# Exercises in Linear System Theory

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*Abstract*

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# 1. State Space Basics

## 1.1 Basic concepts

**1.1** Read the Matlab Control Toolbox Tutorial.

**1.2** Some people like the notation

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) := D + C(sI - A)^{-1}B = G(s)$$

Write state equations, using the notation above, for

**a.** series connection $G_1G_2$,

**b.** parallel connection $G_1 + G_2$,

**c.** feedback, with $A_1, B_1, C_1, D_1$ in the forward loop and $A_2, B_2, C_2, D_2$ in the backward loop,

**d.** system inversion (for the case of invertible $D$)

**e.** $G^T(-s)$.

**1.3** Show that the relative degree $k = \deg(a) - \deg(b)$ of a strictly proper system $b(s)/a(s) = c(sI - A)^{-1}b$ equals the smallest $k$ such that $cA^{k-1}b \neq 0$

**1.4** Assume that $\bar{x} = Tx$ where

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \qquad \begin{cases} \dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}u \\ y = \bar{C}\bar{x} + Du \end{cases}$$

Show that the following two systems then also are equivalent (the *adjoint* systems)

$$\begin{cases} \dot{z} = -A^Tz + C^Tu \\ y = B^Tz + D^Tu \end{cases} \qquad \begin{cases} \dot{\bar{z}} = -\bar{A}^T\bar{z} + \bar{C}^Tu \\ y = \bar{B}^T\bar{z} + D^Tu \end{cases}$$

(Hint: $\bar{z} = T^{-T}z$)

**1.5** Let

$$\begin{array}{l} \dot{x} = Ax + Bu \\ y = Cx + Du \end{array}, \quad u = e^{\lambda t}g \quad t \geq 0$$

Show that

$$\left\{ \exists x_0 : \; y(t) \equiv 0, t \geq 0 \right\} \iff \left\{ \exists x_0 : \begin{pmatrix} \lambda I - A & -B \\ C & D \end{pmatrix} \begin{pmatrix} x_0 \\ g \end{pmatrix} = 0 \right\}$$

If $\{A, C\}$ is observable then one can show that also $\Rightarrow$ holds.

**1.6** Let $G(s) = c(sI - A)^{-1}b$. Show that if $v$ is not an eigenvalue of $A$ then there exists an initial state $x_0$ such that the response to $u(t) = e^{vt}, t \geq 0$ is $y(t) = G(v)e^{vt}, t \geq 0$. What is this initial condition ?

**1.7** Show that the zeros of $(A, b, c, d)$ can be computed by solving

$$\begin{pmatrix} \lambda I - A & -b \\ c & d \end{pmatrix} \begin{pmatrix} x_0 \\ g \end{pmatrix} = 0$$

Note that since

$$\begin{pmatrix} \lambda I - A & -b \\ c & d \end{pmatrix} = \lambda \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & b \\ -c & -d \end{pmatrix} = \lambda E - F$$

this is a generalized eigenvalue problem for which there are good numerical routines. How does Matlab compute zeros (see `tzero`)?

## 1.2 System parameters

**1.8** Write a Matlab function `markov(b,a,m)` that calculates the $m+1$ first Markov parameters, $h_0, \ldots, h_m$

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^n + \ldots + b_n}{s^n + \ldots + a_n} = h_0 + h_1 s^{-1} + h_2 s^{-2} + \ldots + h_m s^{-m} + \ldots$$

Calculate $h_{100}$ for

$$\frac{1}{(1 - s^{-1})(1 - s^{-5})(1 - s^{-10})}$$

(Hint: Use `filter`)

**1.9** Show that $h_{k+1} = cA^k b = h^{(k)}(0)$, $k = 0, 1, \ldots$, where $h(\cdot)$ is the impulse response.

**1.10** Write a Matlab function `moment(b,a,m)` that calculates the $m + 1$ first moments $m_0, \ldots, m_m$

$$G(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^n + \ldots + b_n}{s^n + \ldots + a_n} = m_0 + m_1 s + \ldots + m_m s^m + \ldots$$

It is assumed that $G(s)$ is analytic at $s = 0$. Calculate $m_{100}$ for $1/(s + 1)(s + 2)$.

**1.11** Show that if the system is stable then

$$m_k = \int_0^\infty \frac{(-t)^k}{k!} h(t)\, dt = \begin{cases} -cA^{-1}b + d & k = 0 \\ -cA^{-k-1}b & k > 0 \end{cases}$$

**1.12** Show that (if $b_0 = 0$)

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = \begin{pmatrix} 1 & 0 & \ldots & 0 \\ a_1 & 1 & 0 & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ a_{n-1} & \ldots & a_1 & 1 \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_n \end{pmatrix} = \begin{pmatrix} m_0 & m_1 & \ldots & m_{n-1} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \ldots & m_0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

(Hint: Compare coefficients in two series)

# Part 1

## Exercises

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Show that

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(Hint: Compare coefficients in two series)

**1.13** Show that (if $b_0 = 0$)

$$h_{n+k} + a_1 h_{n+k-1} + \cdots + a_n h_k = 0, \quad k = 1, 2, \cdots$$
$$m_k + a_1 m_{k+1} + \cdots + a_n m_{k+n} = 0, \quad k = 0, 1, \cdots$$

Describe how the moments and Markov parameters can be produced using a linear shift register.

## 1.3 Canonical forms

**1.14** Verify by direct calculation that

$$c(sI - A)^{-1} b = \frac{b(s)}{a(s)}$$

for the four canonical forms.

(b.) Controllability

$$\begin{pmatrix} h_1 & \dots & h_n \end{pmatrix} \begin{pmatrix} s & & & a_n \\ -1 & s & & \\ & \ddots & \ddots & \vdots \\ & & -1 & s + a_1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \dots$$

$$= \begin{pmatrix} h_1 & \dots & h_n \end{pmatrix} \frac{1}{a(s)} \underline{A}^T \begin{pmatrix} s^{n-1} \\ \vdots \\ 1 \end{pmatrix} = (1.12) = b(s)/a(s)$$

(c,d.) Dual with a and b.

**1.15** Write Matlab functions for transformation from transfer function form to any of the four canonical forms.

Ex. `[A,b,c,d]=tf2observer(num,den)`

**1.16** Write Matlab functions for transforming a state-space realization to any of the four canonical forms.

**1.17** Find realizations in controller, observer, controllability, observability, and Jordan canonical forms of the transfer function

$$H(s) = \frac{4s^3 + 25s^2 + 45s + 34}{s^3 + 6s^2 + 10s + 8}$$

**1.18** Transform

$$G(s) = \frac{1}{(s+1)(s+2)\cdots(s+20)}$$

to controller form. How is the poles changed if $\varepsilon = 10^{-6}$ is added to $A(1, 20)$. Is the controller form a numerically reliable realization of this $G(s)$ ?

**1.19** Check the diagram on p. 129 for the system

$$\frac{b_1 s + b_2}{s^2 + a_1 s + a_2}$$

**1.20**

**a.** Write a Matlab function `ladder(b,a)` that calculates the continued-fraction represention

$$G(s) = \frac{b_1 s^{n-1} + \ldots + b_n}{s^n + a_1 s^{n-1} + \ldots + a_n} = \cfrac{1}{g_1 s + \cfrac{1}{g_2 + \cfrac{1}{g_3 s + \cfrac{1}{g_4 + \ldots}}}}$$

**b.** Show that $G(s)$ can be realized by a RC-network as in Fig 1. using the ladder form.

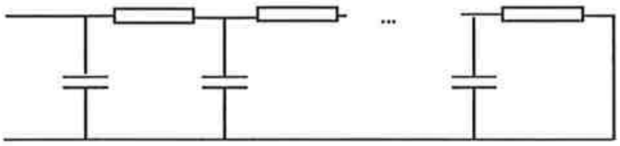

**Figure 1.1** Ladder form

**c.** Show that $G(s)$ can be realized by the triple

$$A = \begin{pmatrix} -1/g_1g_2 & 1/g_1g_2 & 0 & \cdots \\ 1/g_2g_3 & -1/g_2g_3 - 1/g_3g_4 & 1/g_3g_4 \ldots & \\ 0 & \ddots & \ddots & \ddots \end{pmatrix} \quad B = \begin{pmatrix} 1/g_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix}$$

Note that the $A$ matrix is tridiagonal.

**d.** Show that if $g_i > 0$, $\forall i$ in the ladder form then $G(s)$ is positive real (that is $\mathrm{Re}(G(s)) \geq 0$ for $\mathrm{Re}(s) \geq 0$).

## 1.4 Initial conditions, Controllability, Observability

**1.21** Suppose that

$$Y(s) = \frac{b_1 s^{n-1} + \cdots b_n}{s^n + a_1 s^{n-1} + \cdots + a_n}$$

Show that the vector of initial values can be calculated by

$$\begin{pmatrix} y(0+) \\ \vdots \\ y^{(n-1)}(0+) \end{pmatrix} = \underline{A}^{-1} \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

**1.22** Show that when $A$ is nonsingular the moments enter naturally into the problem of determining $x^{(n)}(t)$ from $y(t)$ and its derivatives. What is the most natural canonical form for this problem ? Compare also with discrete time.

**1.23** Suppose that $\{A, b, c\}$ is a realization for which $\mathcal{C}(A, b) = I$. Show that this information completely determines $\{A, b\}$. How about general invertible $\mathcal{C}(A, b)$ ?

**1.24** Show that the pair $\{A, b\}$, where $A$ is diagonal is controllable iff

1) the eigenvalues are distinct,

2) all components of $b$ are non zero.

Give a physical explanation to why repeated eigenvalues cause a loss of controllability.

**1.25** Let

$$\mathcal{C}_k = \begin{pmatrix} b & Ab & \cdots & A^{k-1}b \end{pmatrix}$$

Show that if $\text{Im}(\mathcal{C}_{k+1}) = \text{Im}(\mathcal{C}_k)$, then $\text{Im}(\mathcal{C}_{k+i}) = \text{Im}(\mathcal{C}_k)$, $\forall i \geq 1$.

**1.26** With

$$(sI - A)^{-1}b = \frac{1}{a(s)} \begin{pmatrix} p_{n-1}(s) \\ \vdots \\ p_0(s) \end{pmatrix}$$

show that $\{A, b\}$ is controllable iff the matrix $P$ defined by

$$\begin{pmatrix} p_{n-1}(s) \\ \vdots \\ p_0(s) \end{pmatrix} = P \begin{pmatrix} s^{n-1} \\ \vdots \\ 1 \end{pmatrix}$$

is nonsingular. (Hint: What is the relation of $P$ to the matrix that transforms to controller form)

**1.27** Prove that for an observable system one can determine the state ***without*** knowledge of the input iff the transfer function has no (finite) zeros.

**1.28** Show that the state $x$ of a discrete time system can be reached from the origin in $n$ steps while $y(k) \equiv 0$ iff $\exists p$ such that

$$x = \begin{pmatrix} A^{n-1}b & \cdots & Ab & b \end{pmatrix} p$$
$$Tp = 0$$

where $T$ is the lower Toeplitz matrix of Markov parameters.

**1.29** Show how to find a minimal realization $\{A, b, c\}$ of a system given the impulse response $\{h_1, h_2, \ldots\}$ and the system order. (Hint: Note that $M(1, n-1) = \mathcal{O}(c, A)\mathcal{C}(A, b)$ and $M(2, n-1) = \mathcal{O}A\mathcal{C}$ and try to find the controllability form.)

**1.30** Let

$$G_1(s) = \frac{1}{s-1}, \qquad G_2(s) = \frac{s-1}{s+1}$$

Give simple physical explanations to the loss of observability resp. controllability of $G_2G_1$ and $G_1G_2$.

**1.31** Consider a realization with

$$A = \text{block diag}\left\{\begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right\} b = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$c = \begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix}$$

Check its controllability in as many ways as you can.

**1.32** Consider the equation

$$y(t) + a_1 y(t-1) + \cdots + a_n y(t-n) = b_1 u(t-1) + \cdots + b_n u(t-n)$$

Assume that the associated polynomials $a(z)$ and $b(z)$ are relatively prime, and introduce as state vector

$$x(t) = \begin{pmatrix} y(t) \\ \vdots \\ y(t-n+1) \\ u(t-1) \\ u(t-n+1) \end{pmatrix}$$

Write the state equations for the system and show that it is not minimal. What are the hidden modes ?

## 1.5 The PBH-test and Minimality

**1.33**

a. Show that $\{A, b\}$ is controllable iff $\{I + Ah, bh\}$ is ($h \neq 0$).

b. Prove that $\{A, B\}$ is controllable iff $\{A - BL, B\}$ is controllable for all $L$.

c. Prove that $\{A, B\}$ is controllable iff $\{A, BB^T\}$ is controllable.

**1.34** Use the PBH test to show that the controller form realization of $b(s)/a(s)$ is observable iff $\{a(s), b(s)\}$ are coprime.

**1.35** Let $\{A_i, b_i, c_i\}$ be minimal realizations of order $n_i$ of the transfer functions $G_i(s) = b_i(s)/a_i(s)$, $i = 1, 2$.

a. Show that the series connection $G_2 G_1$ is controllable iff $b_1$ and $a_2$ are coprime. What about observability ?

b. Show that the parallel combination is observable (controllable) iff $a_1$ and $a_2$ are coprime.

c. Show that the feedback configuration with $G_1$ in the forward path and $G_2$ in the feedback path is observable (controllable) iff $b_1$ and $a_2$ are coprime.

d. Extend the results to the case with direct feedthrough from input to output.

**1.36** Suppose $\{A, e_1\}$ is controllable where

$$A = \begin{pmatrix} a_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Show that $\{A_{22}, A_{21}\}$ is controllable.

**1.37**

**a.** Show that $\{c, A\}$ is observable iff the $n$ polynomials $\{q_i(s)\}$ defined by

$$c(sI - A)^{-1} = \begin{pmatrix} q_1(s) & \dots & q_n(s) \end{pmatrix} / a(s)$$

have no common factors.

**b.** Use the result to obtain an alternative proof to the claim that observability of the series connection $G_2G_1$is equivalent to $b_2(s)$ and $a_1(s)$ being coprime.

