma \leq \gamma_+$. Now, since $\lambda(\gamma_*) \geq 0$ and $\lambda(\gamma_+) < 0$, again by the continuity of eigenvalues there must exist some $\gamma_0 \geq \gamma_*$ such that $\lambda(\gamma_0) = 0$, i.e.,

$$H(\gamma_0) = \begin{pmatrix} A & \frac{1}{\gamma_0^2} BB^T \\ -C^T C & -A^T \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} L(\gamma_0)$$

has zero eigenvalue. But this is a contradiction by Lemma 14 as $\|CA^{-1}B\| = \|G(0)\| < \gamma_* \leq \gamma_0$. Thus, $\|G\|_\infty \leq \|G(0)\|$, and so

$$\|G(0)\| \leq \sup_\omega \|G(i\omega)\| = \|G\|_\infty \leq \|G(0)\|$$

and the conclusion follows. □

6. Conclusions

In this paper, we have explored cone-preserving (possibly nonsymmetric) solutions to Lyapunov and Riccati equations and inequalities for the purpose of stability and performance verification. Overall, these results apply to LTI systems with zero direct term that are monotone w.r.t. self-dual cones, and most of the results appear to be novel also in the special case of positive systems. Given the latter's success, the main purpose of the paper has been to complement it with a new angle as well as to pin down the structure it hinges on: the self-duality of the nonnegative orthant. By contrast, previous results indicate that the celebrated diagonal solution property and its many consequences are generated by the symmetric cones, a subset of the self-dual ones. One concrete benefit of this distinction is that we may now attribute properties such as the H_∞ norm being determined by the static gain to the former structure rather than the latter, instead of collapsing the two.

Although the main contribution of the present paper is arguably one of understanding, it also hints at potential usage in areas such as controller synthesis. In particular, synthesis based on diagonal solutions as in [Tanaka and Langbort, 2011] gives H_∞ optimality only for the restricted set of stabilizing closed-loop positive controllers. In a worst-case scenario, this optimal value may be very far from the optimal value over the entire set of stabilizing controllers. By contrast, in Subsection 4.1, we see how closed-loop positive controllers are featured quite naturally as optimal also within this wider framework.

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References

Angeli, D. and E. D. Sontag (2003). "Monotone control systems". *IEEE Trans. Autom. Control* **48**:10, pp. 1684–1698.

Berman, A. and R. J. Plemmons (1994). *Nonnegative Matrices in the Mathematical Sciences.* SIAM. ISBN: 9780120922505.

Boyd, S., L. El Ghaoui, E. Feron, and V. Balakrishnan (1994). *Linear Matrix Inequalities in System and Control Theory.* Vol. 15. SIAM. ISBN: 9780898713343.

Boyd, S. P. and L. Vandenberghe (2004). *Convex Optimization.* Cambridge University Press. ISBN: 9780521833783.

Briat, C. (2013). "Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: L_1-gain and L_∞-gain characterization". *Int. J. Robust Nonlinear Control* **23**:17, pp. 1932–1954.

Cantoni, M., E. Weyer, Y. Li, S. K. Ooi, I. Mareels, and M. Ryan (2007). "Control of large-scale irrigation networks". *Proc. IEEE* **95**:1, pp. 75–91.

Dalin, O., A. Ovseevich, and M. Margaliot (2024). "On special quadratic lyapunov functions for linear dynamical systems with an invariant cone". *IEEE Trans. Autom. Control* **69**:9, pp. 6435–6441.

Doyle, J., K. Glover, P. Khargonekar, and B. Francis (1988). "State-space solutions to standard H_2 and H_∞ control problems". In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 1691–1696.

Ebihara, Y., D. Peaucelle, and D. Arzelier (2014). "LMI approach to linear positive system analysis and synthesis". *Syst. & Control Lett.* **63**, pp. 50–56.

Farina, L. and S. Rinaldi (2000). *Positive Linear Systems: Theory and Applications.* Vol. 50. John Wiley & Sons. ISBN: 9780471384564.

Gahinet, P. and P. Apkarian (1994). "A linear matrix inequality approach to H_∞ control". *Int. J. Robust Nonlinear Control* **4**:4, pp. 421–448.

Horn, R. A. and C. R. Johnson (2012). *Matrix Analysis.* Cambridge University Press. ISBN: 9780521548236.

Lu, X., Y. Chen, B. Zhu, J. Shen, B. Du, Y. Chen, and J. Lam (2024). "Kyp lemma for cone-preserving systems and its applications to controller design". *IEEE Trans. Autom. Control* **69**:12, pp. 8812–8819.

Rantzer, A. (2015a). "On the kalman-yakubovich-popov lemma for positive systems". *IEEE Trans. Autom. Control* **61**:5, pp. 1346–1349.

Rantzer, A. (2015b). "Scalable control of positive systems". *Eur. J. Control* **24**, pp. 72–80.

Rantzer, A. and M. E. Valcher (2018). "A tutorial on positive systems and large scale control". In: *Proc. IEEE Conf. Decis. Control (CDC)*, pp. 3686–3697.

Rugh, W. J. (1996). *Linear System Theory*. Upper Saddle River, NJ, USA: Prentice-Hall, Inc. ISBN: 9780134412054.

Schneider, H. and B.-S. Tam (2006). "Matrices leaving a cone invariant". *Handb. Linear Algebra, ed. L. Hogben, Chapman & Hall.*

Schneider, H. and M. Vidyasagar (1970). "Cross-positive matrices". *SIAM J. Numer. Anal.* **7**:4, pp. 508–519.

Shen, J. and J. Lam (2016). "Some extensions on the bounded real lemma for positive systems". *IEEE Trans. Autom. Control* **62**:6, pp. 3034–3038.

Shen, J. and J. Lam (2017). "Input–output gain analysis for linear systems on cones". *Automatica* **77**, pp. 44–50.

Tanaka, T. (2012). *Symmetric formulation of the Kalman-Yakubovich-Popov lemma and its application to distributed control of positive systems*. PhD thesis. University of Illinois at Urbana-Champaign.

Tanaka, T. and C. Langbort (2011). "The bounded real lemma for internally positive systems and H-infinity structured static state feedback". *IEEE Trans. Autom. Control* **56**:9, pp. 2218–2223.

Tanaka, T., C. Langbort, and V. Ugrinovskii (2013). "DC-dominant property of cone-preserving transfer functions". *Syst. & Control Lett.* **62**:8, pp. 699–707.

Vladu, E. and A. Rantzer (2022). "On decentralized H-infinity optimal positive systems". *IEEE Control Syst. Lett.* **7**, pp. 391–394.

Vladu, E. and A. Rantzer (2025). "A cone-preserving solution to a nonsymmetric Riccati equation". *Linear Algebra Appl.* **709**, pp. 449–459. DOI: 10.1016/j.laa.2025.01.020.

Zhou, K. and J. C. Doyle (1998). *Essentials of Robust Control*. Vol. 104. Upper Saddle River, NJ, USA: Prentice Hall. ISBN: 9780135258330.

Paper V

On Integral Linear Constraints on Convex Cones

Emil Vladu **Alexandre Megretski** **Anders Rantzer**

Abstract

In this paper, we consider integral linear constraints and the dissipation inequality with linear supply rates for certain sets of trajectories confined pointwise in time to a convex cone which belongs to a finite-dimensional normed vector space. Such constraints are then shown to be satisfied if and only if a bounded linear functional exists which satisfies a conic inequality. This is analogous to the typical situation in which a quadratic integrand over the entire space is related to a linear matrix inequality. A connection is subsequently drawn precisely to linear-quadratic control: by proper choice of cone, the main results can be applied to produce a known L_1-gain analogue to the bounded real lemma in positive systems theory, as well as a non-strict version of the Kalman-Yakubovich-Popov Lemma in linear-quadratic control.

1. Introduction

In systems theory and control, various constraints on the dynamics of a linear time-invariant (LTI) system can be verified by solving certain algebraic matrix equations or inequalities. Important examples include verifying stability by means of Lyapunov equations, establishing an optimal quadratic cost over all input signals (LQR) [Kalman et al., 1960] and verifying L_2-gain bounds by solving Riccati equations [Doyle et al., 1988] or linear matrix inequalities (LMIs) [Gahinet and Apkarian, 1994]. Importantly, symmetry is a recurring feature of these equations and their associated solutions, and the costs featured in the problem formulations are generally quadratic, thus giving rise to the dominating linear-quadratic paradigm in systems theory and control.

By contrast, more recently an area called positive systems theory has gained popularity. Positive systems, which are characterized by nonnegative inputs producing nonnegative outputs, occur naturally in such areas as biology, economy or Markov models, and they possess remarkable properties useful especially for analysis and control of large-scale systems, e.g., [Farina and Rinaldi, 2000][Rantzer and Valcher, 2018] and the references therein. Examples of such properties include the existence of positive diagonal Lyapunov solutions as both necessary and sufficient for stability, the existence of positive vectors solving entrywise inequalities to verify upper bounds on the L_1/L_∞-gain [Briat, 2013][Ebihara et al., 2011] and positive diagonal solutions [Tanaka and Langbort, 2011] or nonsymmetric solutions to LMIs [Ebihara et al., 2014] to certify a given upper bound on the H_∞ norm. Note in particular the recurring absence of symmetry, as well as the occassional occurrence of linear cost functions rather than quadratic ones.

In general, the relationship between the many kinds of equations and inequalities certifying stability and performance for positive systems, and those doing the same for general systems, is complex. However, arguably the most straightforward connection can be seen by considering the following equivalent condition for stability of a Metzler matrix $A \in \mathbb{R}^{n\times n}$

$$\exists p > 0 \text{ such that } Ap < 0$$

as well as the one for general matrices

$$\exists P \succ 0 \text{ such that } A^T P + PA \prec 0,$$

where $>$ and $\succ$ are induced by the nonnegative orthant and positive semidefinite cone, respectively. The common denominator here is an operator L on either $\mathbb{R}^n$ or $\mathcal{S}^n$ such that $\frac{d}{dt}x = L(x)$ leaves the above cones invariant, and under this assumption, asymptotic stability is equivalent to the existence of a positive vector solving a conic inequality. This idea is well-known, e.g., [Gowda and Tao, 2009] for a formalization in a finite-dimensional Hilbert space framework. The assumption is sometimes known as cross-positivity in the literature and has been studied thoroughly over the years, e.g., [Schneider and Vidyasagar, 1970].