**1.38** Suppose that $\{A, B, C\}$ is minimal and $\det(sI - A)$ has a repeated root. Prove that $A$ can not be diagonalized by a similarity transform.

**1.39** Another test of coprimeness is that the controllability form realization of $b(s)/a(s)$ is observable. Show that this leads to a Hankel matrix test for coprimeness.

## 1.6 The Diophantine Equation

**1.40** Show that the Sylvester matrix

$$S = \mathrm{Res}_{m-1,n-1}(a, b)$$

is invertible if and only if $a$ and $b$ are relative prime ($\deg(a) = n, \deg(b) = m$) (Hint: A square matrix $S$ is invertible iff $\mathrm{Ker}(S) = \emptyset$ or iff $\mathrm{Im}(S) =$ whole space. $a$,$b$ are relative prime iff $\exists r, S$ of degree $< m$ resp. $< n$ such that $ar + bs = 1$).

**1.41** Use Matlab to solve the DAB-equation

$$(s + 1)^5 R(s) + (s + 2)^4 S(s) = (s + 3)^9$$

where $\deg(R) = \deg(S) = 4$.

**1.42** Suppose $a(s) = \prod_1^n (s - \alpha_i)$ and $b(s) = b_0 \prod_1^m (s - \beta_i)$ ($b_0 \neq 0$). Show that

$$\det S(a, b) = b_0^n \prod_{i,j} (\alpha_i - \beta_j)$$

where $S(a, b)$ is the Sylvester matrix of $\{a(s), b(s)\}$.

**1.43** Let $a(s)$ and $b(s)$ have degree $n$. Show that by elementary row and column operations one can reduce the Sylvester matrix to the form block diag $\{I_n, \mathcal{O}(b, A_c)\}$, where $A_c$ is a companion matrix with $-\begin{pmatrix} a_1 & \dots & a_n \end{pmatrix}$ as the first row. Use this to give another proof of the Sylvester coprimeness test.

**1.44** Prove that the Bezoutian matrix

$$B = \tilde{I}(\mathcal{A}_+\mathcal{B}_- - \mathcal{B}_+\mathcal{A}_-)$$

is symmetric.

## 1.7 Solutions of State Equations

**1.45** Evaluate $\exp(At)$ for

**a.**

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

**b.**

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

**c.** `diag(ones(n,1),1)` (Matlab-notation)

**d.** `diag(lambda*(ones(n,1)))+diag(ones(n-1,1),1)` (i.e. a Jordan block)

**e.** A nilpotent matrix (a matrix such that $A^k = 0$).

**1.46** Use Macsyma or another program for symbolic calculations to calculate $\exp(A)$ for

**a.**

$$A = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

**b.**

$$A = \begin{pmatrix} \sigma & \omega \\ \omega & -\sigma \end{pmatrix}$$

**1.47**

**a.** Establish that $\exp A(t_1 + t_2) = \exp At_1 \exp At_2$ by using the series expansion.

**b.** Show that $(\exp At)^{-1} = \exp(-At)$ by using a)

**c.** Show Liouvilles formula

$$\det(\exp At) = \exp(\text{trace} At)$$

**1.48** The formula

$$\exp(A + B)t = \exp At \exp Bt \quad \forall t$$

is only valid iff $AB = BA$. Show that in the general case

$$\exp(A + B)t = \exp At \exp Bt \exp C_2t^2 \exp C_3t^3 \cdot \dots$$

Describe a method to calculate $C_i$, and calculate especially $C_2$.

## 1.8 The Lyapunov Equation

**1.49** If $P = P^T$ satisfies the linear (Lyapunov) equation

$$AP + PA^T + Q = 0$$

show that

$$P = e^{At}Pe^{A^Tt} + \int_0^t e^{As}Qe^{A^Ts}\,ds$$

**1.50** Show that the discrete time Lyapunov equation can be rewritten as

$$(I - A^T \otimes A^T)p = q$$

Hence show that the Lyapunov equation has a unique solution iff $1 - \lambda_i\lambda_j \neq 0$, all $i, j$ where $\{\lambda_i\}$ are the eigenvalues of $A$. ( That $A$ is discrete time stable is thus a sufficient condition).

**1.51** Let $a(z) = a_0z^n + \ldots + a_n$ and form the matrix

$$S = [s_{ij}] \quad s_{ij} = \sum_{0}^{\min(i,j)} (a_{i-k}a_{j-k} - a_{n+k-i}a_{n+k-j}), \quad 0 \leq i, j \leq n-1$$

or, more compactly,

$$S = LL^T - \tilde{L}\tilde{L}^T = \mathcal{A}_-\mathcal{A}_-^T - \mathcal{A}_+^T\mathcal{A}_+$$

where $L$ is a lower Toepliz matrix with first column $[a_0, \ldots, a_{n-1}]^T$ and $\tilde{L}$ also lower Toepliz with first column $[a_n, \ldots, a_1]^T$. Take as discrete-time Lyapunov function the quadratic form

$$V(x(k)) = x^T(k)Sx(k)$$

Show that $S$ is positive definite iff

$$\begin{aligned} &V(x(k+1)) - V(x(k)) = \\ &= -\left((1 - a_n^2)x_1(k) + (a_1 - a_{n-1}a_n)x_2(k) + \cdots + (a_{n-1} - a_1a_n)x_n(k)\right)^2 \end{aligned}$$

is negative and not identically zero except when $x(\cdot) \equiv 0$. Deduce a (so-called Shur-Cohn) test for checking if $a(z)$ has all roots strictly inside the unit circle. (See, e.g., A. Vieira and T. Kailath, IEEE Trans. Circuits Systems, April 1977, pp. 218-220)

## 1.9 'General' Linear Systems

**1.52** Assume that

$$\begin{pmatrix} z \\ y \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} w \\ u \end{pmatrix}$$

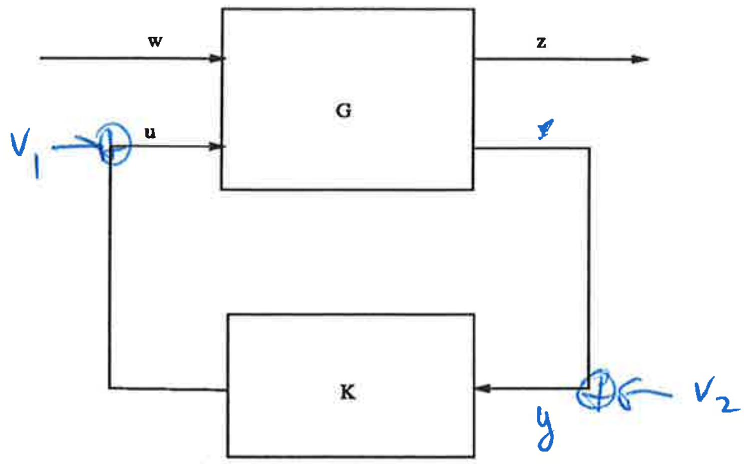


**Figure 1.2** **A Linear System**

where $G$ is real-rational and proper. To define what it means for the controller $U = K(s)Y$ to stabilize $G$ we introduce additional inputs $v_1$ and $v_2$ as in the figure.

**a.** Describe the equations relating the three inputs $w, v_1, v_2$ and the three signals $z, u, y$.

**b.** Assume that $G_{22}$ is strictly proper. Describe the closed loop transfer function from $w$ to $z$.

a.

# 2. Examples

**2.1** In IEEE AC, Dec 84, there is an article about control of a wind tunnel. Compute the transfer function from $u$ to $y = \begin{pmatrix} \partial M & \partial\Theta \end{pmatrix}^T$, (Hint: see eq. (3))

**2.2** In Int. J. Control, 1984 pp 1351-1365, there is a description of a triple inverted pendulum. Find a control law that stabilizes the system. Simulate it.

**2.3** An inverted pendulum, of mass $m$, is hinged at $A$. A gyro with spin angular momentum, $h$, is attached to the pendulum but is free to rotate about the pendulum axis (angle $\phi$) as shown in Fig. 2.

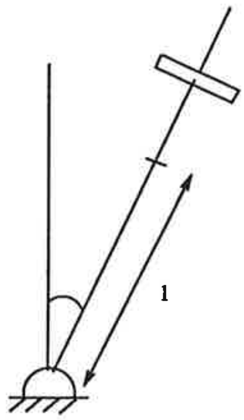


**Figure 2.1** Inverted pendulum with gyro

A control torque, $Q$, can be applied to the gyro from the pendulum. The equations of motion are

$$I\ddot{\theta} = mgl\theta - h\dot{\phi}$$
$$J\ddot{\phi} = h\dot{\theta} + Q$$

Where $I$ is the moment of inertia of pendulum plus gyro about $A$, $J$ is the moment of inertia of the gyro about axis AC, and $C$ is the mass center of the pendulum plus gyro.

a. Compute the transfer functions form $u$ to $\phi$ and $\Theta$

b. Show that the system is controllable by $Q$, observable with $\phi$, and unobservable with $\Theta$.

c. Show that the open loop system is always unstable.

# 3. State Feedback and Observers

## 3.1 State Feedback

**3.1** Show that the relative order of a linear system is not affected by state-variable feedback.

**3.2** Let

$$(sI - A)^{-1}b = \frac{1}{a(s)} \begin{pmatrix} p_{n-1}(s) \\ \vdots \\ p_0(s) \end{pmatrix}$$

Show that the common roots of the $n+1$ polynomials $\{p_{n-1}(s), \ldots, p_0(s), a(s)\}$ are exactly the uncontrollable natural frequencies of $\{A, b\}$.

**3.3** Let $b(s)/a(s)$ be an irreducible transfer function and write

$$\begin{aligned} a(s)\xi(s) &= u(s) \\ y(s) &= b(s)\xi(s) \end{aligned}$$

**a.** Show that the knowledge of $\xi$ and its derivatives determines the state variables of any minimal realization of $b(s)/a(s)$. (Hint: Controller-form). Therefore $\xi(s)$ is often called the *partial state* of the system.

**b.** Show that constant state-feedback corresponds to polynomial feedback of the partial state $\xi(s) : u(s) = v(s) - l(s)\xi(s)$ for some polynomial $l(s)$ of degree $\leq n - 1$.

**c.** Show that with such a feedback the new transfer function is $b(s)/(a(s) + l(s))$

**3.4** Show that if $\mathcal{C}(A, b)$ has rank less than $k$, state $x_0$ can be taken to $x_1$ in not more than $k$ steps if $x_1$ and $x_0$ lie in the space spanned by the columns of $\mathcal{C}(A, b)$.

### Examples

**3.5** Solve some state feedback problems using the functions `place` and `acker` in Matlab. When and why do these functions not work reliably ?

**3.6** The linearized equations of a free pendulum are $\ddot{\Theta} + \omega_0^2\Theta = u$. Show that output feedback $-k\Theta$ will not stabilize the system but that this can be done using a system $(s + a)/(s + b)$ in the feedback path. What are the conditions on $a, b$ ? Do this example in as many ways you can.

**Figure 3.1** Inverted pendulum and cart

**3.7** Find a stabilizing state-feedback law for an inverted pendulum with cart. Use the velocity of the cart as input signal, assume you can measure $z, \phi, \dot{\phi}$ and that $m << M$.

**3.8** A helicopter near hover can be described by the equations

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -0.02 & -1.4 & 9.8 \\ -0.01 & -0.4 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 9.8 \\ 6.3 \\ 0 \end{pmatrix} u$$

where

$$\begin{aligned} x_1 &= \text{horizontal velocity} \\ x_2 &= \text{pitch rate} \\ x_3 &= \text{rotor tilt angle} \end{aligned}$$

a) Find the open-loop poles

b) Find a state-feedback law that moves the poles to $s = -2, -1 \pm i$.

c) Design a velocity-command controller for this system

**3.9** Consider the helicopter example above with an extra additive disturbance term $\begin{pmatrix} -0.02 & -0.01 & 0 \end{pmatrix}^T w$ describing headwind. Design a velocity-command controller that incorporates integral-error feedback using state-space feedback. Put the new pole in $-1$.

**3.10** The approximate equations of motion in pitch and plunge for an airplane in nearly steady, horizontal flight (see Fig. 6) can be shown to be

$$\begin{aligned} \tau\dot{\gamma} &= \alpha \\ \ddot{\theta} &= -\omega_0^2(\alpha - Qu) \\ \dot{h} &= V\gamma \end{aligned}$$

where

$\gamma = \theta - \alpha =$ flight path relative to horizontal

$\theta =$ pitch angle pertubation

$\alpha =$ angle-of-attack pertubation

$V =$ magnitude of vertical velocity (assumed constant)

$u =$ elevator deflection (control)

$Q =$ elevator effectiveness

$\tau =$ lift time constant

$\omega_0 =$ natural frequency in pitch plunge

$h =$ height above reference altitude

An autopilot is to be designed to keep $h \approx 0$ in the presence of the vertical wind disturbances.

a. Show that proportional error feedback from altitude, $u = -kh$ can not stabilize the system.

b. Using a gyro to also measure $\theta$, show that a stablilizing control law $u = -k_1 h - k_2 \theta$ can be found.

**3.11** Use Matlab to reproduce the example 3.3.3 on p.212-214

# 4. Observers and Combined Design

## 4.1 Observers

**4.1** Obtain a formula for $k$ in terms of $\{A, c\}$ and the coefficients $\{\alpha_i\}$ of the desired characteristic polynomial $\alpha(s) = \det(sI\text{-}A\text{+}kc)$ (Ackerman)

**4.2** If

$$\dot{x} = Ax(t) + Bu(t), \quad x(t_0) = x_0$$
$$y(t) = Cx(t)$$

let $\hat{x}(t)$ obey

$$\dot{\hat{x}} = F\hat{x} + Gu + Hy, \quad \hat{x}(t_0) = \hat{x}_0$$

A natural requirement on an observer is that $\hat{x}_0 = x_0 \Rightarrow \hat{x}(t) = x(t), t \geq t_0$. Show that a necessary and sufficient condition for this is that $F = A - KC, H = K, G = B$, where $K$ is an arbitrary matrix. What if there is a direct term $d$ from $u$ to $y$ ?

**4.3** Given a realization

$$\dot{x} = Ax + bu$$
$$y = cx$$

consider an observer

$$\dot{\hat{x}} = \tilde{A}\hat{x} + \tilde{b}u + \tilde{b}l(y - \hat{y})$$
$$\hat{y} = \tilde{c}\hat{x}$$

where the $\{\tilde{A}, \tilde{b}, \tilde{c}\}$ are estimates of $\{A, b, c\}$. Show that

$$\dot{\epsilon} = (\tilde{A} - \tilde{b}\tilde{c})\epsilon + (\delta A - \tilde{b}l(\delta c))x + (\partial b)u$$

where $\epsilon = x - \hat{x}$, $\partial A = A - \tilde{A}$ etc. Further analysis allows one to relate these results to questions of system sensitivity. (Ref: W.A. Porter IEEE Feb. 77 pp. 144-146)

**4.4** For the station-keeping satelite of Example 3.3-2, design an observer using measurements $y$ of azimuthal position pertubation. Place the observer poles at $s = (-2\omega, -3\omega, -(3 \pm i)\omega)$, which means that the estimate errors will decay in about $2\frac{1}{2}$ days.

**4.5** Develop observer designs for discrete-time systems and show how to design an observer for which the error will go to zero in no more than $n$ steps, where $n$ is the number of states.

**4.6**

a. Design an observer, using plant augmentation, so that the transfer function from measurement noise to output has a notch at frequency $\omega$.

b. Design an observer, using plant augmentation, so that the transfer function from input disturbance to output has a notch at frequency $\omega$.

## 4.2 Combined Observers and State-feedback

**4.7** Describe the state-feedback observer configuration using $x$ and $\tilde{x} = x - \hat{x}$ as state variables. Calculate $H_{o-c}(s)$ and $a_{o-c}(s)$.

**4.8** The realisation in Fig 4.2.1 is not minimal.

a) Show that the realisation in not controllable. What are the uncontrollable modes ?

b) Show that the realization is nonobservable iff at least one of the following holds

(1)$\{k, A - lc\}$ is not observable

(2)$\{c, A - bk\}$ is not observable

(3) A pole of the observer cancels a zero of the augmented system

It is assumed that $\{A, b, c\}$ is minimal.

**4.9** In the combined controller-observer design we select $k$ and $l$ so that $a_c(s)$ and $a_o(s)$ both have stable poles in the left half plane. Is it true that the resulting design must be stable even if the loop is broken open at $y$, for instance ?

**4.10** Simulate the observer/state-feedback design for the inverted pendulum described in TFRT 7405 using Simnon and Matlab. Note especially the slow observation on the mode describing the position of the wagon.

### Reduced Order Observers

**4.11** Obtain a reduced order observer for the tank system used in our AK-labs. Measure the level of tank two.

**4.12** Write a Matlab function `Luenberg(A,B,C,D)` calculating a reduced order observer.

**4.13** Show that the subsystem $\{c_r, A_r\}$ defined by K. (4.3-3) will be observable if $\{c, A\}$ is observable. Do this

a. By direct manipulation of $\mathcal{O}(c_r, A_r)$

b. By using the PBH test

**4.14** An error model of the east-velocity channel of an inertial navigator is

$$\begin{pmatrix} \dot{v} \\ \dot{\phi} \\ \dot{\epsilon} \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} v \\ \phi \\ \epsilon \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}$$

where $v$ = east-velocity error, $\phi$ = platform tilt about north axis, $\epsilon$ = north-gyro drift, and $w$ = gyro drift rate of change. Construct a second-order observer using $z = v$ as the observation, placing the estimation poles at $\begin{pmatrix} -0.1 & -0.1 \end{pmatrix}$

## 4.3 Polynomial methods

**4.15** A system is described by the transfer function

$$Y(s) = \frac{(s+1)^2}{s^2} U(s)$$

Find a dynamic compensator so that the closed loop transfer function becomes

a.

$$Y(s) = \frac{(s+8)(s+1)}{(s+2)(s^2+2s+4)} Y_r(s)$$

b.

$$Y(s) = \frac{(s+1)}{(s^2+2s+4)} Y_r(s)$$

c. Is the system controllable from $y_r$ ? Observable from $y$ ?

**4.16** Given $(s-1)/s^2$ find a compensator giving closed loop characteristic polynomial $(s+5)(s^2+s+1)$. Is there a unique solution to this problem. If not, give some criteria for chosing among the different possible solutions.

**4.17** Show that for a system with closed-loop poles at $\{-p_i\}$ and zeros at $\{-z_i\}$ the steady-state error in the response to a unit ramp is

$$\sum \frac{1}{p_i} - \sum \frac{1}{z_i}$$

**4.18** In Eqs. (4.5-4) assume that $\delta(s) = 1$. Show that in Fig 4.5.1 the numerators $n_u(s)$ and $n_y(s)$ can be found from the linear equation

$$S(a,b)\begin{pmatrix} n_y \\ n_u \end{pmatrix} = \begin{pmatrix} 0 \\ \alpha - a \end{pmatrix}$$

where $S(a,b) = \text{Res}(a,b)$ is the Sylvester matrix of $\{a,b\}$, $n_y$ is the coefficient vector corresponding to $n_y(s)$ and similarly for $n_u, \alpha$ and $a$.

**4.19** Suppose the original transfer function is realised in controller form. Calculate the feedback gain $L$ required to give new characteristic polynomial $\alpha(s)$, and assume that the states $x(t)$ are provided by an ideal observe. Calculate $H_u(s)$ and $H_y(s)$ for this compensator and show that they coincide with $n_y(s)$ and $n_u(s)$ found in the previous exercise.

**4.20** Show that a necessary condition for *strong stabilisation*, i.e. stabilisation using a stable compensator, is that between every pair of unstable zeros ($\infty$ is considered a zero if the system is strictly proper) there is an even number of poles. (Hint: Use a root-loci argument) One can show that this is also a sufficient condition.

**4.21** Consider the undamped harmonic oscillator

$$\dot{x} = \begin{pmatrix} 0 & \omega_0 \\ -\omega_0 & 0 \end{pmatrix} x + \begin{pmatrix} 0 \\ \omega_0 \end{pmatrix} u, \quad \omega_0 = 1$$
$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} x$$

Let $a_m(s) = s^2 + 2\zeta\omega s + \omega^2$, $\omega = 1.5, \zeta = 0.7$. Compare different observer designs. Study step response, input disturbance $v$ and ouput measurement noise $e$. (Plot e.g. Bode diagrams of $H_v$ and $H_e$.)

a) No integral action, $a_o(s) = s^2 + 2\zeta\omega_{obs}s + \omega_{obs}^2$, vary $\omega_{obs} = 3, 4, 8$

b) Integral action, reduced order, $a_o(s) = s^2 + 2\zeta\omega_{obs}s + \omega_{obs}^2$, vary $\omega_{obs} = 3, 4, 8$

c) Integral action, reduced order, $a_o(s) = (s + a_1)(s + a_2)$. Try to find $a_1$ and $a_2$ that give smaller effect of the measurement noise.

# 5. Brockett

## 5.1 Linear Independence and Linear Mappings

**5.1** Br 1.3

**5.2** Br 1.4

**5.3** Br 1.12

**5.4** Br 1.13

## 5.2 Uniqueness of Solutions Given the State: Linearization

**5.5** Br 2.1

**5.6** Br 2.9

## 5.3 The Transition Matrix

**5.7** Br 3.2

**5.8** Br 3.5

**5.9** Br 3.6

**5.10** Br 3.7

**5.11** Br 3.9

**5.12** The Wronski-determinant of the functions $u(t)$ and $v(t)$ is definied by

$$W(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$$

The functions are called *independent* in an interval if $W \neq 0$ in this interval. Let $u$ and $v$ be solutions to a second order homogenuous linear equation

$$y'' + a(t)y' + b(t)y = 0$$

**a.** Find a first order differential equation for $W$

**b.** Show that $u$ and $v$ are independent if $W$ is non-zero for a single $t$-value.

**c.** Show that if $a(t) \equiv 0$ then $W$ is constant.

**d.** Rewrite the differential equation to state space form. Calculate trace and determinant of the system matrix.

**5.13** The Wronski-determinant of $n$ functions is defined by

$$W(u_1, u_2, \ldots, u_n) = \begin{vmatrix} u_1 & u_2 & \cdots & u_n \\ u_1' & u_2' & \cdots & u_n' \\ \vdots & \vdots & \ddots & \vdots \\ u_1^{(n-1)} & u_2^{(n-1)} & \cdots & u_n^{(n-1)} \end{vmatrix}$$

The funtions are said to be independent on an interval if $W$ is non-zero there. Show that if they are solutions to a homogenuous linear differential equation

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y^{(1)} + a_0(t)y = 0$$

and if the Wronski-determinant is zero for one time $t$ then they are independent.

**5.14** Show Libris theorem: If $n$ independent solutions $y_1, y_2, \ldots, y_n$ to the linear homogenuous differential equation

$$y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y^{(1)} + a_0(t)y = 0$$

are known then the coefficients of the equation can be determined uniquely. How ?

**5.15** Describe a system of ODE for $\alpha_i(t), i = 0, \ldots, n-1$ such that

$$e^{At} = \sum_{i=0}^{n-1} \alpha_i(t) A^i$$

Show that $\alpha_i(t)$ are linearly independent. (Hint: Use the characteristic equation)

## 5.4 Three properties of $\Phi$

**5.16** Br 4.3

**5.17** Show that if $f$ is a divergence free vector field, i.e. $\text{div} f \equiv \text{tr}(f_x') = 0$ then the flow of $f$ preserves volume:

$$\det\left(\frac{\partial \Phi_{t,t_0}}{\partial x_0} \equiv 1\right)$$

## 5.5 Matrix exponentials

**5.18** Br 5.2

**5.19** Br 5.5

**5.20** Br 5.7

**5.21** Br 5.9

**5.22** Br 5.11

## 5.6 Inhomogenous Linear Equations

**5.23** Br 6.3 (but disregard the Neymann-Pearson lemma)

## 5.7 Adjoint Equations

**5.24** Br 7.1

**5.25** Br 7.2

## 5.8 Periodic Homogenous Equations:Reducibility

**5.26** Br 8.11

## 5.9 Periodic Inhomogenous Equations

**5.27** Br 9.2

**5.28** Br 9.5

## 5.10 Some Basic Results of Asymptotic Behavior

**5.29** Br 10.6

## 5.11 Linear Matrix Equations

**5.30** Br 11.3

## 5.12 Structure of Linear Mappings

**5.31** Br 12.5

**5.32** Br 12.8

## 5.13 Controllability

**5.33** Br 13.4

**5.34** Br 13.9

**5.35** Br 13.19

## 5.14 Observability

**5.36** Br 14.5

## 5.15 Weighting Patterns and Minimal Realizations

**5.37** Br 15.1

## 5.16 Stationary Weighting Patterns: Frequency Response

**5.38** Br 16.1

**5.39** Br 16.2

**5.40** Br 16.7

## 5.18 McMillan Degree

**5.41** Br 18.1

**5.42** Br 18.3

**5.43** Br 18.4

# 6. State-Space and MFD:s

## 6.1 Direct Realizations

**6.1** Draw pedagogical figures of realizations 6.1.1. and 6.1.2 on p. 348 and the Gilbert realization on p. 349.

**6.2** Make a Gilbert realization of the (MacFarlane-) system

$$G(s) = \frac{1}{(s+1)(s+2)} \begin{pmatrix} s-1 & s \\ -6 & s-2 \end{pmatrix}$$

**6.3** Use Macsyma (or Maple) to find $G(s)$ for

$$A = \begin{pmatrix} -4 & -4 & 0 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & -5 & -2 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

## 6.2 Observability and Controllability; MFD

**6.4** Check that the proofs for the scalar PBH-tests carry over without a change to 6.2.5 and 6.2.6 on p. 366.

**6.5** K. 6.2.6

**6.6** K. 6.2.9

**6.7** K 6.2.10

**6.8** Assume that we have a right MFD of the system,

$$y = B(s)A^{-1}(s)$$

and a left MFD of the controller,

$$R(s)u = -S(s)y + T(s)u_c$$

Show that the closed loop satisfies

$$y = B(s)(R(s)A(s) + S(s)B(s))^{-1}T(s)u_c$$

Note that the order of the polynomial matrices are important.

## 6.3 Some Properties of Polynomial Matrices

**6.9** Use Holmbergs Macsyma package to find the Hermite form in Example 6.3-1.

**6.10** K. 6.3.2

**6.11** K. 6.3.12

**6.12** K. 6.3.13

**6.13** K. 6.3.17

**6.14** Determine a left MFD from the right MFD (48) on p.369. Use lemma 6.3.8 or Holmberg's package.

**6.15** What are the invariant polynomials $\lambda_i(s)$ for a polynomial matrix $sI - A$ where $A$ is on block Jordan form ?

**6.16** Instead of polynomial matrices some people prefer to work with matrices having rational elements that are proper and stable. For example

$$\frac{s-1}{s-2} = NM^{-1}, \qquad \text{where } N = \frac{s-1}{s+\alpha}, M = \frac{s-2}{s+\alpha}, \alpha > 0$$

The Bezout identity is then easier to understand and work with: Assume that a MIMO $G(s) = [A, B, C, D]$ (minimal) is given. Check that (or at least check some of the idenities):

$$G = NM^{-1} = \tilde{M}^{-1}\tilde{N}$$

$$\begin{pmatrix} \tilde{X} & -\tilde{Y} \\ -\tilde{N} & \tilde{M} \end{pmatrix} \begin{pmatrix} M & Y \\ N & X \end{pmatrix} = I$$

where $M, N, X, Y, \tilde{M}, \tilde{N}, \tilde{X}, \tilde{Y}$ are obtained as follows: Choose a feedback vector $L$ such that $A - BL$ is stable and a $K$ such that $A - KC$ is stable (this is possible since the system were assumed minimal, see chap. 7). Then

$$\begin{aligned} M(s) &= [A - BL, B, -L, I] \\ N(s) &= [A - BL, B, C - DL, D] \\ X(s) &= [A - BL, K, C - DL, I] \\ Y(s) &= [A - BL, K, -L, 0] \\ \tilde{M}(s) &= [A - KC, -K, C, I] \\ \tilde{N}(s) &= [A - KC, B - KD, C, D] \\ \tilde{X}(s) &= [A - KC, -(B - KD), -L, I] \\ \tilde{Y}(s) &= [A - KC, K, -L, 0] \end{aligned}$$

## 6.4 Basic State-Space Realizations

**6.17** K 6.4.1, use Holmberg's package or Matlab.

**6.18** K 6.4.2

**6.19** K 6.4.8

**6.20** K 6.4.11

## 6.5 Some properties of Rational Matrices

**6.21** K. 6.5.1

**6.22** K. 6.5.3

**6.23** K. 6.5.12

**6.24** K. 6.5.13

**6.25** K. 6.5.14

# 7. State Feedback & Compensator Design

## 7.1 Analysis of Feedback, State-Space

**7.1** K 7.1.1

**7.2** Consider the linear constant coefficient DAE

$$E\dot{x} = Ax + f \qquad (*)$$

If we let $x = Qy$ and premultiply by $P$ where $P, Q$ are nonsingular matrices (*) becomes

$$PEQ\dot{y} = PAQy + Pf$$

and the pencil $\lambda E - A$ is changed to $\lambda PEQ - PAQ$. The nature of the solutions to (*) are thus determined by the canonical form of the pencil $\lambda E - A$ under the transformations $P, Q$. The DAE is 'solvable' (a definition we will not go into detail about) iff $\lambda E - A$ is a regular pencil. Assume that $\lambda E - A$ is regular in the following.