The above way of identifying linear-conic structure in standard Lyapunov stability analysis has very recently been exploited also in other areas of control to provide a unifying framework for various results. For example, [Bamieh, 2024] leverages a result on linear-cone duality on general Banach spaces in order to connect differential Riccati equation solutions to various finite-horizon linear-quadratic phenomena such as LQR. Another example is given by [Pates and Rantzer, 2024] which instead passes through the Bellman equation to draw parallels between a recent positive systems result [Rantzer, 2022] and LQR. Matrix dynamical systems associated normally with covariance matrices are central to both works, and have a long history in control, e.g., [Gattami, 2009] where they were used for stochastic control. Note finally the related yet distinct body of literature which also exploits cone theory in the context of control: generalizations of positive systems theory which establish precisely how and for which type of cone its appealing properties manifest in a wider context, e.g., [Angeli and Sontag, 2003] [Papusha and Murray, 2015][Shen and Lam, 2016][Tanaka et al., 2013][Shen and Lam, 2017].

The main purpose of the present paper is to identify a similar linear-conic structure in some already published results on both positive systems and linear-quadratic theory. More precisely, the satisfaction of various integral linear constraints and the dissipation inequality with linear supply rate for trajectories confined to a cone is shown to be equivalent to the existence of a bounded linear functional satisfying a conic inequality. The latter is perhaps most fruitfully compared to [Willems, 1971], in which a similar connection is made between LMIs and linear systems satisfying the dissipation inequality with quadratic supply rate. Further, there is a great literature on integral quadratic constraints, e.g., [Megretski and Rantzer, 1997] and the references therein, and not unexpectedly integral linear constraints appear also in positive systems theory, e.g., [Briat, 2013]. There is a solid theoretical framework for dissipative systems [Willems, 1972a], and in particular when the system is linear and the supply rate is quadratic [Willems, 1972b]. Similarly, there exists a parallel dissipation theory subsequently developed for positive systems [Haddad and Chellaboina, 2005] in which the supply rate is linear. However, the present paper considers the dissipation inequality only to see what can be said without any nonnegativity requirement on the storage function as in standard dissipativity.

The main results of the present paper are subsequently exploited to derive first a non-strict variant of a known result in positive systems theory on L_1-gains [Briat, 2013][Ebihara et al., 2011] (Proposition 6), and second a non-strict version of the Kalman-Yakubovich-Popov (KYP) Lemma [Kalman, 1963][Yakubovich, 1962][Popov, 1961] (Proposition 7). Various versions, generalizations and proofs to the latter have been presented over the years, some of which are algebraic in their nature and others dynamical, e.g., [Rantzer, 1996][Megretski, 2010]. In this paper, the aim is not swift theorem verification but rather an attempt at shedding additional light on an important result in the control literature, this time from a linear-cone perspective. To this end, a corresponding linear-cone analog to the non-strict KYP Lemma (Theorem 22) is presented, in which the desired connection between dynamical constraints and conic inequalities is observed in a more basic setting. In connection to this, controllability on a cone, or K-controllability in short, is defined. Note that such notions already exist in the positive systems literature, where their relationship to standard controllability has been studied extensively, e.g., [Coxson and Shapiro, 1987][Ohta et al., 1984][Valcher, 1996][Valcher, 2009] and the references therein. The KYP Lemma now readily follows from Theorem 22 after an application of a crucial rank one decomposition (Theorem 23). This latter result is novel to the best of the authors' knowledge and breaks down trajectories on the positive semidefinite cone which satisfy the matrix dynamical system in [Bamieh, 2024] to components satisfying standard LTI system dynamics, thereby bridging the gap between the linear-cone and the linear-quadratic domain.

The outline of the paper is as follows: in Section 2, we define and recall relevant mathematical notions. Section 3 provides the results of the paper and Section 4 the associated proofs. Section 5 subsequently concludes the paper.

2. Preliminaries

In this section, we explain the notation used throughout the paper and recall basic functional analytical concepts.

We denote by $\mathbb{R}$ ($\mathbb{C}$) the set of real (complex) numbers, and by $\mathbb{R}^n$ ($\mathbb{C}^n$) and $\mathbb{R}^{n\times m}$ ($\mathbb{C}^{n\times m}$) the set of n-dimensional vectors and $n \times m$-matrices, respectively, with entries in $\mathbb{R}$ ($\mathbb{C}$). The identity matrix will be denoted by I, with context determining its dimension. A square matrix A is said to be Hurwitz if all its eigenvalues have negative real part, and Metzler if all its offdiagonal elements are nonnegative. We denote by $\mathcal{S}^n$ the set of symmetric matrices in $\mathbb{R}^{n\times n}$ and by $\mathcal{S}^n_+$ the positive semidefinite cone therein which induces the partial order $\succeq$. Similarly, $\mathbb{R}^n_+$ is the set of entrywise nonnegative vectors which induces the entrywise partial order $\geq$. For Metzler matrices A, it is well known that being Hurwitz is equivalent to $-A^{-1} \geq 0$ as well as the existence of a $p > 0$ such that $Ap < 0$, e.g., [Rantzer and Valcher, 2018].

In this paper, we shall refer to a vector space X over $\mathbb{R}$ equipped with a norm $\|\cdot\|_X$ as a normed space. Convergence, limits and continuity are defined in the usual $\varepsilon - \delta$ sense as in real analysis. For a linear transformation $L : X \to Y$ with normed spaces X, Y, we say that L is bounded if there exists an $M > 0$ such that $\|L(x)\|_Y \leq M\|x\|_X$ for all $x \in X$. We denote by X^* the dual of X, i.e., the set of all bounded linear functionals ($x^* : X \to \mathbb{R}$), and by $B(X, Y)$ the set of all bounded linear transformations $L : X \to Y$. The adjoint L^* corresponding to L is defined as the transformation $L : Y^* \to X^*$ such that $L^*(y^*)(x) = y^*(L(x))$ for all $x \in X$. When the norm is induced by an inner product, it is well known by the Riesz representation theorem that X^* can be identified with X. For an excellent introduction to the theory of normed spaces, see e.g., [Luenberger, 1997].

A cone $K \subseteq X$ is a set for which $x \in K$ implies $\alpha x \in K$ for all $\alpha \geq 0$. A convex (and pointed, i.e., $K \cap -K = \{0\}$) cone induces a preorder (partial order) $\succeq_K$ such that $x \succeq_K y$ if and only if $x - y \in K$; $x \succ_K y$ if and only if $x - y \in \text{Int}(K)$, where Int denotes the interior. The associated dual cone is defined as $K^* = \{x^* \in X^* \mid x^*(x) \geq 0 \ \forall x \in K\}$, and in finite dimensions for closed K the interior of the dual is given by $\text{Int}(K^*) = \{x^* \in X^* \mid x^*(x) > 0 \ \forall x \in K, x \neq 0\}$, noting that $\text{Int}(K^*)$ is nonempty if in addition K is also pointed, e.g., [Boyd and Vandenberghe, 2004, p. 64]. Examples of closed, convex, pointed cones with nonempty interior – so-called proper cones – are the nonnegative orthant in $\mathbb{R}^n$ and the positive semidefinite cone in $\mathcal{S}^n$. For more on finite-dimensional cones, see e.g., [Berman and Plemmons, 1994][Barker, 1981].

A normed space X for which every Cauchy sequence converges is called a Banach space; this is assumed from this point on. Finite-dimensional normed spaces are Banach spaces. Given an open interval $I \subseteq \mathbb{R}$, we say that $f : I \to X$ is differentiable at $t_0 \in I$ if the limit $\lim_{h \to 0} \frac{f(t_0+h)-f(t_0)}{h}$ exists, and we denote it variously by $\dot{f}(t_0)$ or $\frac{d}{dt}f(t_0)$; if the limit exists for all $t_0 \in I$ we say that f is differentiable. In the interest of simplicity, we interpret $\int_I f(t)\ \text{dt}$ in the Riemann sense, for which integration on closed intervals I generalizes in a natural way to Banach-valued functions. Improper integrals are carried out in the principal value sense, i.e., $\int_{-\infty}^{\infty} f(t)\ \text{dt} = \lim_{T \to \infty} \int_{-T}^{T} f(t)\ \text{dt}$, provided the limit exists. We note that standard intuition applies well, as most of the basic results from real analysis persist in this setting, e.g., [Gordon, 1991]. Two such examples of which we shall make use include the fundamental theorem of calculus and interchanging the order of integration with a linear operator. Finally, given $E, L \in B(X, Y)$, we say that a piecewise continuous f satisfies the differential equation $\frac{d}{dt}E(f) = L(f)$ on some interval I if $E(f(t)) - E(f(t_0)) = \int_{t_0}^{t} L(f(\tau))\ \text{d}\tau$ for all $t, t_0 \in I$ or, equivalently, if in addition to being continuous, $E(f)$ is differentiable and the differential equation holds on those open intervals on which f is continuous. Unless otherwise stated, we assume in the remainder of the paper that all functions of time are locally bounded piecewise smooth, meaning simply that for any closed interval $[a,b] \subseteq I$, $[a,b]$ can be partitioned into a finite number of intervals on the interior of which f is continuous, bounded and has bounded derivatives of all orders. We denote this set by $C_p^\infty(I)$.

We close this section by recalling the following fact.

LEMMA 15
Given a matrix $Q \in \mathcal{S}^{n+m}$ such that $Q \succeq 0$ with corresponding partition

$$Q = \begin{pmatrix} Q_{nn} & Q_{nm} \\ Q_{nm}^T & Q_{mm} \end{pmatrix},$$

then $\mathrm{Im}(Q_{nm}) \subseteq \mathrm{Im}(Q_{nn})$.

Proof. Suppose on the contrary that this were not the case. Then there exists a $z \in \mathbb{R}^m$ such that $Q_{nm}z \notin \mathrm{Im}(Q_{nn}) = \left(\mathrm{Im}(Q_{nn})^\perp\right)^\perp$. As such, there is $w \in \mathbb{R}^n$ such that $w^T Q_{nn} v = 0$ for all $v \in \mathbb{R}^n$, yet $w^T Q_{nm} z \neq 0$. But this would imply that

$$\begin{aligned} \begin{pmatrix} w \\ z \end{pmatrix}^T Q \begin{pmatrix} w \\ z \end{pmatrix} &= w^T Q_{nn} w + 2w^T Q_{nm} z + z^T Q_{mm} z \\ &= 2w^T Q_{nm} z + z^T Q_{mm} z < 0 \end{aligned}$$

by choosing the sign of w properly and making it sufficiently large, a contradiction as $Q \succeq 0$. □

3. Results

In this section, we present some general results on integral linear constraints on cones. At the heart lies the following finite-dimensional phenomenon for which a non-strict and a strict version are provided.

THEOREM 19
Let the finite-dimensional normed spaces Z, X and the convex cone $K \subseteq Z$ be given. Suppose now that $L(K) = X$. Then for any given $m^* \in Z^*$ and $L \in B(Z,X)$, the following conditions are equivalent:

(i) There exists $p^* \in X^*$ such that

$$L^*(p^*) - m^* \preceq_{K^*} 0. \tag{1}$$

(ii) $m^*(z_0) \geq 0$ for every $z_0 \in K$ such that $L(z_0) = 0$.

THEOREM 20
Let the finite-dimensional normed spaces Z, X and the closed, convex and pointed cone $K \subseteq Z$ be given. For any given $m^* \in Z^*$ and $L \in B(Z,X)$, the following conditions are equivalent:

(i) *There exists* $p^* \in X^*$ *such that*

$$L^*(p^*) - m^* \prec_{K^*} 0.$$

(ii) $m(z_0) > 0$ *for every nonzero* $z_0 \in K$ *such that* $L(z_0) = 0$.

Proof. See Section 4. □

Although the nature of Theorem 19 and Theorem 20 is finite-dimensional, it does have bearing on functions of time. In order to see this, let L, X, K be given as above, define the three sets

$$\begin{aligned} H_l &= \cup_{[t_0,t_1]}\{z{:}[t_0,t_1]\to K \mid z(t)=z_0 \text{ for some } z_0\in K \text{ such that } L(z_0)=0\} \\ H_u &= \cup_{[t_0,t_1]}\{z{:}[t_0,t_1]\to K \mid \exists\ x{:}[t_0,t_1]\to X \text{ such that } \dot{x}=L(z)\} \\ H_v &= \cup_{[t_0,t_1]}\{z{:}[t_0,t_1]\to K \mid \exists\ x{:}[t_0,t_1]\to X \text{ such that } \dot{x}=L(z) \text{ and } x(t_0)=x(t_1)\}\end{aligned}$$

and denote by H_z any subset of all x such that $\dot{x} = L(z)$ given $z \in H_u$. The following then holds.

THEOREM 21
Let the finite-dimensional normed spaces Z, X *and the convex cone* $K \subseteq Z$ *be given. Suppose now that* $L(K) = X$. *Then for any set* H *constrained as* $H_l \subseteq H \subseteq H_u$ *with an associated family* H_z, *and any* $m^* \in Z^*$ *and* $L \in B(Z, X)$, *the following conditions are equivalent:*

(i) *There exists* $p^* \in X^*$ *such that*

$$L^*(p^*) - m^* \preceq_{K^*} 0.$$

(ii) *There exists a continuous* $V : X \to \mathbb{R}$ *with* $V(0) = 0$ *such that for all* $z \in H$ *and* $x \in H_z$,

$$V(x(t_0)) + \int_{t_0}^{t_1} m^*(z(t))\, \mathrm{dt} \geq V(x(t_1)).$$

If in addition $H_l \subseteq H \subseteq H_v$, *then the following condition is also equivalent:*

(iii) *For all* $z \in H$,

$$\int_{t_0}^{t_1} m^*(z(t))\, \mathrm{dt} \geq 0.$$

Proof. See Section 4. □

Thus far, there has been no mention of dynamics. This changes now as we apply Theorem 21 to sets H constrained by differential equations. First, however, we make the following systems theoretical definition in keeping with [Willems, 1971].

Definition 2
Let the finite-dimensional normed spaces Z, X and the convex cone $K \subseteq Z$ be given. We say that the pair $E, L \in B(Z, X)$ satisfies the dissipation inequality on K w.r.t. $w : K \to \mathbb{R}$ if there exists a continuous function $V : X \to \mathbb{R}$ with $V(0) = 0$ such that

$$V(E(z(t_0))) + \int_{t_0}^{t_1} w(z(t))\,\mathrm{dt} \geq V(E(z(t_1))) \tag{2}$$

for all $t_1 \geq t_0$ and all $z \in C_p^\infty[t_0, t_1]$ such that $z(t) \in K$ and $\frac{d}{dt}E(z) = L(z)$ whenever $t \in [t_0, t_1]$.

A consequence of Theorem 21 is the following.

Corollary 7
Let the finite-dimensional normed spaces Z, X and the convex cone $K \subseteq Z$ be given. Suppose now that $L(K) = X$. Then for any $m^* \in Z^*$ and $E, L \in B(Z, X)$, the following conditions are equivalent:

(i) There exists $p^* \in X^*$ such that

$$L^*(p^*) - m^* \preceq_{K^*} 0.$$

(ii) (E, L) satisfies the dissipation inequality on K w.r.t. m^*.

In addition, if the dissipation inequality holds for some function V as in Definition 2, then it holds also for some function in X^*.

Proof. See Section 4. □

Finally, we consider what can be said for sets of trajectories converging to the origin. For this purpose, we define the following cone analog to the standard concept of controllability in keeping with e.g., [Valcher, 1996].

Definition 3
Let the finite-dimensional normed spaces Z, X and the convex cone $K \subseteq Z$ be given. $E, L \in B(Z, X)$ is then said to be controllable on K if for every $x_0, x_1 \in E(K)$ there is $t_1 \geq 0$ and a continuous $z \in C_p^\infty[0, t_1]$ such that $z(t) \in K$, $\frac{d}{dt}E(z) = L(z)$ and $E(z(0)) = x_0, E(z(t_1)) = x_1$.

The following can be said to constitute a cone analog to the KYP Lemma, cf. Proposition 7.

Theorem 22
Let the finite-dimensional normed spaces Z, X and the convex cone $K \subseteq Z$ be given. Suppose now that the pair $E, L \in B(Z, X)$ is controllable on K and that $E(K)$ has nonempty interior. Then for any given $m^* \in Z^*$, the following conditions are equivalent:

(i) There exists $p^* \in X^*$ *such that*

$$L^*(p^*) - m^* \preceq_{K^*} 0.$$

(ii) For all $z \in C_p^\infty(\mathbb{R})$ *such that* $z(t) \in K$, $\frac{d}{dt}E(z) = L(z)$ *and* $\int_{-\infty}^{\infty} \|z(t)\|_Z \mathrm{dt} < \infty$,

$$\int_{-\infty}^{\infty} m^*(z(t))\,\mathrm{dt} \geq 0.$$

(iii) $m^*(z_0) \geq 0$ *for every* $z_0 \in K$ *such that* $L(z_0) = 0$.

Proof. See Section 4. □

We next give some remarks on the above results.

Remark 14
The heart of both Theorem 19 and Theorem 20 is a separating hyperplane argument. Although such statements exist also for infinite-dimensional spaces (Hahn-Banach), one of the sets involved is then required to possess nonempty interior in the case of non-strict inequality, something that does not generally hold in this setting. Similarly, compactness becomes too severe an assumption in the strict case.

Remark 15
The above results are fundamentally about connecting the existence of elements in the dual that satisfy a conic inequality to integral linear constraints on sets of trajectories confined to a cone. Surprisingly, however, dynamics and differential constraints appear not to play an essential role in this phenomenon: if (E, L) *satisfies the dissipation inequality, then a solution to (1) proves that (2) holds for many other sets* H *of trajectories due to Theorem 21, so long as* $H_l \subseteq H \subseteq H_u$. *Examples include the set of all* $z : [t_0, t_1] \to K$ *such that there exists an* $\hat{E} \in B(Z, X)$ *such that* $\frac{d}{dt}\hat{E}(z) = L(z)$, *or indeed the dynamically disconnected set* H_u *itself.*

Remark 16
When $Z = \mathbb{R}^{n+m}$, $X = \mathbb{R}^n$, $L(x, u) = Ax + Bu$ *and* $E(x, u) = x$, *Corollary 7 collapses to a statement about standard LTI systems* $\dot{x} = Ax + Bu$ *relevant to systems theory and control. The statement can then be compared to [Willems, 1971], which connects quadratic supply rates on all of* $\mathbb{R}^{n+m}$, *as opposed to linear ones on cones, to an LMI, corresponding to the conic inequality (1). Similarly, Theorem 21 is comparable to part of [Willems, 1971, Theorem 2] but additionally includes the algebraic condition (i).*

Remark 17
The above results concerning trajectories all involve non-strict inequalities and rely on Theorem 19. However, analogous results with strict inequality can be obtained in a similar fashion from Theorem 20 instead. The main benefit in this case is that the assumption $L(K) = X$ vanishes, but in exchange additional assumptions on the cone are incurred.

3.1 Applications

Next, we proceed to apply the above results in order to obtain and thereby connect two seemingly unrelated results in the control literature. In the first result, we choose K as the nonnegative orthant in $\mathbb{R}^{n+m}$ and regain a non-strict variant of a known result for positive systems on L_1-gains reminiscent of the bounded real lemma [Briat, 2013][Ebihara et al., 2011].

Proposition 6
Let $\gamma > 0$ be given and consider the system $\dot{x} = Ax + Bu$ with A Metzler and $B \in \mathbb{R}_+^{n\times m}$ such that $Bu > 0$ for some $u \geq 0$. The following conditions are equivalent:

(i) A is Hurwitz and

$$\sup_{\substack{u\in L_1^m[0,\infty), u\geq 0\\ u\neq 0, x(0)=0}} \frac{\|x\|_1}{\|u\|_1} \leq \gamma$$

(ii) There exists $p > 0$ such that

$$\begin{aligned} p^T A + \mathbf{1}_n^T &\leq 0 \\ p^T B - \gamma \mathbf{1}_m^T &\leq 0 \end{aligned}$$

Proof. See Section 4. □

In the second result, we choose K as the positive semidefinite cone in $\mathbb{S}^{n+m}$ and recover a non-strict version of the Kalman-Yakubovich-Popov (KYP) Lemma. For this purpose, we shall require the following crucial decomposition of trajectories on $\mathcal{S}_+^{n+m}$.