**a.** Show, using the Kronecker form, that there exists $P, Q$ such that

$$PEQ = \begin{pmatrix} I & 0 \\ 0 & N \end{pmatrix}, \quad PAQ = \begin{pmatrix} C & 0 \\ 0 & I \end{pmatrix}$$

where $N$ is a nilpotent matrix, $N^k = 0$. The smallest such $k$ is called the **index**.

**b.** After the coordinate change, (*) becomes

$$\begin{aligned} y_1' + Cy_1 &= f_1 \\ Ny_2' + y_2 &= f_2 \end{aligned}$$

Show that

$$y_2 = \sum_{i=0}^{k-1} (-1)^i N^i f_2^{(i)}$$

Note that

1. The solution of (*) involve derivatives of order $k-1$ of the forcing function $f$.

2. Not all initial conditions admit a smooth solution if $k \geq 1$. Those that do admit solutions are called *consistent initial conditions*

3. Higher index DAE's can have hidden algebraic constraints.

c. What is the index for the problem

$$\begin{aligned} x_1' + x_3 &= f_1 \\ x_2' + x_1 &= f_2 \\ x_2 &= f_3 \end{aligned}$$

This equation has one explicit algebraic constraint but also two implicit.

d. Show that the system of equations in c) can be described by the ODE

$$\begin{aligned} x_1' &= f_2' - f_3'' \\ x_2' &= f_3' \\ x_3' &= f_1' - f_2'' + f_3''' \end{aligned}$$

*Remark.* The index of this problem is 0. Note that initial conditions for $x_i$ are needed for equivalence. Note also that this system requires that $f_3$ be three times differentiable but that the system above only required that $f_3$ be twice differentiable. The DAE-'index' is a formulation dependent entity, the same problem can have different index when modeled differently. There are some working numerical methods for index 1 models, higher index models can be numerically very difficult to simulate. A high index model often needs remodeling.

e. Suppose $E\dot{x} = Ax + f$ is a solvable linear constant DAE with index $k \geq 1$. Show that $(E - \mu A)^{-1}$ has a pole of order $k$ at $\mu = 0$.

## 7.2 Analysis of Feedback, Transfer Function

**7.3** To further discuss the freedom in choice of feedback matrix for MIMO systems, consider the open loop system ([Moore, AC Oct 1976, pp. 691])

$$A = \begin{pmatrix} -1.25 & 0.75 & -0.75 \\ 1 & -1.5 & -0.75 \\ 1 & -1 & -1.25 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has controllable eigenvalues -1.25, -2.25 and an uncontrollable eigenvalue at -0.5. Assume that the control objective is to shift the eigenvalues to -5,-6,-0.5 and to obtain a reasonable response for the initial condition $\begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T$, which represents a disturbance in $x_3(t)$.

a. Check that $L = \begin{pmatrix} 4.53 & -4.53 & 6.69 \\ 1.0784 & -1.0784 & 4.04 \end{pmatrix}$ gives the correct poles and simulate the system (hint: matlab and `lsim`).

b. The design in a) is not satisfactory since the slow uncontrollable mode is present in $y_1$. Find a state feedback so that the eigenvector corresponding to the uncontrollable mode is $X_3 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$, and hence is unobservable in the output.

7.4

a. Show that if $G(s) = N(s)D^{-1}(s) = C(sI - A)^{-1}B$ with $D(s)$ column reduced and $(A, B, C)$ controllable, then the column indices equals = the Scheme 2 indices = the Kronecker indices of $[sI - A \;\; B]$.

b. Consider the two integrator chains

$$s^2 y_1 = u_1$$
$$s^2 y_2 = u_2$$

Is it possible to find a realization $G = \tilde{N}\tilde{D}^{-1}$ where $\tilde{D}$ has invariant polynomials $\psi_1 = s^3, \psi_2 = s$? Is it possible to find a state space realization where $sI - A$ has $s^3, s$ as invariant polynomials ?

c. Find a state space feedback matrix $L$ such that $sI - A_c + B_c L$ has invariant polynomials $s^3, s$. Use

## 7.3 State Observers

7.5 K 7.3.1

7.6 Make a matlab function `luenberger.m` that calculates a reduced order observer for MIMO-systems.

7.7 K 7.3.2

7.8 Consider the flexible servo with two tachometers and two motors

$$x_1 = \omega_1$$
$$x_2 = \omega_2$$
$$x_3 = \theta_1$$
$$x_4 = \theta_2$$

$$\dot{x} = \begin{pmatrix} 0 & 0 & -a_1 & a_1 \\ 0 & 0 & a_2 & -a_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} x + \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u$$

$$y = \Theta = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x$$

a. Find a MFD for the system on the form $(Js^2 + Ds + K)\Theta = M$.

b. Calculate the invariant polynomials for this MFD.

c. Calculate the controllability indices.

d. Choose state feedback$u = v - Lx$ to obtain invariant polynomials $\psi_1 = \psi_2 = s^2 + 2\zeta\omega s + \omega^2$. (If symbolic computations are problematic use numerical values $a_i = b_i = 1, \zeta = 0.7, \omega = 1$)

e. Calculate a reduced order observer and place the two observer poles in $s^2 + 2\zeta_o\omega_o s + \omega_o^2$. $(\zeta_o = 0.7, \omega_o = 2)$.

## 7.5 Transfer Function Design of Compensators

**7.9** Interpret the design in the flexible servo problem above as a transfer function design. Find $P(s)$ and $\Delta(s)$ and calculate their invariant polynomials.

**7.10** K 7.5.1

**7.11** K 7.5.2

**7.12** K 7.5.4

**7.13** K. 7.5.6.

**7.14** (hard) Generalize the popular SISO pole placement procedure to MIMO. What is the corresponding condition to 'no cancellation of unstable zeros' ?

## 7.6 Geometrical Theory

**7.15** Prove that the orthogonal complement $\mathcal{A}^{\perp}$ has the representations $\mathcal{A}^{\perp} = \text{Im}(A_2^T) = \ker(A_1^T)$.

**7.16** Show that $A^{-1}\mathcal{B} = \text{Ker } (\mathcal{B}^{\perp}A)$.

**7.17** Show that $\sigma$=observabilityindex of $(A, C)$. How is $\dim(\mathcal{V}_i)$ related to all the observabilityindices and the "Crate 2"-diagram in [Kailath] ?

**7.18** Write a Matlab function for Kalman decomposition.

**7.19** Show that

$$\mathcal{V}_i = \{x \mid \exists u_0, \ldots, u_{i-1} \text{ such that } x_t \in \mathcal{S} \ \forall t = 0, \ldots, i\} \tag{7.1}$$

Conclude that if $\mathcal{S} =$ ker $(C)$ then $x \in \mathcal{V}_i \iff \mathcal{O}_i x \subset$ Im $(T_i)$ where

$$\mathcal{O}_i = \begin{pmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^i \end{pmatrix} \qquad T_i = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ CB & 0 & \cdots & 0 \\ CAB & CB & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ CA^{i-1}B & \cdots & CAB & CB \end{pmatrix} \tag{7.2}$$

(see p. 80 [Kailath] or [AK]).

**7.20** Construct an exercise that illustrates theorem 5 in [BoB] on general feedforward feedback.

**7.21** Show that $\langle A + BF \mid \mathcal{B} \rangle = \langle A \mid \mathcal{B} \rangle$ (Feedback do not change controllability.)

**7.22**

If $\{A, B\}$ is controllable and $b \in \text{Im}(B)$ then there is a $F$ such that $\{A+BF, b\}$ is controllable. (It is theoretically only necessary with one control signal + feedback.)

**7.23** Show that every controllability subspace is a subspace of the controllable subspace $\langle A \mid \mathcal{B} \rangle$.

**7.24** Show that if $CB$ is invertable then Ker$C$ is $(A, B)$ invariant. What will the poles of $(A + BL)|_{\text{ker } C}$ be ?

# 8. PMD:s

## 8.1 PMD and System Matrices

**8.1** Regard a single input single output state space system $S(A, B, C)$ on controller form. Introduce its system matrix **P(s)**, and describe operations to obtain the system matrix of a corresponding right MFD.

**8.2** Regard the single input single output PMD $\{P, Q, R, W\}$

$$\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} s + \begin{pmatrix} 1 & 1 \\ 1 & \epsilon \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \end{pmatrix}\right\}$$

for $\epsilon \neq 0$and $\epsilon = 0$. Determine operations on the system matrix to form a corresponding PMD $\{A(s), B(s), 1, J(s)\}$, with a strictly proper $B/A$.

## 8.2 State-Space PMDs and System Equivalence

**8.3** Find, if possible, matrices that will relate the system matrices (SISO)

$$\begin{pmatrix} a(s) & b(s) \\ c(s) & 0 \end{pmatrix} \quad \text{to} \quad \begin{pmatrix} a(s) & c(s) \\ b(s) & 0 \end{pmatrix}$$

Under what conditions are 8.2.20 fulfilled ?

**8.4** K 8.2.3a

**8.5** K 8.2.4

**8.6** K 8.2.9

**8.7** Transform a reasonable sized PMD to state-space form.

**8.8** Check (16-18) on page 561.

**8.9** Check (23) on page 562.

## 8.3 Properties and Applications of System Equivalence

**8.10** K 8.3.1

**8.11** K 8.3.9

**8.12** One way to analytically investigate the poles after high gain output feedback is directly using a minimal MFD:

$$A(s)y(s) = B(s)u(s), \qquad u(s) = k\,(v(s) - y(s))$$

If we now regard both $y$ and $u$ as output we get the resulting MFD:

$$\begin{pmatrix} -A(s) & B(s) \\ kI & I \end{pmatrix} \begin{pmatrix} y(s) \\ u(s) \end{pmatrix} = \begin{pmatrix} 0 \\ kv(s) \end{pmatrix}$$

or after unimodular row-operations

$$\begin{pmatrix} 0 & B(s) + \frac{1}{k}A(s) \\ kI & I \end{pmatrix} \begin{pmatrix} y(s) \\ u(s) \end{pmatrix} = \begin{pmatrix} A(s)v(s) \\ kv(s) \end{pmatrix}$$

so that nonzero $y$ and $u$ may exist for zero $v$, if $y(t) = y_0 e^{s_i t}$ and $u(t) = u_0 e^{s_i t}$, where the closed loop poles $s_i$ are the zeros of $B(s) + A(s)/k$, and the poles that don't go to infinity go to the zeros of $B(s)$. For $B(s)$ with less than full column rank we have to be careful. The arbitrary zeros will show up somewhere.

Regard the example

$$A(s) = \begin{pmatrix} s^2 & 2c^2 \\ 0 & s^2 \end{pmatrix} \qquad B(s) = (s+1)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Show that one pole goes off to infinity as $-2k$, while the other three approach $\pm c$ (unexpected) and $-1$.

# Part 2

## The Course 1989

# Linear Systems

## Course Program fall 89

### Meeting time:

Monday 13.15–15.00 and Thursday 15.15–17.00 (problem sessions). First meeting is on Sept 4.

### Examination:

- Take home exam, Dec 18 to 21
- Homework problems handed in every week
- Problem solution using our program packages

### Participants:

Stefan Diehl

Anders Hansson

Klas Nilsson

Anders Persson

Stéphane Sallé

### Literature:

KAILATH, T. (1980): *Linear Systems*, Prentice-Hall Inc., Englewood Cliffs, NJ.

BERNHARDSSON, B. (1989): "Exercises in Linear System Theory TFRT-xxxx, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden,".

BROCKETT, R. W. (1970): *Finite Dimensional Linear Systems*, (out of print), John Wiley & Sons, New York.

ÅSTRÖM, K. J. (1976): *Reglerteori (2nd ed)*, Almqvist & Wiksell Förlag AB, Stockholm.

### Course outline:

We start with the first five chapters of Kailath while brushing up our knowledge of AK (Åström Reglerteori). This should be finished around Oct 1. There are few new concepts introduced. The main thrust is the exercises and maybe an introduction to Kailath's way of writing.

Then comes Brockett and some handouts on General System Theory and systems as linear maps between function spaces. Time varying linear systems are introduced and played with. There are several recommended exercises, and this part is conceptually heavy, but it has to be finished by Nov 1.

The core of the course is the material in chapters 6-8 of Kailath describing systems using polynomial matrices in the Laplace or shift operator. Concepts like zeros are thus generalized to multivariable systems. The relation to "natural statespace realizations" is the best part of Kailath, while the connection with geometrical theory (Wonham) may need further reading, eg. a report by Gunnar Bengtsson.

# List of Exercises Recommended 1989

**Note that the numbering has been changed. The following list uses the new numbering used in part1.**

**1** **Do (at least 20 of) 1.1-1.32 and 1.46. Most important are 1.14, 1.19, 1.26 and Matlab and Maple exercises like 1.15 and 1.46.**

**2** **Do (at least 15 of) 1.30-2.3**

**3** **Do (at least 15 of) in chapter 3 and 4.1-4.10**

**4** **Do (at least 9 of) 4.11-4.21**

**5** **5.1-5.11**

**6** **5.16-5.23**

**7** **5.24-5.29**

**8** **5.30-5.36**

**9** **5.37-5.43**

**10** **6.1-6.8**

**11** **6.9-6.15, 6.16-6.18**

**12** **6.19-6.25**

**13** **7.1-7.12**

**14** **7.15-7.24,8.1-2**

**15** **8.3-8.12**

# Translations of Exercise Numbering

**The notation used in part 2 (1989) is not the same as in part 1. Some renumbering was necessary when the material was worked through after the course using the recommendations from the students. I would like to thank Stefan Diehl, Anders Hansson, Klas Nilsson and Anders Persson, for these suggestions of improvement.**

| Old Number (1989) | New Number (as in Part 1) |
|---|---|
| 1.31 | 1.35 |
| 1.32 | 1.31 |
| 1.33 | 1.32 |
| 1.34 | 7.18 |
| 1.35 | 1.54 |
| 1.36 | 1.34 |
| 1.37 | 1.33 |
| 1.38 | 1.37 |
| 1.39 | 1.36 |
| 1.40 | 1.38 |
| 1.41 | - |
| 1.42-55 | 1.39-52 |
| 1.56-57 | 2.1-2 |
| 2.1-2.11 | 3.1-3.11 |
| 2.12-2.32 | 4.1-4.21 |
| 2.33 | - |
| Brockett | Chapter 5 |

## First session

During the first session an outline of the course is given, and the books are presented. The course is a continuation to AK, and that material should be fresh in mind. This means SISO, time-invariant, finite dimensional systems treated using $S(A, B, C, D)$ or $G(s)$, i.e. using internal or external models.

The main extension in the course is to MIMO-systems, and this implies that a lot of structure is added to the problem. $G(s)$ will now be a matrix of transfer functions.

As known from SISO systems, the transfer function description contains no information about modes that are uncontrollable or unobservable. In AK we often regard $G(s)$ as the quotient between the two polynomials $B(s)$ and $A(s)$, and those two polynomials start to live their own life. We introduce representations like

$$A(s)y(s) = B(s)u(s) \tag{8.1}$$

and we study the properties in terms of poles, zeros, and relative degree. When we investigate how common factors correspond to unobservable or uncontrollable modes, we normally do this in the state space internal descriptions, but by introducing an 'internal state' $z(s)$ we can rewrite the equation as

$$\begin{cases} A(s)z(s) = B(s)u(s) \\ y(s) = z(s) \end{cases} \tag{8.2}$$

or as

$$\begin{cases} A(s)z(s) = u(s) \\ y(s) = B(s)z(s) \end{cases} \tag{8.3}$$

Actually common modes in the first representation correspond to uncontrollable modes, while they correspond to unobservable modes in the second representation. A natural extension is a mixed representation

$$\begin{cases} A(s)z(s) = B(s)u(s) \\ y(s) = C(s)z(s) \end{cases} \tag{8.4}$$

Motivation for the polynomial representation is e.g. an easy understanding of interconnected systems composed of first and second order subsystems and a good intuition when approximating fast time-constants using direct transmission terms. The polynomials are often given as products of polynomials of low order. Another reason would be the direct connection with 'asymptotes' of Bode-diagrams and root-loci.

We generalize the polynomial viewpoint to MIMO-systems, and in doing so a number of important tools have to be introduced. The two 'semi-internal' representations above (2) and (3) get a special name, Matrix Fraction Descriptions, and the $A$'s and $B$'s for the two representations of a certain $G(s)$ now normally differ. The mixed formalism (4) is often called the Rosenbrock system matrix description. A more neutral name is Polynomial Matrix Description.

The way to keep the ground solid is to consistently relate to 'natural' state space realizations. Actually we also introduce some new linear algebra tools to facilitate the understanding of the state space world as well. Some people call them the geometric linear system theory.

A recent version of (2)-(4) is to use proper stable transfer function matrices $A(s)$ and $B(s)$ instead of polynomial matrices. At the expense of some insight we are able to easily formulate things like "any stabilizing controller" for a given system.

One formalism suitable also for infinite-dimensional systems is the convolution of the input signal with the impulse response, and some insight will be given as to how one can view linear systems as maps between function spaces. This view is then extended also to the time-variable case.

A secondary objective of the course is to provide familiarity with the computer packages available at the department by using them in the solution of exercises and small projects.

### Reading assignment

This week pages 27-120 of Kailath should be read. They are mainly a repetition of AK apart from a few new concepts and some new notation that will be used in the sequel. Chapter 1 does not give an intuitive feeling for the setting of the book, so therefore skip sections 1.1-1.3. It dwells on subtleties in the definition of linear systems and the Laplace transform.

In Chapter 2 you are introduced to the world of analog-computer simulation as a way to discuss the realization of linear systems. The example in section 2.0 should be compared with the discussion on pp237 of the AK-book. One could also relate to the two alternative polynomial representations above (2) and (3).

Section 2.1 contains the canonical block-diagrams with $1/s$ as basic element, while section 2.2 shows the corresponding statespace forms. On p42 is the first new notation, the *Markov parameters*, and it reappears on p70, where they are used to form Hankel matrices $M[i, j]$. Notice also the philosophical statement on p55 that state variables are easily chosen as state variables for the elementary systems that make up a complicated system. Natural statevariable choices are thus important, eg. capacitor voltage and inductor current, the variables in which you normally specify initial conditions for a network (p56-p59). On p67 is a reference to the Appendix for resolvent identities and the adjugate matrix. Remember that the $S_i$'s were used in the AK-book (p46-p53) to construct transformation matrices to canonical forms. Be sure to consult the appendix whenever you have forgotten your linear algebra.

In section 2.3 the observability treatment actually follows the AK-book (p250). Eq (2) and (5.10) are the same. The treatment on p80 is simplified and complicated by the notation with the observability matrix and the lower triangular Toeplitz matrix. Notice how space is saved by using the transpose of a block-matrix of transposed blocks. The controllability is dual, but "the dual of a derivative is an impulse", so it becomes more complicated. Much more on duality later.

For discrete time systems we know that the mathematics becomes easier. Regard p101-2 as an exercise in the new notation.

Worked examples is really the time when the mathematics should prove useful. Both the pendulum and the air balloon could be used later on. It is nice to keep some physical examples in mind. On p108 is used the trick that a matrix times a vector is a linear combination of the columns forming the matrix. See also p29. Be sure to understand this. The trick will be used several times, sometimes without special reference. In this specific case the obtained expressions are used solve for $t_i$ recursively, and then use those results to form $T$. The $At_n$-expression is never used.

### Suitable exercises

Do (at least 20 of) 1.1–1.29 and 1.49. Some of the exercises are difficult. Do not get stuck on one exercise too long. Discussion is allowed and encouraged. Most important are 1.14, 1.19, 1.26 and Matlab and Maple exercises like 1.15 and 1.49.

### Hand in problems

The following problems should be handed in by Sept 15, 1989.

1.1 **8.13** Introduce a minimal state space realization for the transfer function

$$G(s) = \frac{A}{T_1 s + 1}\left(\frac{B}{T_2 s + 1} + \frac{C}{T_3 s + 1} + \frac{Ds}{T_3 s + 1}\right)$$

1.2 **8.14** Formulate and solve a reasonably sized modeling task resulting in a state representation with no less than two state variables. Be sure to keep physically relevant variables and parameters.

## Second session

### Reading assignment

This week pages 120-196 of Kailath should be read. For p121-122 think also in terms of Jordan blocks as in Exercise 2.3-16. The diagram on p125 summarizes the wellknown theorems 2.4-5 and 2.4-6 proven on p126-127. The proofs are made by splitting up into too many Lemmas that are "proven" by reference to old examples. Notice however the proof of Lemma 2.4-2, using the Hankel matrix $M[1, n-1]$. Thus both $M[1, n-1]$ and $M[1, n]$ are rank $n$ for an $n$-th order minimal system. Notice also the formula used for

$$H(s) = c(sI - A)^{-1}b = c\left(\sum A^{i-1}/s^{i}\right)b$$

and the corresponding formula for the impulse response

$$h(t) = ce^{At}b$$

The proof of the uniqueness in Theorem 2.4-7 needs one more line:

$$\mathcal{O}(c_1, A_1)\widetilde{T} = \mathcal{O}(c_2, A_2) = \mathcal{O}(c_1, A_1)T$$

The diagram on p129 is nice and deserves some second thoughts especially for the transformation matrices.

The whole discussion on p129-134 is leading to the Kalman decomposition theorem. If you are not well-acquainted with block matrices that might be an excuse for all the pages. It is now time to consult the Matlab Control-Toolbox manual. A controllable - uncontrollable decomposition can be made using numerically nice orthogonal transformations and similarly for unobservable - observable. However there is in general no orthogonal transformation that gives the combination, the Kalman decomposition.

The pages 135-138 are very important, and will be extended even further as coprimeness of matrix polynomials. When considering theorem 2.4-8.2 think of an eigenvector as the simpliest $A$-invariant subspace, that is when starting at $x_0 = v$ the derivative lies along $v$ and we remain along $v$. So if $cv = 0$, then the output will remain silent.

When considering Theorem 2.4-9.1 remember that $[sI - A]$ looses rank for the eigenvalues, so they are the only $s$-values for which the rank might possibly be less than $n$. The question is now if $b$ can make up for the lost rank. What about a noncyclic $A$?

The pages 141-4 will be extended to matrix polynomials in the difficult Chapter 6. Notice eq. (37). All six examples are nice. The row and column operations in example 2.4-4 will be used a lot in the future, and the technique to pick out columns in matrices is heavily used in example 2.4-5.

Another proof of the converse of Example 2.5-1, p166, would be to set up $x(0^+)$ by $\delta$-functions as in the proof of controllability, and then use the fact that $e^{AT}$ is invertible.

Concerning p192 some of you might like to try to find stabilizing $G_f(s)$ or $G_c(s)$ that in themselves are stable for the proposed example. We know that it is possible if we allow internal feedback in the controller, i.e. both $u$ and $y$ are considered as inputs to the controller we require stability of.

**Suitable exercises**

Do (at least 15 of) 1.30–1.58.

**Homework problems**

Solutions to the following problems should be handed in by Sept 21, 1989.

**8.15** Show that

$$x(t) = e^{At}b$$

if the input to the system

$$\frac{dx}{dt} = Ax + bu, \qquad x(0) = 0$$

is

$$u(t) = \delta(t)$$

Hint: Use

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-s)}bu(s)ds$$

What happens if

$$u(t) = \delta^{(i)}(t)$$

Perform the corresponding calculations using Laplace transforms by writing $\mathcal{L}\{x(t)\}$ as a sum of terms formed by nonegative powers of $s$ and terms with denominator $det(sI - A)$ respectively.

**8.16** Solve 2.3-4 and 2.3-16 using PBH.

**8.17** An electrical circuit consists of three circuits each with a resistor and an inductance in series. Assume also coupling between the inductances. Show that the system can be described by

$$(sL + R)I(s) = U(s)$$

where $L$ is a symmetric matrix of inductances and $R$ is a diagonal matrix of resistances. Discuss conditions for controllability from $u_1$ in terms of the elements of $L$ and $R$ using both PBH and $C(A, b)$. This is a difficult problem. Try at least to formulate and solve special cases. Use your physical insight. This problem can provide both trivial and nontrivial examples of limitted controllability.

# Third session

### Reading assignment

This week the following pages of Kailath should be read: 197-218, 237-243, 259-281. Section 3.2 contains three different methods to calculate the pole-placement state feedback. Notice that $K$ is used instead of $L$ and vice versa. The first approach (Bass-Gura) is what we use in AK. Transform to controllable form and keep track of what you do. Eq. (3.2-15) is also quite nice. The Ackermann formula is covered in Digital Control (CCS), while the Mayne-Murdoch formula is different. Notice that it describes how you may move individual eigenvalues, and especially what amounts of gain the movements require.

The section 3.2.2 shows that polynomial methods can do what state space methods do, and that it requires fewer pages. The question is only if that is obtained at the expense of long gaps between the statements or not.

In this course we cover no LQG, so the corresponding pages are left out. From the discrete time section 3.5 notice the nilpotent matrix in eq (10) and the corresponding characteristic polynomial (11). What about the minimal polynomial?

The treatment of observers follows AK quite well. Don't forget to think in terms of intermittent process disturbances that act on the plant, while it takes some time until the observer has managed to adjust. The robustness to model error and measurement noise is the other design constraint. Notice that you might introduce noise amplification also when you try to make the observer estimation slower than the the corresponding modes of the system.

Concerning the combined observer-controller problem it should be remembered that the response to set point changes could be improved considerably by feedforward. The real test for a regulator is the response to disturbances on the process. Then feedforward does not help, unless you can measure the disturbances, and then some of the dynamics of the observer might enter the response. One type of disturbance that is easily analyzed is a sudden change of some state variables. Such a change would preferably occur along the $B$-vector of the disturbance. Investigate the transfer function from the output to the input, of the controller, for the controller on p 278-280. Is it stable?

### Suitable exercises

Solve at least 15 of the exercises 2.1-2.21.

### Homework problems

Hand in the solution to 1.44, 1.56 or 1.57, and 2.11.

# Fourth session

## Reading assignment

This week the following pages of Kailath should be read: 281-292, 297-344. The first topic is reduced order observers. You have already read two different ways of introducing the Luenberger observer (AK 250-270) and (CCS 209-210). Kailath starts off with a third one and continues with a fourth approach. It is good to remember the two basic ways to observe the state, to use derivatives of inputs and outputs or to adjust, based on the measurements, the states of a full system model. To make the derivatives feasible you need filters, and for a $p$-output system you may need derivatives of order $n-p$. It is reasonable to assume that you can modify the full observer to a reduced observer of order $n-p$.

The full state observer is a strictly proper dynamical system. The order reduction is obtained by allowing a direct feed-through from the output, so that the reduced observer is only proper. One can actually obtain the order reduction by requiring infinitely fast convergence of the estimation error $C\tilde{x}$.