Theorem 23
For every $Q : \mathbb{R} \to \mathcal{S}_+^{n+m}$ in $C_p^\infty(\mathbb{R})$, the following conditions are equivalent:

(i) Q satisfies

$$\frac{\mathrm{d}}{\mathrm{dt}} \begin{pmatrix} I & 0 \end{pmatrix} Q \begin{pmatrix} I \\ 0 \end{pmatrix} = \begin{pmatrix} A & B \end{pmatrix} Q \begin{pmatrix} I \\ 0 \end{pmatrix} + \begin{pmatrix} I & 0 \end{pmatrix} Q \begin{pmatrix} A^T \\ B^T \end{pmatrix}. \tag{3}$$

(ii) There exist $n+m$ functions $x_i : \mathbb{R} \to \mathbb{R}^n$ and $u_i : \mathbb{R} \to \mathbb{R}^m$ in $C_p^\infty(\mathbb{R})$ such that

$$Q = \sum_{i=1}^{n+m} \begin{pmatrix} x_i \\ u_i \end{pmatrix} (x_i^T \; u_i^T)$$

and either $x_i = 0$ on $\mathbb{R}$ or $\dot{x}_i = Ax_i + Bu_i$ on $\mathbb{R}$ for all i.

Proof. See Section 4. □

As a first application of Theorem 23, we connect controllability to K-controllability on $\mathcal{S}_+^{n+m}$.

COROLLARY 8
(A, B) is controllable if and only if (E, L) defined by (3) is controllable on $\mathcal{S}_+^{n+m}$.

Proof. See Section 4. □

Theorem 22, Theorem 23 and Corollary 8 now together give the following result [Willems, 1971][Rantzer, 1996][Megretski, 2010].

PROPOSITION 7—KYP
Given $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $M = M^T \in \mathbb{R}^{(n+m)\times(n+m)}$ with (A, B) controllable, the following statements are equivalent:

(i) There exists a matrix $P \in \mathbb{R}^{n\times n}$ such that $P = P^T$ and

$$M + \begin{pmatrix} A^TP + PA & PB \\ B^TP & 0 \end{pmatrix} \preceq 0.$$

(ii) For all $x \in \mathbb{C}^n$ and $u \in \mathbb{C}^m$ such that either $x = 0$ or $i\omega x = Ax + Bu$ for some $\omega \in \mathbb{R}$,

$$\begin{pmatrix} x \\ u \end{pmatrix}^T M \begin{pmatrix} x \\ u \end{pmatrix} \le 0.$$

(iii) For all $x \in L_2^n(-\infty, \infty)$ and $u \in L_2^m(-\infty, \infty)$ such that either $x = 0$ on $\mathbb{R}$ or $\dot{x} = Ax + Bu$ on $\mathbb{R}$,

$$\int_{-\infty}^{\infty} \begin{pmatrix} x \\ u \end{pmatrix}^T M \begin{pmatrix} x \\ u \end{pmatrix} \, \mathrm{dt} \le 0.$$

(iv) For all $\omega \in \mathbb{R}$ such that $i\omega$ is not an eigenvalue of A,

$$\begin{pmatrix} (i\omega I - A)^{-1}B \\ I \end{pmatrix}^* M \begin{pmatrix} (i\omega I - A)^{-1}B \\ I \end{pmatrix} \preceq 0.$$

Proof. See the end of this section. □

Before we close the section with a proof to Proposition 7, we provide the following remarks.

REMARK 18
Traditionally, the KYP Lemma is regarded as a bridge between state space formalism and the frequency domain, and is usually phrased in terms of an equivalence between the LMI in condition (i) and the set of frequency inequalities in condition (iv). From the perspective of the linear-cone theory advanced in Section 3, however, the frequency inequality is perhaps best viewed as part of a transition from the vector quadratic constraint in condition (ii) to the integral quadratic constraint in condition (iii), which incidentally manifests itself through the frequency domain and Parseval's theorem, an L_2-specific phenomenon. The cone analog to Proposition 7 is given by Theorem 22, and the above two conditions then correspond to the equilibrium point condition (iii) and the integral linear constraint condition (ii), respectively.

REMARK 19
The nontrivial part of the KYP Lemma in Proposition 7 occurs in the unexpected weakening of the nonpositivity of a linear functional $m^*(Q) = \mathrm{tr}(MQ)$ on the positive semidefinite cone to the not only necessary but also sufficient nonpositivity thereof on part of its boundary. Since this latter part is the set of positive semidefinite rank one matrices, the quadratic functional form follows. Moreover, this central step corresponding to the direction (iii) $\Rightarrow$ (i) is enabled precisely by Theorem 23. Note that this step is frequency independent.

REMARK 20
As a complement to Remark 19, we observe also a second dimension further enforcing it: there appears to exist a correspondence between complex vectors and real vector-valued functions of time. This can be observed already in the parallel conditions (ii) and (iii). In fact, the supplementary results in the well-known proof for the KYP Lemma in [Rantzer, 1996] can be used to obtain a complex vector analog to the crucial Theorem 23, in which a real positive semidefinite matrix satisfying an equilibrium point condition, as opposed to a matrix-valued function of time satisfying the corresponding dynamics, is decomposed into a sum of rank one matrices, the components of which are complex vectors satisfying the constraint in condition (ii) in Proposition 7. This subsequently offers a parallel algebraic proof of the KYP Lemma in which the nontrivial weakening of the nonpositivity of a linear functional on $\mathcal{S}_+^{n+m}$ to its boundary as in Remark 19 occurs instead over the field of complex numbers. This corresponds to the direction (ii) $\Rightarrow$ (i) and is not seen in the proof given by [Rantzer, 1996], as the same fundamental components are executed in a different order.

Remark 21
Theorem 23 says that trajectories confined to $\mathcal{S}_+^{n+m}$ that satisfy system (3) can be decomposed into sums of rank one matrix trajectories, the corresponding vectors of which satisfy $x = 0$ or $\dot{x} = Ax + Bu$. Thus, in some sense, nothing new happens in the interior of the cone for the extended system (3), and dynamics on $\mathcal{S}_+^{n+m}$ can essentially be expressed in terms of the original system. A consequence is also that trajectories satisfying the original dynamics correspond to rank one matrix trajectories on the boundary of $\mathcal{S}_+^{n+m}$.

Remark 22
Theorem 23 appears to constitute a bridge between positive systems and cone intuition on the one hand, and linear-quadratic intuition on the other. For instance, it is the phenomenon that transfers linear functionals onto quadratic ones, as in the KYP Lemma. Another example is given by the transfer of standard controllability onto K-controllability as in Corollary 8.

Proof. **Proposition 7**

(i) ⇒ (ii): Multiply the matrix in the LMI by $(x, u) \in \mathbb{C}^{n+m}$ from the right and $(x, u)^*$ from the left and note that the expression in condition (ii) follows immediately if $x = 0$, and otherwise also since

$$\begin{pmatrix} x \\ u \end{pmatrix}^* \begin{pmatrix} A^TP + PA & PB \\ B^TP & 0 \end{pmatrix} \begin{pmatrix} x \\ u \end{pmatrix} = (Ax + Bu)^*Px + x^*P(Ax + Bu)$$
$$= -i\omega x^*Px + i\omega x^*Px = 0.$$

(ii) ⇒ (iv): Take any such $\omega \in \mathbb{C}$ and multiply the matrix in condition (iv) by $u \in \mathbb{C}^m$ from the right and u^* from the left and note that the quadratic form in condition (ii) is obtained by setting $x := (i\omega I - A)^{-1}Bu$. If $x = 0$, condition (ii) can be invoked directly and if $x \neq 0$, note that the latter implies $i\omega x = Ax + Bu$ so that condition (iv) follows from (ii).

(iv) ⇒ (iii): Take any $x \in L_2^n(-\infty, \infty)$ and $u \in L_2^m(-\infty, \infty)$ such that $\dot{x} = Ax + Bu$ and note that as a consequence, the Fourier transforms satisfy $\hat{x} \in L_2^n(i\mathbb{R})$ and $\hat{u} \in L_2^m(i\mathbb{R})$. Thus, Parseval's theorem can be applied to the integral-quadratic form in condition (iii) to obtain a corresponding expression with $\hat{x}$ and $\hat{u}$ (offset by scaled identity if $M \succeq 0$ fails to hold). Now, since $\dot{x} = Ax + Bu$ implies $\hat{x}(i\omega) = (i\omega I - A)^{-1}B\hat{u}(i\omega)$ a.e. (and in particular not at those finite number of ω which may correspond to imaginary axis eigenvalues of A), by condition (iv) the integrand will be nonpositive a.e. and hence also the integral. The case $x = 0$ follows immediately by noting that the lower-right $m \times m$ block of M is negative semidefinite (let $\omega \to \infty$ in condition (iv)).

(iii) ⇒ (i): Take any $Q(t) \succeq 0$ that satisfies (3) with entries in $L_1(-\infty, \infty)$ and invoke Theorem 23 to obtain a rank one decomposition of Q such that the vector corresponding to each term satisfies either $\dot{x}_i = Ax_i + Bu_i$ or

$x_i = 0$. Note now that since all entries in Q are in $L_1(-\infty,\infty)$ and the diagonal entries are sums of squares of entries of x_i and u_i, the integral of each such term must be convergent and it follows that $x_i \in L_2^n(-\infty,\infty)$ and $u_i \in L_2^m(-\infty,\infty)$. Consequently, condition (iii) gives

$$\begin{aligned}\int_{-\infty}^{\infty} \operatorname{tr}(MQ(t))\,\mathrm{dt} &= \sum_{i=1}^{n+m} \int_{-\infty}^{\infty} \operatorname{tr}\left(M \begin{pmatrix} x_i(t) \\ u_i(t) \end{pmatrix} (x_i(t)^T u_i(t)^T)\right) \mathrm{dt} \\ &= \sum_{i=1}^{n+m} \int_{-\infty}^{\infty} \begin{pmatrix} x_i(t) \\ u_i(t) \end{pmatrix}^T M \begin{pmatrix} x_i(t) \\ u_i(t) \end{pmatrix} \mathrm{dt} \leq 0\end{aligned}$$

for all such Q. With $Z = \mathcal{S}^{m+n}$, $X = \mathcal{S}^n$, $K = \mathcal{S}_+^{n+m}$, (E, L) as in (3) and $m^*(Q) = \operatorname{tr}(-MQ)$, this means exactly that condition (ii) in Theorem 22 is satisfied. To be clear, note that we equip X with the norm induced by the standard trace inner product and Z with the L_1-norm, i.e., $\|\cdot\|_Z$ sums the absolute valued matrix entries together. Note also that any functional $f^*(Q) = \operatorname{tr}(CQ)$ for some $C \in S^{n+m}$ is clearly linear and bounded (with $\sup|f^*(Q)|$ over $\|Q\|_Z = 1$ equal to the greatest entry in C in absolute value) so that $m^* \in Z^*$. Invoke now Corollary 8 to obtain K-controllability and note that $E(K) = \mathcal{S}_+^n$ so that $E(K)$ has nonempty interior. Theorem 22 may thus be applied to obtain a $p^* \in X^*$, therefore on the form $p^*(Q) = \operatorname{tr}(PQ)$ for some $P \in \mathcal{S}^n$, such that the conic inequality (1) holds. But with $U = (A\ B)$ and $V = (I\ 0)$, since by the linearity and permutation properties of the trace operator we have

$$\begin{aligned}&L^*(p^*)(Q) = p^*(L(Q)) = \operatorname{tr}\big(P(UQV^T + VQU^T)\big) = \operatorname{tr}\big(PUQV^T\big) + \\ &\operatorname{tr}\big(PVQU^T\big) = \operatorname{tr}\big(V^T PUQ\big) + \operatorname{tr}\big(U^T PVQ\big) = \operatorname{tr}\big((U^T PV + V^T PU)Q\big),\end{aligned}$$

the conic inequality (1) means that $\operatorname{tr}((U^T PV + V^T PU + M)Q) \leq 0$ for all $Q \succeq 0$. But this is equivalent to condition (i), as $\operatorname{tr}(CQ) \geq 0$ for all $Q \succeq 0$ if and only if $C \succeq 0$. □

4. Proofs

In this section, we provide proofs to the rest of the results in the previous sections.

Proof. **Theorem 19** and **Theorem 20**

(i) ⇒ (ii): Take any $z_0 \in K$ such that $L(z_0) = 0$ and note that $L^*(p^*)(z_0) = p^*(L(z_0)) = 0$ so that $m^*(z_0) \geq 0$ and $m^*(z_0) > 0$ follow in the non-strict and strict case, respectively.

(ii) ⇒ (i): In order to find a desired p^*, we show the existence of a separating hyperplane. We do this first in the case of non-strict inequality, and then in the strict case.

Non-strict inequality: Define the two convex sets

$$Q = \{(L(z), -m^*(z)) \mid z \in K\} \subseteq X \times \mathbb{R}$$

and

$$R = \{(0, v) \mid v > 0\} \subseteq X \times \mathbb{R},$$

both clearly convex and nonempty, and suppose that Q and R are not disjoint. Then there exists $\hat{z} \in K$ such that $0 = L(\hat{z})$ and $m^*(\hat{z}) < 0$, a contradiction by condition (ii). As such, there exists a hyperplane separating the two sets, i.e., there exist a $c \in \mathbb{R}$ and a nonzero $\hat{p}^* \in (X \times \mathbb{R})^*$, with $\hat{p}^*(x, r) = p^*(x) + q(r)$ for some nonzero pair $(p^*, q) \in X^* \times \mathbb{R}$ by identification, such that $qv \geq c$ for all $v > 0$ and $p^*(L(z)) - qm^*(z) \leq c$ for all $z \in K$. Now, if $c > 0$, then a sufficiently small $v > 0$ can be chosen so as to violate $qv \geq c$, and if $c < 0$, then $z = 0$ can be chosen in the second inequality to give $0 \leq c < 0$, and so $c = 0$ must hold. Further, if $q < 0$ then any $v > 0$ will violate $qv \geq c = 0$, and if $q = 0$, then $p^* \neq 0$ and $p^*(L(z)) \leq 0$ must hold for all $z \in K$, a contradiction since $L(K) = X$ by assumption. It follows that $q > 0$, and so division by q gives, after the relabeling $\frac{1}{q}p^* \to p^*$ and usage of the definition of adjoints, $L^*(p^*)(z) - m^*(z) = p^*(L(z)) - m^*(z) \leq 0$ for all $z \in K$, which is equivalent to condition (i).

Strict inequality: Define the two sets

$$Q = \{(L(z), -m^*(z)) \mid z \in K \cap B\} \subseteq X \times \mathbb{R},$$

where $B = \{z \in Z \mid \|z\|_Z = 1\}$, and

$$R = \{(0, v) \mid v \geq 0\} \subseteq X \times \mathbb{R},$$

both clearly nonempty, and suppose that $\operatorname{conv}(Q)$ and R are not disjoint. Then there exist k elements $w_i \in Q$, $v_c \geq 0$ and $\alpha_i \geq 0$ with $\sum_{i=1}^k \alpha_i = 1$ such that

$$(0, v_c) = \sum_{i=1}^{k} \alpha_i w_i = (L(z_c), -m^*(z_c)), \tag{4}$$

where $z_c = \sum_{i=1}^k \alpha_i z_i$ for some $z_i \in K \cap B$. Now, being the convex combination of nonzero elements in a convex cone, preservation of nonnegative linear combinations along with pointedness implies that z_c belongs to K and is nonzero. Condition (ii) thus gives $m^*(z_c) > 0$, a contradiction due to (4), and it follows that $\operatorname{conv}(Q)$ and R are disjoint.

In the next step, we note first that $\operatorname{conv}(Q)$ is compact. This follows since $K \cap B$ is closed and bounded and hence compact in finite dimensions, implying that Q is compact as the image of $K \cap B$ under a continuous transformation, noting also that the convex hull preserves compactness in finite dimensions.

Now, since in addition R is closed, there must exist a strictly separating hyperplane between the two convex sets and hence between R and Q, i.e., there exist $c \in \mathbb{R}$ and nonzero $\hat{p}^* \in (X \times \mathbb{R})^*$, with $\hat{p}^*(x, r) = p^*(x) + q(r)$ for some nonzero pair $(p^*, q) \in X^* \times \mathbb{R}$ by identification, such that $qv > c$ for all $v \geq 0$ and $p^*(L(z)) - qm^*(z) < c$ for all $z \in K \cap B$. If $c > 0$, then $v = 0$ causes a contradiction so that $p^*(L(z)) - qm^*(z) < c \leq 0$. Similarly, if $q < 0$, then a sufficiently large $v > 0$ will contradict $qv > c$. Finally, if $q = 0$, then $p^* \neq 0$ and $p^*(L(z)) < 0$ for all $z \in K \cap B$. Since the latter is a compact set and p^* and L are continuous, $p^*(L(z))$ achieves its maximum in the image, which must therefore be negative. Thus, properly scaled by a constant $\beta > 0$, the maximum of $-m^*(z)$ over $K \cap B$ can be added to $p^*(L(z)) < 0$ without changing the negativity so that $p^*(L(z)) - \beta m^*(z) < 0$ for all $z \in K \cap B$. It follows after dividing by either q or β depending on if $q > 0$ or $q = 0$ that, after relabeling, $p^*(L(z)) - m^*(z) < 0$ for all $z \in K \cap B$, which must in fact hold for all nonzero $z \in K$ since $\alpha z \in K \cap B$ for a suitable $\alpha > 0$. Altogether, this means exactly that $p^*(L(z)) - m^*(z)$ belongs to the interior of $-K^*$, see Section 2. □

Proof. **Theorem 21**

(i) ⇒ (ii): Since $L^*(p^*)(z) = p^*(L(z))$, we have $p^*(L(z)) - m^*(z) \leq 0$ for all $z \in K$. Taking any trajectory $z \in H$ and any associated $x \in H_z$ for which $\dot{x} = L(z)$, by integrating we obtain

$$p^*\left(\int_{t_0}^{t_1} L(z(t))\ \mathrm{dt}\right) - \int_{t_0}^{t_1} m^*(z(t))\ \mathrm{dt}$$
$$= p^*(x(t_1)) - p^*(x(t_0)) - \int_{t_0}^{t_1} m^*(z(t))\ \mathrm{dt} \leq 0$$

and so V can be chosen as p^*.

(ii) ⇒ (i): Because $H_l \subseteq H$, H contains $\hat{z}(t) = z_0$ for a given $z_0 \in K$ such that $L(z_0) = 0$. Choosing an interval with $t_0 \neq t_1$, for any $x \in H_{\hat{z}}$ we thus have

$$0 = V(x(t_1)) - V(x(t_0)) \leq \int_{t_0}^{t_1} m^*(\hat{z}(t))\ \mathrm{dt} = m^*(z_0)(t_1 - t_0)$$

because $\dot{x} = L(z_0) = 0$ so that $x(t_0) = x(t_1)$. Theorem 19 now gives condition (i).