The following treatment of reduced order observers as the limit of a full order observer, where the 'output estimation' is infinitely fast, will help your intuition, I believe.

Assume without loss of generality that

$$C = \begin{pmatrix} 0 & 1 \end{pmatrix}, \qquad \text{ie} \qquad x = \begin{pmatrix} z \\ y \end{pmatrix}$$

The full order observer would be

$$\frac{d}{dt}\begin{pmatrix} \hat{z} \\ \hat{y} \end{pmatrix} = A\begin{pmatrix} \hat{z} \\ \hat{y} \end{pmatrix} + Bu + \begin{pmatrix} k_1 \\ k_2 \end{pmatrix}(y - \hat{y})$$

To obtain the reduced order observer we would like to make the estimation $\hat{y}$ of $y$ faster and faster. We want a direct feed-through from $y(t)$ to $\hat{y}(t)$. That mode naturally shows up also in $\hat{z}$, so there will be a direct term from $y$ to $\hat{z}$ as well. Now, introduce $k_2 = 1/\varepsilon$, then

$$\frac{d}{dt}\hat{y} = a_{22}\hat{y} + A_{21}\hat{z} + B_2 u + (y - \hat{y})/\varepsilon$$

and

$$(\varepsilon s - \varepsilon a_{22} + 1)\hat{y} = \varepsilon A_{21}\hat{z} + \varepsilon B_2 u + y$$

For $\varepsilon = 0$ this means $\hat{y} = y$, but in order to get any feedback from $y$ to $\hat{z}$ you have to "increase" $k_1$ "at the same speed", i.e. we can introduce $k_1 = k/\varepsilon$ giving

$$\frac{1}{\varepsilon}(y - \hat{y}) = (s - a_{22})\hat{y} - A_{21}\hat{z} - B_2 u$$

and by insertion into the $\hat{z}$-equation

$$s\hat{z} = A_1\hat{z} + A_{12}\hat{y} + B_1 u + k\left\{(s - a_{22})\hat{y} - A_{21}\hat{z} - B_2 u\right\}$$

or using $y = \hat{y}$:

$$\{sI - (A_1 - kA_{21})\}\,\hat{z} = (B_1 - kB_2)\,u + \{k\,(s - a_{22}) + A_{12}\}\,y \qquad (*)$$

Notice that there is a direct term from $y$ to $\hat{z}$. From

$$\left(\begin{pmatrix} 0 & I \end{pmatrix}, A\right)$$

observable it follows, for instance by PBH, that

$$(A_{21}, A_1)$$

is also observable, and we can place the poles of the $\hat{z}$-observer $(*)$ where we want by the freedom in $k$. For the actual implementation of the filter $(*)$ we first separate the direct term

$$\hat{z} = \zeta + ky$$

and then implement the filter with $\zeta$ as state variable

$$\{sI - (A_1 - kA_{21})\}\,\zeta = (B_1 - kB_2)\,u + \{(A_{12} - ka_{22}) + (A_1 - kA_{21})\,k\}\,y$$

Section 4.5 then interprets the state space results on the combined controller - (maybe reduced order) observer design in terms of polynomial manipulations. Thus the structure eq (4a,b) is assumed and found to imply eq (5), so that we can obtain an arbitrary denominator $p(s)$ of degree $2n$ provided that $a(s)$ and $b(s)$ are relatively prime. The resulting polynomial equation (9) can be rewritten as

$$\begin{aligned} p(s) - s^n a(s) = {} & a(s)\left[(\delta_1 + \gamma_1)s^{n-1} + \cdots + (\delta_n + \gamma_n)\right] \\ & + b(s)\left[\beta_1 s^{n-1} + \cdots + \beta_n\right] \end{aligned}$$

where the left-handside is now of degree $n-1$. There are some printing errors. On the second line below eq (9) it says that we solve for polynomials $n_u$ and $n_y$ of degree $n$. This should be $n-1$. Similarly $p_n$ should be replaced by $p_{n-1}$ on the next page on the line below eq (14). This $p$ is now a totally different $p$. I don't care too much about this 'independent proof' of the coprime requirement. For the reduced-order compensator notice how (9) is simply replaced by (17) and how the degrees are exactly the same as in the rewritten equation above. Notice also how polynomial methods are much easier in a case like this with direct feedthrough.

The solutions using the Sylvester matrix are actually what we use for our calculations in CCS, but we are more used to the almost identical direct polynomial formulation in the next subsection, where (23) is used to get a proper or strictly proper compensator from an arbitrary solution of the Diophantine equation without any assumption of the structure (4a,b). The final approach starting with a derivating observer for the partial state $\xi$ is aiming forward.

In chapter 5 we only read through the first section dealing with equivalence classes of inputs using impulse responses, Markov parameters, and Hankel matrices, while section 5.2 contains some very basic material. Actually I don't understand how anybody can read ch 2-4 without an active knowledge of p329-339. The next two pages about invariant subspaces is material that I mean is very important. You should read about the Schur form (A.47). The transformation matrix to this form can actually be chosen to be othogonal. Regard $AU = US$ where $S$ is upper triangular and $U$ orthogonal. The first $i$ columns of $U$ form an orthogonal basis for an $A$-invariant subspace of dimension $i$.

### Suitable exercises

Solve at least 9 of the exercises 2.22-2.33.

### Homework problems

Solutions to the following problems should be handed in by Oct 5, 1989.

**8.18** Regard the system

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad B = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad b_2 = 0$$

with one controllable pole at $\lambda_1$=1 and one uncontrollable pole at $\lambda_2$=-1. Could you find a feedback $k$ such that we get one pole at $\mu_1$=-2 and one at $\mu_2$=-1? Use *Mayne-Murdoch*, but give an interpretation also in terms of *Bass-Gura.*

**8.19** Show that if $[A, b]$ is controllable then $A - bk$ is cyclic for all $k$, that is if $A - bk$ has equal eigenvalues then it can't be diagonalized.

**8.20** Obtain a reduced order observer for the tank system used in our AK-labs. Measure the level of tank two. Let the speed by which this level is estimated by a full order observer increase to infinity. Determine the estimator gains using Matlab.

## Fifth session

### Reading assignment

This week sections 1, 2, and 3 (p1-27) of Brockett should be read, as well as the very special chapter 2.12 in AK. The pages 7-9 should be emphasized, and note especially on p7 (ii) but also (iii) and similarly on p8 (ii)-(iii) but also (iv). Maps in function spaces are nice descriptions of timevariable systems and also infinite dimensional systems.

If you don't have a good background in ODE don't dwell too much on existence and uniqueness in section 2. Theorem 1 is one thing special for timevarying linear systems as compared to general nonlinear systems. A typical tool used in timevarying systems is the Grönwall lemma found in exercise 2.9.

When obtaining the general Peano Baker series (PB) for the transition matrix $\Phi(t, t_0)$ it is important to remember that the intuitive expression in Cor 1 does not hold in general, only in the scalar case and in the timeinvariant case. A simple example to investigate would be

$$\frac{d}{dt}x_1 = tx_2, \qquad \frac{d}{dt}x_2 = -x_2$$

If however the commutativity requirement in the proof of Cor 1 holds for some other case, then we have the simple expression here too. Notice that with

$$I(t) = \int_0^t A(s)ds$$

we get

$$\frac{d}{dt}(I(t)I(t)) = A(t)I(t) + I(t)A(t)$$

so commutativity between A(t) and I(t) would mean that

$$\int_0^T A(t) \int_0^t A(s)dsdt = \frac{1}{2}\left[\int_0^T \frac{d}{dt}I(t)^2 dt\right] = \frac{1}{2}I(t)^2$$

and the other terms of (PB) would be obtained similarly.

### Some comments on linear maps in function spaces

The representation of a linear map (operator) in function spaces using an integral and a kernel is actually discussed in the math course Linear Systems.

$$x(t) = (Hu)(t) = \int_{t_0}^{t_1} h(t,s)u(s)ds$$

You have also met the scalar product in function spaces several times. At least you have seen the corresponding $L_2$-norm. The difficult thing is to combine to get the adjoint operator. First you introduce the scalar product between a function and the image of the operator.

$$< y, Hu >= \int_{t_0}^{t_1} y^T(t)(Hu)(t)dt = \int_{t_0}^{t_1} y^T(t) \int_{t_0}^{t_1} h(t,s)u(s)dsdt$$

Then you try to find a function $z$ such that

$$< z, u >=< y, Hu > \qquad (*)$$

Of course $z$ depends on $y$, and we can normally find a linear map giving $z$ out of $y$. This operator is also closely related to $H$ and we call it the adjoint operator $H^*$

$$z = H^* y$$

By introducing the integral representation for $H$ into the scalar product integral and changing the order of integration we can identify $z$ as

$$z^T(s) = \int_{t_0}^{t_1} y^T(t)h(t,s)dt$$

so that the integral representation of the adjoint operator is

$$z(s) = (H^* y)(s) = \int_{t_0}^{t_1} h^T(t,s)y(t)dt$$

Notice that the integration is now with respect to the first argument of the kernel $h$. If $H$ is an operator from a space $\mathcal{U}$ to $\mathcal{X}$, the first scalar product of (*) is in the space $\mathcal{U}$ while the second one is in $\mathcal{X}$. Functions in $\mathcal{X}$ have in this derivation running variable $t$, while functions in $\mathcal{U}$ have running variable $s$, but don't be sure that such a distinction is always made. Actually we often use a different convention, namely that the image function of a map has running variable $t$, while the integration variable is $s$.

Sometimes we discuss maps from a function space to a finite dimensional space, like from the input signal during an interval to the value of the state at the end of the interval. Of course we can talk about adjoint maps here as well. Notice that one scalar product is then in the function space, and the other one is in the standard Euclidian space.

One reason for the introduction of adjoint operators is when finding solutions of linear operator equations like

$$x = Hu$$

in the case when $H$ is not invertible but $x$ lies in the range space of $H$. We will see that one solution, with nice properties, is

$$u = H^* (HH^*)^{-1} x$$

as discussed later in section 13.

We see that dynamical systems can be represented by integral operators, but some of the discussions later on will be in terms of solutions to differential equations. In section 7 we will actually discuss adjoint differential operators.

**Suitable exercises**

1. 3, 4, 12, 13
2. 1, 9
3. 2, 5, 6, 7, 9

**Homework problems**

Hand in solutions to exercises 1.14 and 2.11 of Brockett by Oct 12. If you can't solve them in general, try a special case.

# Sixth session

## Reading assignment

This week sections 4, 5, and 6 (p27-43) of Brockett should be read. The property that $\Phi$ is invertible is not true in general if we generalize to infinite dimensional linear systems, like a pure time-delay. The functional equation on p29 is the same as in AK2.12, and it is often called the half group property. Concatenation of time-segments form a semigroup. We need to regard backwards time-segments to get inverse elements. For finite-dimensional systems $\Phi$ is invertible and we have a group.

The last part of Section 4 contains timevarying state-variable transformations. Notice how you get and additive term from the derivative of the transition matrix in the new $A$-matrix. This can sometimes be utilized to eliminate the time dependence of the $A$-matrix.

After a discussion on the product of two exponential matrices, we again discuss the commutativity between $A(t)$ and its integral, before some well-known properties of the timeinvariant problem are listed.

Section 6 contains the forced systems and results in the variations of constants formula (Th 1). Derivation under the integral is an easy method to check its validity.

## Suitable exercises

4. 1, 3
5. 2, 5, 7, 9
6. 3 (but disregard the N-P lemma)

## Homework problems

Consider two well-mixed tanks in series with timevarying flow $q(t)$ and constant volumes $V_1$ and $V_2$. Regard the inlet concentration of a tracer element as input and the tank concentrations as state variables. Determine the fundamental matrix as a function of $q$, $V_1$, and $V_2$. What is the weighting function from the inlet to the outlet concentration?

# Seventh session

## Reading assignment

This week sections 7 - 10 (p43-58) of Brockett should be read. In section 7 adjoint operators pop up again. The adjoint differential equation is given for the homogenous case. For the nonhomogenous case exercise 7.2 contains the full treatment, and one solution with some comments was handed out separately.

One reason for all this emphasis on adjoint operators is the (ii)-part of Theorem 4 in section 1. It is used several times to characterize the "image" of an operator or the solvability of an operator equation. A reformulation of the proof of Theorem 1 in section 9 was handed out. The adjoint criterion is here used to go from a nonhomogenous problem to criteria in terms of a homogenous adjoint solution.

It was also commented that the reformulation of $\Phi$ in terms of $\exp Rt$ in the Floquet-Liapunov Theorem, p47, is independent of the choice of $R$. The special choice is used to obtain the periodicity of $P(t)$. As seen in Theorem 3 of section 10, this means that $P(t)$ and its inverse are bounded as well, so that stability of the periodic system is totally determined by the $R$.

Also read the handouts on Brockett 7.2 and Th. 9.1.

## Suitable exercises

7. 1, (2)
8. 11
9. 2, 5
10. 6

## Homework problems

See special handout for session 7 and 8.

# Eighth session

## Reading assignment

This week sections 11 - 14 (p58-91) of Brockett should be read. The first theorem of section 11 is very appealing and intuitive. Remember it. The evaluation of the state loss $\eta$ on the middle of p60 is an important application of matrix differential equations, and there is actually an important connection with controllability to come.

In section 12 the first theorem is useful tool for heuristic reasoning about singular matrices. Another more modern tool would be the singular value decomposition of a matrix. The signature of a matrix is however something that is rarely used in the literature, and we actually skip the proof in Brockett that uses it. The introduction of Gramian matrices to characterize linear independence of columnvectors forming a matrix, ie the nullspace of the matrix, will be the main tool to discuss observability and controllability.

Brockett starts off to formulate the reachability and controllability in terms of a map from inputs during $[t_0, t_1]$ to the state at time $t_1$ but he never really mentions that Lemma 1 shows that $R(L) = R(LL^*)$, and that $LL^*$ is represented by the matrix $W$. Some separate handouts provides more comments. The $W$'s are in fact Gramians for the maps, and since the maps are integral maps associated with differential equations, one might supect that the $W$'s satisfy closely related matrix differential equations.

The observability and reconstructability problems are introduced using a different map $L$ in Lemma 14.1, and we find that the interesting space is now the nullspace of $L$ and thus that of $L^*L$.