(i) ⇔ (iii): One direction follows via condition (ii) as x can be chosen such that $x(t_1) = x(t_0)$ by assumption; the other follows as in the direction *(ii) ⇒ (i)*. □

Proof. **Corollary 7**

By setting $H = \cup_{[t_0,t_1]}\{z : [t_0,t_1] \to K \mid \frac{d}{dt}E(z) = L(z)\}$ and $H_z = \{E(z)\}$ and noting that clearly $H_l \subseteq H \subseteq H_u$, this follows immediately from Theorem 21. Further, if the dissipation inequality holds for some V, then the conic inequality in condition (i) holds and a new V can be chosen as in the proof of Theorem 21 as $V = p^* \in X^*$. □

Proof. **Theorem 22**

Suppose first $L(K) \neq X$. Then, because $L(K)$ is a convex cone, there must exist $p^* \in X^*$ such that $p(L(z)) \leq 0$ for all $z \in K$. Choose now an interior point $x_0 \in E(K)$, which exists by assumption, and another sufficiently close point $x_1 \in E(K)$ so that $p(x_1 - x_0) > 0$. Next, use K-controllability to find $t_1 \geq 0$ and $z(t) \in K$ satisfying $\frac{d}{dt}E(z) = L(z)$ such that $E(z(0)) = x_0$ and $E(z(t_1)) = x_1$. Now, by the linearity and continuity of p^*, we have

$$\begin{aligned} 0 < p^*(x_1 - x_0) = p^*(E(z(t_1))) - p^*(E(z(0))) &= \int_0^{t_1} \frac{d}{dt}p^*(E(z(t)))\,\mathrm{dt} \\ &= \int_0^{t_1} p^*\left(\frac{d}{dt}E(z(t))\right)\mathrm{dt} = \int_0^{t_1} p^*(L(z(t)))\,\mathrm{dt} \leq 0, \end{aligned} \tag{5}$$

a contradiction. As such, $L(K) = X$ and Corollary 1 and Theorem 1 may be invoked below.

(i) ⇒ (ii): Take any z as in condition (ii). By Corollary 7, (E, L) satisfies the dissipation inequality (2), and so for all restrictions of z to the interval $[-T, T]$, where $T > 0$, we have

$$\begin{aligned} V(E(z(t_1))) - V(E(z(t_0))) &\leq \int_{-T}^{T} m^*(z(t))\,\mathrm{dt} \leq \int_{-T}^{T} \left|m^*(z(t))\right|\mathrm{dt} \\ &\leq \|m^*\|_{Z^*} \int_{-T}^{T} \|z(t)\|_Z\,\mathrm{dt}. \end{aligned}$$

Condition (ii) now follows from the continuity of V by letting $T \to \infty$, as $z(\infty) = z(-\infty) = 0$ and the improper integral in question converges since by assumption $\int_{-\infty}^{\infty}\|z(t)\|_Z \mathrm{dt} < \infty$.

(ii) ⇒ (iii): Suppose on the contrary that $m^*(z_0) < 0$ for some $z_0 \in K$ such that $L(z_0) = 0$. Invoke K-controllability to construct a trajectory z which is zero for $t \leq 0$ and satisfies $z(t) = z_0$ for all $t \in [1, t_2]$ and $z(t) = 0$ again for all $t \geq t_2 + 1$. Choose finally a large enough t_2 so as to make the improper integral negative and violate condition (ii) so that in fact $m^*(z_0) \geq 0$ and condition (iii) follows.

(iii) ⇒ (i): This follows directly from Theorem 19. □

Proof. **Proposition 6**

Choose $Z = \mathbb{R}^{n+m}$, $X = \mathbb{R}^n$, $L(x,u) = Ax + Bu$, $E(x,u) = x$, $m^*(x,u) = \gamma \mathbf{1}_m^T u - \mathbf{1}_n^T x$ and $K = \mathbb{R}_+^{n+m}$. Note also that since $L(\mathbb{R}_+^{n+m})$ is a convex cone which contains part of $\mathrm{Int}(\mathbb{R}_+^n)$ by assumption, as well as the nonpositive orthant $-\mathbb{R}_+^n$ (A is Metzler Hurwitz so that $-A^{-1} \geq 0$), we must have $L(\mathbb{R}_+^{n+m}) = X$, so that Theorem 19 and Corollary 7 can be invoked below.

(i) ⇒ (ii): Note first that $\int_0^\infty m^*(x,u)\ \mathrm{d}t \geq 0$ for all nonnegative $u \in L_1^m[0,\infty)$ is equivalent to the supremum part of condition (i). Suppose now that there is some $z_0 = (x_0, u_0) \geq 0$ with $L(z_0) = Ax_0 + Bu_0 = 0$ such that $m^*(z_0) < 0$. We note that the system behaves like an unforced system with equilibrium point at x_0 when $u(t) = u_0$, as

$$\dot{\tilde{x}} = \dot{x} = Ax + Bu_0 = A(x - x_0) + Ax_0 + Bu_0 = A\tilde{x}$$

where $\tilde{x} = x - x_0$. Thus, since A is Hurwitz, the trajectory starting at the origin ($\tilde{x} = -x_0$) will converge to x_0 ($\tilde{x} = 0$) and will additionally be confined to the nonnegative orthant by positivity/monotonicity. The integral over $m^*(x,u)$ can thus be made arbitrarily negative by letting $u(t) = u_0$ sufficiently long due to $m^*(x_0,u_0) < 0$, after which it can be completed into a nonnegative $L_1[0,\infty)$-trajectory by setting $u(t) = 0$, a contradiction by condition (i). Hence, $m^*(z_0) \geq 0$ and so by Theorem 19, there is $\hat{p}^* \in X^*$, i.e., $\hat{p}^*(x) = p^T x$, such that for all $z = (x,u) \in \mathbb{R}_+^{n+m}$,

$$\begin{aligned} 0 &\geq \hat{p}(L(x,u)) - m^*(x,u) = p^T(Ax + Bu) - (-\mathbf{1}_n^T x + \gamma \mathbf{1}_m^T u) \\ &= \begin{pmatrix} p^T A + \mathbf{1}_n^T & p^T B - \gamma \mathbf{1}_m^T \end{pmatrix} z \end{aligned} \tag{6}$$

and the two inequalities in condition (ii) follow since $\mathbb{R}_+^{n+m}$ is self-dual. Finally, $p > 0$ follows from the upper equation, as $p^T \geq -\mathbf{1}_n^T A^{-1} > 0$, since being invertible, A^{-1} can have no zero columns.

(ii) ⇒ (i): Note first that $A^T p \leq -\mathbf{1}_n < 0$ for A Metzler and $p > 0$ implies that A is Hurwitz, e.g., [Rantzer and Valcher, 2018]. Take now any nonnegative $u \in L_1^m[0,\infty)$ with a corresponding $x \in L_1^n[0,\infty)$ (since A is Hurwitz) such that $\dot{x} = Ax + Bu$, and note that $x(t) \geq 0$ since the system is positive. In light of (6), invoke Corollary 7 to find a continuous V such that the restriction of (x,u) to an interval $[0,T]$ with $T > 0$ satisfies the dissipation inequality (2). As a result, by letting $T \to \infty$ and noting that $x(0) = x(\infty) = 0$, we have $\int_0^\infty m^*(x,u)\ \mathrm{d}t \geq 0$ and therefore condition (i). □

Proof. **Corollary 8**

Define E and L through (3) so that $\frac{d}{dt}E(z) = L(z)$. First, for any pair $X_0, X_1 \in E(\mathcal{S}_+^{n+m}) = \mathcal{S}_+^n$, we can perform a spectral decomposition on X_0 and X_1 and exploit controllability to find control inputs connecting the vectors in each rank one term to one another over finite time, say $t_1 \geq 0$. Stacking the resulting state trajectories and their corresponding control inputs, forming rank one matrices and summing up gives a desired z by Theorem 23 and thus controllability on $\mathcal{S}_+^{n+m}$.

For the converse, take any nonzero $x \in \mathbb{R}^n$ and invoke K-controllability to obtain a time $t_1 \geq 0$ and a trajectory z such that $E(z(0)) = 0$ and $E(z(t_1)) = xx^T \succeq 0$. By Theorem 23, z can be expressed as a sum of rank one terms with corresponding vectors satisfying either $x_i = 0$ or $\dot{x}_i = Ax_i + Bu_i$. Since $E(z(t_1)) = xx^T \neq 0$, at least one of the latter kind must exist with $x_i \neq 0$ at $t = t_1$, and if several exist they must be proportional to some vector which by extension must also be proportional to x. Any of the corresponding u_i may then be taken with appropriate scaling, thus proving reachability and therefore controllability. □

Proof. **Theorem 23**

<u>$(i) \Rightarrow (ii)$</u>: Given the partition

$$Q = \begin{pmatrix} Q_{nn} & Q_{nm} \\ Q_{nm}^T & Q_{mm} \end{pmatrix},$$

with r_{max} denoting the maximal rank of $Q_{nn}(t)$ over $t \in \mathbb{R}$, the proof outline is as follows: in the first step, we show the existence of a sum

$$S(t) = \sum_{i=0}^{r_{max}} \begin{pmatrix} x_i \\ u_i \end{pmatrix} (x_i^T \; u_i^T)$$

such that a) each term satisfies (3) over $\mathbb{R}$, b) $Q(t) - S(t) \succeq 0$ for all $t \in \mathbb{R}$, c) $Q_{nn} = S_{nn} = \sum_{i=0}^{r_{max}} x_i x_i^T$ and d) $x_i(t) = 0 \Rightarrow u_i(t) = 0$. In the second step, we show that $\dot{x}_i = Ax_i + Bu_i$ is implied by a) and d), and that b) together with c) give the remaining terms required to complete the sum in condition (ii).

For the first step, we proceed by induction on r_{max}. In the case that $r_{max} = 0$ so that $Q_{nn} = 0$, this follows trivially by choosing $x_0 = 0$ and $u_0 = 0$. Suppose next that the above sum S exists when $r_{max} = k$ for a given $k \geq 0$ and assume that $Q_{nn}(t)$ has maximal rank $r_{max} = k + 1$ over t. Note that by the continuity of eigenvalues, rank $k + 1$ can only happen on open intervals I_i, given some labeling. Below, we construct a special rank one function $\hat{Q} : \mathbb{R} \to \mathcal{S}_+^{n+m}$ with components $(\hat{x}, \hat{u})$ for the purpose of decreasing the maximal rank of Q_{nn}, while preserving the positive semidefiniteness and dynamics satisfaction of Q.

Begin by noting that since $Q(t) \succeq 0$ implies $\text{Im}(Q_{nm}(t)) \subseteq \text{Im}(Q_{nn}(t))$ at each t by Lemma 15, it follows that each column of Q_{nm} must belong to $\text{Im}(Q_{nn})$ and so there exists a function $R : \mathbb{R} \to \mathbb{R}^{n \times m}$ such that $Q_{nm}(t) = Q_{nn}(t)R(t)$. Importantly, exploiting the pseudoinverse or proceeding simply from the piecewise smooth columns of Q_{nn}, R can be chosen to be piecewise smooth on each open interval on which the rank of Q_{nn} is constant, with potential unbounded growth towards the boundary, cf. the scalar case. With this in mind, we now proceed to define $\hat{x}$ and $\hat{u}$ on I_1. For this purpose, choose any $t_0 \in I_1$ and $v_0 \in \mathbb{R}^n$ such that $Q_{nn}(t_0)v_0 \neq 0$, which exists since $r_{max} > 0$, and denote by v the solution to the unforced linear time-varying system $\dot{v} = -\big(A^T + R(t)B^T\big)v$, $v(t_0) = v_0$. Such a solution exists on I_1 since the state matrix is at least piecewise continuous [Rugh, 1996]. Note next the crucial fact that $v(t)^T Q_{nn}(t)v(t)$ is nonzero and constant on this interval as $Q_{nn}(t_0)v_0 \neq 0$ and

$$\begin{aligned} \frac{d}{dt}\Big(v^T Q_{nn} v\Big) &= 2v^T Q_{nn}\dot{v} + v^T \dot{Q}_{nn} v \\ &= 2v^T Q_{nn}\dot{v} + 2v^T\Big(Q_{nn}A^T + Q_{nm}B^T\Big)v = 0, \end{aligned} \tag{7}$$

where $Q_{nm} = Q_{nn}R$ and the dynamics for v and (3) were exploited. As such, $\hat{x} = (v^T Q_{nn} v)^{-\frac{1}{2}} Q_{nn} v$ and $\hat{u} = (v^T Q_{nn} v)^{-\frac{1}{2}} Q_{nm}^T v$ are well-defined on I_1.

Next, we extend $\hat{x}$ continuously over the entire real line by continuing the solution v from the one-sided limits at the boundary, noting that if $\hat{x} \to 0$, then we simply set $\hat{x} = \hat{u} = 0$ for the remaining t or until the next r_{max} interval I_i, at which point the above procedure is repeated. In the rest of this paragraph, we clarify technical details in connection to this, particularly when $R(t)$ grows unbounded in response to a rank change in Q_{nn}. First, given the spectral decomposition $Q_{nn}(t) = U^T(t)D(t)U(t)$, we note that v can grow unbounded only in those entries of Uv which correspond to vanishing entries in D as $\sqrt{Q_{nn}}v = U^T\sqrt{D}Uv$ has constant norm by (7). Moreover, because of (7), another multiplication by $\sqrt{Q_{nn}}$ will remove these unbounded entries in the limit. Thus, with $v_b = U^T b$ where b contains the entries of Uv corresponding to non-vanishing entries in D, we have $\hat{x} \to (\beta)^{-\frac{1}{2}} Q_{nn} v_b$, where clearly $\beta \geq v_b^T Q_{nn} v_b$. On the other side of the limit point, in the adjacent open interval with constant rank, there are two possibilities: if $R(t)$ does not grow unbounded, then choose v to approach v_b in the limit and scale both $\hat{x}$ and $\hat{u}$ with $0 < \alpha = \beta^{-\frac{1}{2}}(v_b^T Q_{nn} v_b)^{\frac{1}{2}} \leq 1$ so as to make $\hat{x}$ continuous. If, however, $R(t)$ grows unbounded, then it must be because a vanishing nonzero entry has appeared in D. This presents additional freedom in the sense that v can be taken to approach b in the corresponding entries of Uv, while making the constant $v^T Q_{nn} v$ large enough to match β. To see why such a v exists, note that v_0 can be taken so that $U(t_0)v_0$ is sufficiently close to b in these entries, with the rest of the entries chosen so as to place v_0 on the correct elliptical level curve of $v^T Q_{nn} v$ in order to match β. If t_0 is then chosen sufficiently close to the limiting time, Uv ends up arbitrarily close to b in the proper entries in the limit. By continuity arguments (e.g., applying Banach fixed-point theorem), the existence of such a v follows.

In summary, $\hat{x}$ and $\hat{u}$ have been extended across the real line so that either $\hat{x} = (v^T Q_{nn} v)^{-\frac{1}{2}} Q_{nn} v$ and $\hat{u} = (v^T Q_{nn} v)^{-\frac{1}{2}} Q_{nm}^T v$, possibly scaled in areas with rank strictly less than $k+1$, or $\hat{x} = \hat{u} = 0$. Now, in the same way that (7) was shown, it is straightforward to verify that $\hat{Q}$ satisfies (3); as a result, so does $Q - \hat{Q}$. Further, by the Cauchy-Schwarz inequality, we have $w^T(Q - \hat{Q})w \geq 0$ for all $w \in \mathbb{R}^{n+m}$, noting that $\alpha \leq 1$. Finally, it is clear that $\text{rank}(Q_{nn} - \hat{Q}_{nn}) = \text{rank}(Q_{nn} - \hat{x}_i \hat{x}_i^T) \leq \text{rank} Q_{nn} - 1 = k$ for all $t \in I_i$, as the corresponding kernel has increased (to see this, multiply by v), and otherwise also since then the rank is k or less since the kernel has not decreased. The induction assumption may thus be invoked on $Q - \hat{Q}$ to obtain property b) for Q, namely

$$Q \succeq \sum_{i=0}^{k} \begin{pmatrix} x_i \\ u_i \end{pmatrix} (x_i^T \ u_i^T) + \hat{Q} = \sum_{i=0}^{k+1} \begin{pmatrix} x_i \\ u_i \end{pmatrix} (x_i^T \ u_i^T)$$

by defining $x_{k+1} = \hat{x}$ and $u_{k+1} = \hat{u}$. Further, property c) derives from the induction assumption: $Q_{nn} - \hat{Q}_{nn} = \sum_{i=0}^{k} x_i x_i^T$ so that $Q_{nn} = \sum_{i=0}^{k+1} x_i x_i^T$. Property a) now follows by noting that each term in S satisfies (3), the last one by construction, and the same holds for property d). Thus, the first main step follows by induction.

In the second step, let a rank one matrix function $M(t) \succeq 0$ with components (x, u) be given. Supposing that $x(t) \neq 0$ and that $\dot{x}(t)$ exists at a given t, then M solves (3) if and only if

$$\dot{x}x^T + x\dot{x}^T = (Ax + Bu)x^T + x(Ax + Bu)^T. \tag{8}$$

Multiplication from the right by x and division by $x^T x \neq 0$, noting that $\dot{x}^T x = x^T \dot{x}$, gives

$$\left(I + (x^T x)^{-1} x x^T\right)(\dot{x} - (Ax + Bu)) = 0.$$

Since the left matrix is invertible, we must have $\dot{x} = Ax + Bu$ at t. Setting $x = x_i$ and $u = u_i$ from the above rank one decomposition, the terms will thus be as in condition (ii), noting that $\dot{x} = Ax + Bu$ also when $x_i(t) = 0$ on an open interval, since then $u_i(t) = 0$ by property d). As for the remaining terms of Q, since

$$\tilde{Q} = Q - \sum_{i=0}^{r_{max}} \begin{pmatrix} x_i \\ u_i \end{pmatrix} (x_i^T \ u_i^T) \succeq 0$$

with $Q_{nn} = \sum_{i=0}^{r_{max}} x_i x_i^T$ from the first main step, it follows that $\tilde{Q}_{nn} = 0$ and so, since $\text{Im}(\tilde{Q}_{nm}) \subseteq \text{Im}(\tilde{Q}_{nn})$ by Lemma 15, $\tilde{Q}_{nm} = 0$. The remaining terms in condition (ii) therefore satisfy $x_i = 0$, as they follow from a spectral decomposition of $\tilde{Q}_{mm}$, and the statement follows.

(ii) ⇒ (i): Each term in condition (ii) which satisfies $\dot{x} = Ax + Bu$ also satisfies (8) and therefore (3); each term with $x_i = 0$ trivially satisfies (3). The same therefore holds for the sum Q, and condition (i) follows. □

5. Conclusions

In this paper, we have considered integral linear constraints on sets of Z-valued trajectories constrained pointwise in time to a cone, where Z is a finite-dimensional normed space. Importantly, the satisfaction thereof was found to be equivalent to the existence of a bounded linear functional p^* satisfying a conic inequality. Notably, the sets of trajectories amenable to this equivalence can but must not in any way be connected to dynamics. Conversely, finding such a solution to the conic inequality establishes an integral linear constraint for a number of trajectory sets, including the ones with differential constraints which are often of interest in the context of dynamical systems. Moreover, parallels were drawn to the control literature by showing that the satisfaction of the conic inequality is equivalent to the satisfaction of the dissipation inequality with linear supply rate on a cone, corresponding to the well-known connection between LMIs and the dissipation inequality with quadratic supply rate.

The above results were subsequently leveraged in order to prove both an L_1-gain analog in positive systems theory to the well-known bounded real lemma, as well as a non-strict version of the KYP Lemma in linear-quadratic control. This contributes to drawing further parallels between and bringing these traditionally different areas together under a linear-conic framework. Furthermore, there is perhaps also a contribution in the above proof of the KYP Lemma in comparison to other already existing proofs. The proof essentially passes through a more basic cone analog of the KYP Lemma in which the characteristic quadratic costs over $\mathbb{R}^{n+m}$ now become linear over a cone, see Theorem 22. The KYP Lemma with its associated quadratic functionals is then obtained by applying a crucial rank one decomposition to matrix trajectories on the positive semidefinite cone, see Theorem 23. Although there are arguably more straightforward and direct ways in which to proceed, this approach additionally provides structure and novel insights as opposed to rote theorem verification, see the remarks in Section 3. For example, it diminishes the role of the frequency inequality in favor of an integral quadratic constraint formulation for the purpose of better understanding the phenomenon mathematically. In addition, algebraic proofs celebrated for their brevity tend to obscure the connection to dynamics by postponing it to the end when the frequency domain enters into the picture. This may lead one to think that the desired connection between LMIs and constraints on the behavior of a system is inaccessible, when in fact in a dynamics proof it is made early on and can be observed clearly in its simplicity on cones.