## Suitable exercises

11. 3
12. 7, 8
13. 4, 9, 19
14. 5 (timevarying A and B, reach. vs reconstr. )

# Ninth session

## Reading assignment

This week the final sections 15-18 of Brockett should be read. Skip pp. 95-97, theorem 16.2 and section 19. In section 15 we now put even more emphasis on the input output map and its kernel, the weighting pattern. Now comes the question: Can we always find a finite dimensional realization of a given weighting function? We know that there is no such realization for the pure timedelay, and the important theorem 1 gives a necessary and sufficient criterion. Such maps have 'separable' kernels. The equality is really only of interest for $t \geq \sigma$. The weighting pattern is always 0 otherwise for causal operators. Notice how the dynamics is included in the time-varying $B$- and $C$-matrices in the degenerate realization with zero $A$-matrix. In the example on page 92 Brockett gives a transformation to a realization with symmetric $A$-matrix.

The intuitive definition of a minimal realization on p94 is important. The relation with controllability and observability is however more complicated. The sufficiency part of theorem 2 is actually wrong: Regard a system $T(t,s) = C(t)B(s)$ with $B(s) = 0$ during the first half of $[t_0, t_1]$ and $C(t) = 0$ during the second half. Then $T(t,s) = 0$ for all $t \geq s$, so it cannot be distinguished from the zero system. Both controllability and observability Gramians are however positive definite over the whole interval if they are positive definite over 'their half intervals'. The necessity part of the theorem, which is true, gives a constructive way to get a minimal realization, but it is complicated. Some other rank factorization of the Gramians, like from a singular value decomposition, would provide a more modern proof. Its use for time-invariant systems is really overkill. Skip p95-97.

The theorem on p99 is an important continuation of theorem 15.1. The sufficency part of the proof assumes that the realization is reduced to minimality using the complicated necessity part of theorem 15.2. It is enough to reduce to a positive definite controllability Gramian by a rank factorization $W = UU^T$. Then $R(U) = R(G(s), s \in [t_0, t_1])$ so that $U\bar{G}(s) = G(s)$ for $\bar{G}(s) = (U^TU)^{-1}U^TG(s)$. This is seen by regarding the Gramian for $U\bar{G} - G$. The norm of the difference is thus zero. Together with $\bar{H}(t) = H(t)U$ we thus get the same weighting pattern, but an invertible new Gramian, $\bar{W} = I$. The proof then proceeds by showing that

$$\dot{H}(t)G(s) + H(t)\dot{G}(s) = 0, \quad s < t$$

Postmultiplication by $G^T(s)$ and integration over a subinterval $[t_0, T]$ less than $t$ gives a differential equation for $H(t)$

$$\dot{H}(t)W(T, t_0) + H(t)W_1(T, t_0) = 0, \quad T < t$$

so that we must have $\dot{H}(t) = H(t)A(T)$, $T < t$ for full rank $W$ and $A = -W_1W^{-1}$. By $H(t) = H(T)e^{A(t-T)}$, $t \geq T$ and $H(t)G(s) = H(t-s)G(0)$ we can extend the expression $H(t)G(s) = H(T)e^{A(t-T-s)}G(0)$ to hold for all $t \geq s$.

Theorem 16.2 is a continuation of theorem 15.3 but the suggested procedure is really not simple. Wait for Kailath. Regard the proof of theorem 16.3

as an exercise in the use of the periodic theory of section 9. The result is of course well known from AK (how was the formula for $x_0$ ?)

In section 17 the two realizations of theorem 17.1 are certainly not minimal in general. In Kailath we will discuss these matters in great detail. The $R_i$'s are the coefficients of the numerator polynomial matrix, that also show up in our usual controllable canonical form, in the single input case, while the $L_i$'s are actually the Markov parameters that show up in the Hankel matrix, Kailath p126. Using this observation you can skip the second half of proof 2 since we know $CA^r B = L_r$.

The theorem 18.1 and the definition of McMillan degree should also be recognized from Kailath. The theorem 18.2 was given in Kailath for the SISO case. The proof here is constructive but a bit complicated. Notice the trick of letting $t = 0$ on line 3 of p114.

The systematic procedures to obtain minimal realizations are tedious. In many situations heuristic reasoning may lead quite far. Regard for instance the satellite example on p115, and introduce a realization given $G(s) = C(sI - A)^{-1}B$ but without any information about the realization on p15. First you recognize that a minimal order would be 4 since this is the maximal degree of any denominator. On the other hand it should not be larger than 12, since this is the sum of the degrees of all the denominators. It might be as low as 4 since all denominators are factors of the denominator of $G_{22}$. Using the theorem 1, we would have a realization of order 8. We see that the factor $s^2 + \omega^2$ is common in all elements, and that the $s$-factors could be regarded as multiplying the second row and the second column. Then there remains a direct term in the (2,2)-element. Introduce $sx_1 = u_2$, $y_2 = x_2$, and $sx_2 = x_1 + z$ so that

$$\begin{pmatrix} y_1 \\ z \end{pmatrix} = \frac{1}{s^2 + \omega^2} \begin{pmatrix} 1 & 2\omega \\ -2\omega & -4\omega^2 \end{pmatrix} \begin{pmatrix} u_1 \\ x_1 \end{pmatrix}$$

Then comes the remarkable fact that the matrix is rank one. The second row is a multiple of the first row, i.e. $z = -2\omega y_1$. Therefore we only need two more state variables for $(s^2 + \omega^2)y_1 = u_1 + 2\omega x_1$,

$$\begin{aligned} sx_3 &= \omega x_4 \\ sx_4 &= -\omega x_3 + (2\omega x_1 + u_1)/\omega \\ y_1 &= x_3 \end{aligned}$$

Notice that an infinitesimal change of one of the matrix elements would give a rank two matrix and require two more state variables. Let us summarize

$$\begin{aligned} sx_1 &= u_2 \\ sx_2 &= x_1 - 2\omega x_3 \\ sx_3 &= \omega x_4 \\ sx_4 &= -\omega x_3 + (2\omega x_1 + u_1)/\omega \\ y_1 &= x_3, \quad y_2 = x_2 \end{aligned}$$

## Suitable exercises

15. 1
16. 1, 2, 7
18. 1, 3, 4

**Homework problems**

Problem 1

The following system is given

$$\begin{aligned}\dot{x}_1 &= \sin(t)u(t)\\ \dot{x}_2 &= \cos(t)u(t)\\ y(t) &= \sin(t)x_1(t) + \cos(t)x_2(t)\end{aligned}$$

Calculate the weight pattern and show that it is stationary. Then give a time invariant realisation.

Problem 2.

Use Brockett theorem 18.1, or maybe easier exercise 18.1, to show that if the elements, $G_{ij}(s) = B_{ij}(s)/A_{ij}(s)$ of the transfer function matrix all have different poles, then the McMillan degree is the sum of the degrees of all the numerator polynomials. This is the generic case if information about internal structure is lacking. All common poles mean 'singular cases'. If you have some information about the system being formed by interconnections of subsystems, such information should be preserved. It may be impossible to retrieve again from real valued parameters of a polynomial representation. You should seek representations that may respect such information, like a careful use of MFD or PMD.

Problem 3.

Perform a continuation of exercise 18.3 using Matlab. First perform a direct realisation of the different elements of the transer function matrix, i.e. with separate states for each of the four denominators. Then use Brockett's two standard realizations, p107-108. Calculate the McMillan degree using theorem 18.1 and Matlab. One way to form the Hankel matrix is via the $A$'s, $B$'s and $C$'s of a maybe nonminimal realization. Use the result to pair common modes of the 'direct realization' to find a minimal realization. We know the minimal dimension, and we know what modes that could possibly be eliminated. Try also to make a minimal realization using the controllable and unobservable subspaces together with the Kalman decomposition theorem. Matlab is probably a necessary tool for this part.

## Tenth session

### Reading assignment

This week the following pages of Kailath should be read pp345-376, including Ex. 6.3-1. The first section 6.1 parallels Brockett ch17-18 and the Matlab homework-problem 9.3, but is much more straight forward.

The Brockett exercise 18.1 can be seen as the Gilbert diagonal realization discussed by Kailath on p349-350, with a proof on p365 using block Vandermonde matrices. There is almost always some structure in a multivariable problem. The same state variable can be influenced by several inputs and observed in several outputs, and then we end up with a matrix transfer function with the same poles in several elements. Thus we have to sort the poles of the elements of the transfer function to form the residue matrices $Z_i$ in Brockett 18.1.

The rank factorization on Kailath p350 is very useful, and PBH directly gives that the subsystems are both controllable and observable:

$$rank\,[sI_{\varrho_i} - \lambda_i I_{\varrho_i}, B_i] = rank B_i = \varrho_i$$

for all $s$, and similarly for observability. Further the parallell connection of two subsystems is minimal if and only if the subsystems are minimal, provided they have no common poles. The matrix

$$\begin{pmatrix} sI - A_1 & 0 & B_1 \\ 0 & sI - A_2 & B_2 \end{pmatrix}$$

has rank $n_1 + n_2$ for all $s$. When the first block column looses rank for $s$ equal to eigenvalues of $A_1$, the second column does not loose rank, since $s$ is not an eigenvalue of $A_2$. What is not spanned by the first column must be accounted for by the third block column, which $B_1$ does and must do in PBH for the subsystem. The case when $s$ is equal to an eigenvalue of $A_2$ is treated similarly.

Thus we have seen that the realization problem is almost trivial in the 'generic' case of distinct poles. There are however some cases where we have a structure giving multiple poles of the transfer function elements. Consider for instance one chain of integrators, where you observe some of the integrators. This must be distinguished from two separate chains of integrators.

A multiple pole criterion for the order of a minimal realization is introduced in Brockett exercise 18.2. His treatment is a little tricky. Only in case of poles being equal to zero the suggested matrix $H_i$ equals the Hankel matrix on p112. For general poles,

$$H(s) = \sum_{j=1}^{\nu} \frac{Z_j}{(s-\lambda)^j}$$

we first find the order of a minimal realization for the related transfer function matrix

$$H_0(s) = \sum_{j=1}^{\nu} \frac{Z_j}{s^j}$$

by Theorem 18.1. To get a minimal realization is actually nontrivial for this case, and much of the future treatment in Kailath is actually needed. Assume that we have obtained such a realization

$$C(sI - A)^{-1}B = \sum_{j=1}^{\nu} \frac{Z_j}{s^j}$$

It then follows for instance by PBH that

$$H(s) = C(sI - (A + \lambda I))^{-1}B$$

is also minimal. Parallel connection gives a condition for the general case. This concludes the solution of Exc 18.2.

Section 6.2 also contains the series expansion at infinity and the Markov parameters. On p356 Kailath introduces controllability (observability) index. This important concept will be further developped to a set of indices later. Keep your eyes open since some of it is hidden in footnotes. The proof of the if-part of Th6.2-3 is unnecessarily complicated. Prove instead that all other realizations of the same transfer function, with the same Hankel-matrix, have dimension greater than or equal to $n$, i.e. that of a controllable and observable realization.

$$n = \text{rank}\mathcal{O}\mathcal{C} = \text{rank}\bar{\mathcal{O}}\bar{\mathcal{C}} \leq \min(\text{rank}\bar{\mathcal{O}}, \text{rank}\bar{\mathcal{C}}) \leq \bar{n}$$

The inequality (33) is of course interesting by itself, but here you only need two simpler facts. You get rank $n$ when you multiply $n$ independent columns by $n$ independent rows, a generalization of a dyad multiplication giving rank one. Further you can never increase the rank by matrix multiplication.

Brockett used weighting-functions and Gramians directly in his Th15.2, with that error in the timevarying case, while this proof is based on the Hankel-matrix. Note that $T$ is unique in Th 6.2-4. Do the proofs of multivariable PBH. The silent space is the largest $A$-invariant subspace in $N(C)$, ie a combination of eigenvectors orthogonal to the rows of $C$.

Claim A and B on p367 are important. Keep your eyes open when we introduce the state space descriptions naturally associated with the MFD's. A singular $D(s)$, with det $D(s) = 0$ for all $s$, is uninteresting, since no $H(s)$ can be represented. For eq (49) we use deg det$(AB)$ = deg (det$A$ $*$ det$B$) = deg det$A$ + deg det$B$.

Later in 6.4.1 and 6.4.3 is introduced the partial states $\xi$ associated with a left or right MFD:

$$y = ND^{-1}u, \quad D\xi = u, \quad y = N\xi$$

$$y = D^{-1}Nu, \quad Dy = \xi, \quad \xi = Nu$$

This relates to pp37-45, but it is good to have seen the above interpretations of the MFD's before you start to manipulate the polynomial matrices in Ch 6.3.

The reduction to Hermite form described on p375 needs a second thought. Ulf Holmberg has made a report on a polynomial package in Macsyma, and a key element here is this algorithm. Work with an example by hand and look at his report.

**Suitable exercises**

Solve 6.1 - 6.8 (of BoB).

**Homework problems**

**8.21** Determine an MFD for the Kailath (De Bra) Example 3.3-3, e.g.

$$\begin{pmatrix} \Psi(s) \\ \Theta(s) \end{pmatrix} = D^{-1}(s)N(s)u(s)$$

or

$$\begin{pmatrix} \Psi(s) \\ \Theta(s) \end{pmatrix} = N_R(s)D_R^{-1}(s)u(s)$$

Introduce the feedback (in normalized units)

$$u(s) = v(s) - \begin{pmatrix} k_1(s) & k_2(s) \end{pmatrix} \begin{pmatrix} \Psi(s) \\ \Theta(s) \end{pmatrix}$$

where

$$k_1(s) = 8.75s \quad +8.403$$
$$k_2(s) = 5.083s \quad +1.042$$

What is the closed loop MFD? Further regard the open-loop system. In the solution of the example row manipulations were performed on $D(s)$ and $N(s)$ in order to find a state-space realization. Do this and show how for the new $\bar{D}(s)$ and $\bar{N}(s)$ we can write $\bar{D}(s) = \bar{D}_h(s) + \bar{D}_l(s)$, where $\bar{D}_h(s)$ is diagonal with $s^{n_i}$ on the diagonal, and where all elements in the corresponding rows of $\bar{D}_l(s)$ and $\bar{N}(s)$ have degree lower than $n_i$.