For future works, it would be interesting to unite additional results under a conic framework, as well as to pursue what may be a cone analog to linear-quadratic theory. This is already achieved for the special case of the dissipation inequality, and suggested in the case of K-controllability. As seen in previous work on the topic, cone-preservance and monotonicity are fundamental assumptions that will likely play an important role to this end.

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References

Angeli, D. and E. D. Sontag (2003). "Monotone control systems". *IEEE Trans. Autom. Control* **48**:10, pp. 1684–1698.

Bamieh, B. (2024). *Linear-quadratic problems in systems and controls via covariance representations and linear-conic duality: finite-horizon case,* arXiv:2401.01422 [eess.SY].

Barker, G. P. (1981). "Theory of cones". *Linear Algebra Appl.* **39**, pp. 263–291.

Berman, A. and R. J. Plemmons (1994). *Nonnegative Matrices in the Mathematical Sciences.* SIAM. ISBN: 9780120922505.

Boyd, S. P. and L. Vandenberghe (2004). *Convex Optimization.* Cambridge University Press. ISBN: 9780521833783.

Briat, C. (2013). "Robust stability and stabilization of uncertain linear positive systems via integral linear constraints: L_1-gain and L_∞-gain characterization". *Int. J. Robust Nonlinear Control* **23**:17, pp. 1932–1954.

Coxson, P. G. and H. Shapiro (1987). "Positive input reachability and controllability of positive systems". *Linear Algebra Appl.* **94**, pp. 35–53.

Doyle, J., K. Glover, P. Khargonekar, and B. Francis (1988). "State-space solutions to standard H_2 and H_∞ control problems". In: *Proc. IEEE Amer. Control Conf. (ACC)*, pp. 1691–1696.

Ebihara, Y., D. Peaucelle, and D. Arzelier (2011). "L_1 gain analysis of linear positive systems and its application". In: *Proc. 50th IEEE Conf. Decis. Control (CDC)*, pp. 4029–4034.

Ebihara, Y., D. Peaucelle, and D. Arzelier (2014). "LMI approach to linear positive system analysis and synthesis". *Syst. & Control Lett.* **63**, pp. 50–56.

Farina, L. and S. Rinaldi (2000). *Positive Linear Systems: Theory and Applications.* Vol. 50. John Wiley & Sons. ISBN: 9780471384564.

Gahinet, P. and P. Apkarian (1994). "A linear matrix inequality approach to H_∞ control". *Int. J. Robust Nonlinear Control* **4**:4, pp. 421–448.

Gattami, A. (2009). "Generalized linear quadratic control". *IEEE Trans. Autom. Control* **55**:1, pp. 131–136.

Gordon, R. (1991). "Riemann integration in banach spaces". *The Rocky Mt. J. Math.* **21**:3, pp. 923–949.

Gowda, M. S. and J. Tao (2009). "Z-transformations on proper and symmetric cones: Z-transformations". *Math. Program.* **117**:1, pp. 195–221.

Haddad, W. M. and V. Chellaboina (2005). "Stability and dissipativity theory for nonnegative dynamical systems: a unified analysis framework for biological and physiological systems". *Nonlinear Anal.: Real World Appl.* **6**:1, pp. 35–65.

Kalman, R. E. (1963). "Lyapunov functions for the problem of lur'e in automatic control". *Proc. Natl. Acad. Sci.* **49**:2, pp. 201–205.

Kalman, R. E. et al. (1960). "Contributions to the theory of optimal control". *Bol. soc. mat. mexicana* **5**:2, pp. 102–119.

Luenberger, D. G. (1997). *Optimization by Vector Space Methods.* John Wiley & Sons. ISBN: 9780471181170.

Megretski, A. (2010). *KYP lemma for non-strict inequalities and the associated minimax theorem* arXiv:1008.2552 [math.OC].

Megretski, A. and A. Rantzer (1997). "System analysis via integral quadratic constraints". *IEEE Trans. Autom. Control* **42**:6, pp. 819–830.

Ohta, Y., H. Maeda, and S. Kodama (1984). "Reachability, observability, and realizability of continuous-time positive systems". *SIAM J. Control Optim.* **22**:2, pp. 171–180.

Papusha, I. and R. M. Murray (2015). "Analysis of control systems on symmetric cones". In: *Proc. 54th IEEE Conf. Decis. Control (CDC)*, pp. 3971–3976.

Pates, R. and A. Rantzer (2024). *Optimal control on positive cones* arXiv:2407.18774 [math.OC].

Popov, V.-M. (1961). "Absolute stability of nonlinear systems of automatic control". *Avtomatika Telemekhanika* **22**:8, pp. 961–979.

Rantzer, A. (1996). "On the kalman—yakubovich—popov lemma". *Syst. & Control Lett.* **28**:1, pp. 7–10.

Rantzer, A. (2022). "Explicit solution to bellman equation for positive systems with linear cost". In: *Proc. 61st IEEE Conf. Decis. Control (CDC)*, pp. 6154–6155.

Rantzer, A. and M. E. Valcher (2018). "A tutorial on positive systems and large scale control". In: *Proc. IEEE Conf. Decis. Control (CDC)*, pp. 3686–3697.

Rugh, W. J. (1996). *Linear System Theory.* Upper Saddle River, NJ, USA: Prentice-Hall, Inc. ISBN: 9780134412054.

Schneider, H. and M. Vidyasagar (1970). "Cross-positive matrices". *SIAM J. Numer. Anal.* **7**:4, pp. 508–519.

Shen, J. and J. Lam (2016). "Some extensions on the bounded real lemma for positive systems". *IEEE Trans. Autom. Control* **62**:6, pp. 3034–3038.

Shen, J. and J. Lam (2017). "Input–output gain analysis for linear systems on cones". *Automatica* **77**, pp. 44–50.

Tanaka, T. and C. Langbort (2011). "The bounded real lemma for internally positive systems and H-infinity structured static state feedback". *IEEE Trans. Autom. Control* **56**:9, pp. 2218–2223.

Tanaka, T., C. Langbort, and V. Ugrinovskii (2013). "DC-dominant property of cone-preserving transfer functions". *Syst. & Control Lett.* **62**:8, pp. 699–707.

Valcher, M. E. (1996). "Controllability and reachability criteria for discrete time positive systems". *Int. J. Control* **65**:3, pp. 511–536.

Valcher, M. E. (2009). "Reachability properties of continuous-time positive systems". *IEEE Trans. Autom. Control* **54**:7, pp. 1586–1590.

Willems, J. (1971). "Least squares stationary optimal control and the algebraic Riccati equation". *IEEE Trans. Autom. Control* **16**:6, pp. 621–634.

Willems, J. C. (1972a). "Dissipative dynamical systems part I: general theory". *Archive for rational mechanics and analysis* **45**:5, pp. 321–351.

Willems, J. C. (1972b). "Dissipative dynamical systems part II: linear systems with quadratic supply rates". *Arch. Ration. Mech. Anal.* **45**, pp. 352–393.

Yakubovich, V. A. (1962). "Solution of certain matrix inequalities in theory of automatic control". *Doklady Akademii Nauk SSSR* **143**:6, pp. 1304–+.

Verifiering av Systembeteende vid Koniska Begränsningar

Emil Vladu
Institutionen för Reglerteknik

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Reglerteknik finns överallt i samhället. Teknologin handlar om att uppnå ett önskat beteende i olika processer utan mänsklig inverkan. Tillämpningar i samhället går att finna överallt: i automatisk styrning av rumstemperaturen, i autopiloten inuti flygplan, i styrning av elnätet eller fjärrvärmen, i robotik, i autonoma bilar samt mycket mer. I grunden har de flesta tillämpningar följande gemensamt: en apparat som tar in mätvärden från omgivningen och omsätter dem till analoga signaler som ska påverka systemet i önskad riktning. Denna apparat följer en matematisk formel, en så kallad styrlag, som är designad av en ingenjör. En viktig del i designprocessen är analysdelen, då man numeriskt eller grafiskt försöker karakterisera systemets beteende. Ett exempel på ett ofta farligt beteende man vill känna till är signaler som skulle börja växa okontrollerat vid minsta störning.

Matematik är ett centralt verktyg i detta gemensamma förfarande. Exempelvis kan fenomenet där signaler växer okontrollerat översättas och preciseras på matematiskt språk, och satser kan därefter tas fram för att likställa sådan tillväxt med existensen av en simpel matris. Eftersom att det senare går att kontrollera och finna numeriskt med hjälp av en dator, så har man med hjälp av matematik uppnått ett simpelt sätt att förutsäga en viss typ av oönskat beteende.

Världen är dock komplex, och ovanstående metoder fungerar bara i matematiskt idealiserade sammanhang. Mer specifikt representeras en verklig process med hjälp av ett matematiskt uttryck som begränsar systemets tillåtna beteende, ofta i form av en eller flera differentialekvationer. Det otroliga är nu att sådana modeller ofta uppvisar stora likheter med hur deras verkliga motsvarigheter beter sig. Detta är inget sammanträffande, utan kommer från att modellerna tas fram med hjälp av experiment eller fysikaliska lagar, varpå förenklingar sker som bevarar de dominanta egenskaperna. Med andra ord spelar inte varje dammkorn på en bil någon större roll för hur den beter sig, och det finns strukturerade matematiska metoder för att kunna extrahera de mest signifikanta dragen. Samtidigt så ökar generellt mängden numeriska och grafiska metoder som finns att tillgå ju simplare modeller man använder. I extremen finner man de så kallade linjära systemen som bevarar de viktigaste dragen och för vilka litteraturen tveklöst är störst.

Bidrag

I denna avhandling studeras just linjära system, och nya algebraiska uttryck tas fram för att karakterisera olika systembeteenden. Det kan handla om stabilitet eller värstafallet-påverkan av externa störningar. Nyhetsvärdet ligger mer specifikt i de applikationsmotiverade begränsningar som läggs på ett system, så kallade koniska begränsningar, samt de resulterande uttrycken som följer genom tillämpning av matematisk konteori, ett ganska nytt inslag i reglertekniken. Syftet är dels att kunna förutsäga gynnsamt och skadligt beteende på nya sätt hos en större mängd verkliga system, samt att uppnå en ökad förståelse för tillsynes vitt skilda fenomen i litteraturen genom att förena dem under samma ramverk och struktur, vilket konteori möjliggör.