**8.22** Regard the simple two-mass model for the azimut axis of a radio-telescope. See below. The damping is disregarded. Determine an MFD for $\begin{pmatrix} y_M \\ y_L \end{pmatrix}$ expressed in $\begin{pmatrix} u \\ z \end{pmatrix}$, the motor-torque $u$ and the disturbance-torque $z$, mainly from the wind. Then introduce motor dynamics $u = 1/(1+Ts)v$ and a wind model

$$z = \bar{z} + x_1 + x_2$$
$$x_1 = \frac{k_1}{1+sT_1}e_1$$
$$x_2 = \frac{k_2(1+\sqrt{3}\,sT_1)}{(1+sT_1)^2}e_2$$

where $e_1$ and $e_2$ are independent white noises. What is the new MFD from $v$, $\bar{z}$, $e_1$, and $e_2$ ?

## Eleventh Session

The Kailath formulation of the Hermite form, Th6.3-2 p375, is not correct for polynomial matrices that are not of full column rank. Column operations are needed to move columns that have zero pivot element to be the last $m - r$ columns. Page 589 of C-T Chen, Linear system theory and design (rev ed 1984) is copied.

### Reading assignment

This week pages 372-422 of Kailath should be read. These are probably the most demanding pages. Now you should learn everything about polynomial matrices and how they can be used in MFD's.

In 6.2 we learnt that we might reduce deg det $D(s)$, the degree of an MFD, by extracting right or left divisors. We also learnt that $D(s)$ and $N(s)$ were coprime if the only divisors were unimodular, i.e. had a nonzero constant as determinant. These concepts are now further developped, and a lot of facts are piled on top of each other, sometimes in dubious order. Linear independence and elementary row and column operations are generalizations from the ordinary vector case.

The Hermite form might be viewed as a special triangular form, or more correctly an "Echelon form", while the Smith form is a very special diagonal form. It is important to remember what type of polynomial matrix transformations that produce the different forms, and what these transformations mean when applied to an MFD.

A general theme in 6.3 is common factors between the numerator and denominator matrices. Before starting with Lemma 6.3-3 think of other possible methods to extract common factors $R(s)$. If det $R(s) = 0$ for some $s$, then det $D(s) = 0$ for the same $s$, i.e. $D(s)$ looses rank for this $s$. The only common factors that have to be investigated are thus those that have a zero determinant for such $s$'s. Further $N(s)$ has to loose rank for the same $s$-value, so just test that for the roots of det $D(s)$. Then comes procedures to find the common matrices, and that is what sometimes could be difficult, but most of the time it is obvious how to take out factors $R(s)$ with a single pole at a time. This rank formulation is what lies behind Lemma 6.3-6, and 6.3-7. The notation "eigenvalue" is here misleading, and the argument $\lambda$ in (10) is really just a way to index a vector that lies in the nullspace of $P(\lambda)$ for a certain distinct latent root $\lambda$. It becomes even worse in Lemma 6.3-7. It would be tempting to interpret (11) as a polynomial vector being equal to a vector of null polynomials, but it just means an equation that might have distinct latent roots, $s = \lambda$, as its solutions.

Many people like the Bezout identities. I have some difficulty to create a good intuition around the formalism. Of course one should go back and consult p141 for the scalar polynomial case. Lemma 6.3-5 is important and you should be prepared to use it in proofs. I try to motivate myself by the following.

The $I$ in (9) is nothing special. It could really be any unimodular matrix, i.e. the important thing is that the left hand side is invertible as a polynomial matrix. The reason for *two* coefficient matrices $X$ and $Y$ is the same as in case of linear dependence, when you deal with coefficients for which division is not defined, as discussed in great detail on p374.

Lemma 6.3-8 requires an example. Use example 6.2-1 and determine left MFD's. They show up later in example 6.4-2.

The concept column-proper or column-reduced is very important. The treatment on p384 is very instructive. The form (22) will show up later to give the ***naturally associated state realization.*** The property that $D$ is column-reduced (row-reduced) is thus almost as important as $D$ being invertible. The lemmas 6.3-10 and 6.3-11 on properness are nice consequences. Notice also that column operations, ie a new choice of internal states $\xi$, is the way to get column reduction in a right MFD, $N(s)D^{-1}(s)$. The formulation, theorem 6.3-12, for rectangular $D$ is of less importance, while the division theorem 6.3-15 could be used also for controller design, just like in the scalar case. It is remarkable that the $k$'s on p388 show up much later as controllability indices.

The Smith-form is fundamental in any treatment of polynomial matrices. You have to understand how the division property works. Almost all Smith-forms have 1's on the diagonal except for the last element. For $sI - A$ it is only in case of noncyclic matrices that that you would get additional invariant factors. See also exercises 6.3-12,13,17 for Smith forms of $sI-A$. The example 6.3-3 gives some further properties. For $r = n$ in Th6.3-18 you need $A = \lambda I$.

The transformations used to obtain the Smith-form would change the MFD's. Row-operations can be used for left MFD's, since they just change the equations $D_L y = N_L u$, while column-operations on a right MFD just means a change of internal variables $\xi$. Both row- and column-operations would be allowed on $P$ in a PMD like $H(s) = R(s)P^{-1}(s)Q(s)$. On the other hand a state realization $C(sI - A)^{-1}B$ is a special case, and here we require similarity transformations, i.e. $T$ from the left of $sI-A$ and $T^{-1}$ from the right in order to preserve the structure. Actually the similarity transformations can be relaxed to general invertible transformations $U$ and $V$ if we introduce the more general form $sE - A$ instead. These are now the operations allowed in (38) for the Kronecker form. The form is difficult to understand from Kailath, but I like it quit much. Start to understand the form for a regular pencil, i.e. with only the two blocks $\{sJ - I, sI - F\}$. The $P$ in a PMD is required to be invertible. The singular pencils are used when we look at block matrices like $[sI - A, B]$, and it is now that the Kronecker indices show up.

It is instructive to see what happens with the Kronecker form, if we allow unimodular instead of constant transformation matrices. We then obtain the Smith-form.

The controller form realization on p404-407 is really *the* result. Again the $k_i$'s appear, but so far we have no proof of any connection.

When getting the state realization of a right MFD as on p404 using the internal states $\xi$ one should remember the influence from $a_0$ in eq. (2)

$$[a_0 s^n + \ldots + a_n]\,\xi(s) = u(s)$$

and compare that with $D_{hc}$ in

$$D(s)\xi(s) = [D_{hc}S(s) + D_{lc}\Psi(s)]\,\xi(s) = u(s)$$

Then $a_0$ might contain all the time constants, so when you neglect a time-constant because it is fast, this would mean $a_0 = 0$, and the degree of the polynomial is reduced. In the multivariable case $D_{hc}$ might be singular in such a case, and we may have to change degrees of $S(s)$, but we may also have to rearrange the internal states using column operations for $D_{hc}$ to remain invertible, i.e. $D$ to remain column proper.

On p405 it is required that the chains are ordered by length, but in some of the examples this is not done. The reason for the sorting is really only to reduce the arbitrariness in the choice of states.

As indicated in fig 6.4-1 the states of the controller form are chosen to be $x(s) = \Psi(s)\xi(s)$. The trick is to apply state feedback around the core realization $x = \Psi S^{-1} u$ and to modify the input matrix by $D_{hc}^{-1}$. Notice the footnote on p406 commenting controllability indices.

Lemma 6.4-1 is so far an exotic way to write a nice result. Much more on the equivalences will come in chapter 8. Generalized Bezouts are however somewhat artificial and hides some of the insight, but they tend to appear more and more in papers, so why not try to understand it.

In case of observer form the states are slightly more complicated to formulate:

$$x(s) = (sI - A_o^0)^{-1} \left[N_{lr} u(s) - D_{lr} y(s)\right]$$

The difficult controllability form on pp419-422 uses $D$ of $ND^{-1}$ in row-reduced form and thus a core realization with block sizes equal to the row indices $\{l_i\}$ and these are not necessarily equal to the controllability indices. In (49) it is shown an easy way to obtain a $C_{co}$ candidate, but the proof that the whole (49) is actually fulfilled for this choice, follows indirectly from p420. Notice the use of the division theorem.

This subsection 6.4.4 is much more difficult than the rest of the section. The tools needed are more complicated, and the text is very compact. Actually I am not sure that there exists a similarity transformation between this controllability realization and the corresponding controller realization, but at present I think it is sufficient to say that my doubts relate to the structure of an unobservable part, ie common factors in the MFD, and to the transformation from column-reduced form to row-reduced form. You really need some system equivalence concept for MFD's. But don't spend all night on this. Skip it at least in the first reading.

### Suitable exercises

Solve 6.9-6.15 and 6.16-6.19 (of BoB)

### Homework problems

**8.23** Regard the transfer function

$$\begin{pmatrix} \frac{1}{s^2+\omega^2} & \frac{2\omega}{s(s^2+\omega^2)} \\ \frac{-2\omega}{s(s^2+\omega^2)} & \frac{s^2-3\omega_2}{s^2(s^2+\omega^2)} \end{pmatrix}$$

for the satellite problem, Brockett p115. Obvious MFD's would be

$$\begin{pmatrix} s(s^2+\omega^2) & 0 \\ 0 & s^2(s^2+\omega^2) \end{pmatrix}^{-1} \begin{pmatrix} s & 2\omega \\ -2\omega s & s^2-3\omega^2 \end{pmatrix}$$

$$\begin{pmatrix} s & 2\omega s \\ -2\omega & s^2 - 3\omega^2 \end{pmatrix} \begin{pmatrix} s(s^2+\omega^2) & 0 \\ 0 & s^2(s^2+\omega^2) \end{pmatrix}^{-1}$$

**Use some methods to obtain irreducible MFD's, i.e. MFD's with lowest possible degree. One method would be Lemma 6.3-3.**

**8.24** **Regard the circuit below. Calculate the transfer function from $V_{in}$ to $y$, and apply the properness condition of Lemma 6.3-11. What happens with the condition when $1/K = 0$ ?**

**8.25** **Regard** $A = diag(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix})$ **and** $B = diag(\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix})$. **Determine the Kronecker form of $[sI - A, -B]$ and the Kronecker indices. Determine also an irreducible right MFD $N(s)D^{-1}(s) = (sI - A)^{-1}B$. Compare the column degrees of $D(s)$ and the Kronecker indices.**

## Twelfth Session

### Reading assignment

This week pages 422-450, 500-501 of Kailath should be read.

First of all, I like the notion of searching the crate. The section 6.4.6 contains some very illuminating material but also a lot of less relevant material that might hide the main message: How should the basal vectors eg. in (56) be chosen to give the best structure of the $A$- and $B$-matrices? Now is the time when the controllability indices really show up. Notice that permutation of the inputs does not change the set of indices, but their pairing to the specific inputs might change.

Intuitively it is clear that Scheme II is "better" than Scheme I. The vectors requiring higher orders of the $A$-matrix would be "less" controllable, would require higher order derivaties of impules for a fast state change. In discrete time it corresponds to the length of the delay chains.

The word controllability form here does not directly relate to the controllability form in section 6.4.4. The property here is that out of $C$ it is possible to choose $n$ vectors forming an identity matrix.

The controller form found by scheme II is not explicitly described here, only by example 6.4-7. On p500 we require that the general procedure is understood, so why not learn it now. From the AK-book and the SISO treatment we know how we should weight the the basis vectors, chosen using scheme II, by coefficients of the characteristic polynomial to form a transformation matrix $T$, so that we get a $C$-matrix with the parameters from the numerator polynomial. Now we want to form a $T$ in the same way, giving $A$ and $C$ matrices from which we can directly read out the coefficients of the $N(s)$ and $D(s)$ matrices of a corresponding MFD. Use the enclosed Matlab macros, from PACLIB, `[larsr.paclib.matlab.linsys]`, to understand the procedure.

It is important to be able to derive the controller form both from an MFD and from an arbitrary state space description. Notic also how easy it is to get an MFD from the controller form using the notation introduced on p406:

$$sx = A_c^0 x - B_c^0 Lx + B_c^0 Gu$$
$$y = Cx$$

and using

$$(sI - A_c^0)^{-1} B_c^0 = \Psi(s) S^{-1}(s)$$
$$x(s) = \Psi(s)\xi(s)$$

we get

$$S\xi = -L\Psi\xi + Gu$$

or

$$G^{-1}[S + L\Psi]\xi = u$$
$$y = C\Psi\xi$$

that is

$$D(s)\xi(s) = u(s)$$
$$y = N(s)\xi(s)$$

In subsection 6.5.1 there is a compact summary of the relevant theorems on minimal MFD's and minimal state space realizations. Subsection 6.5.2 introcuces the Smith-McMillan form of rational matrices, which corresponds to the Smith form for polynomial matrices.

Think twice on the type of transformations required. Remember also that we may have a singular $H(s)$, ie an $H(s)$ that does not have full normal rank. Such transfer functions are quite common and interesting, especially when some of the inputs are disturbances. $D(s)$ should always be nonsingular, but $N(s)$ may have lower than normal rank. This would introduce $\epsilon_i(s) \equiv 0$. By definition they do not introduce any system zeros, although any $s$ really would be a zero. Remember this later in section 7.6. Like it was for the Smith form it is instructive to see how special the $A$-matrix of $C(sI - A)^{-1}B$ is in case $\psi_i \neq 1$ for some $i \geq 2$.

The interpretation of zeros of the state-space realizations on p449 needs some second thought. Remember also the SISO exercise BoB 1.5. Show that the input (23b) and $x(t) = -x_0 e^{s_0 t}$ satisfy the differential equation $\frac{dx}{dt} = Ax + Bu$, if (23a) holds. Then $y(t) = Cx(t)$ and the second equation of (23a) gives (23c). For the first footnote relate to zero $\epsilon_i$ of the Smith-McMillan form. Notice also that it means that $P(s)q(s) = 0$, so that for any $s_0$ there exist zero-directions, but they may be different for different $s_0$. Left and right inverses require full normal column and row rank.

For the zero directions at infinity, we must work by examples. It is to some extent interesting to think also in terms of multivariable root loci, i.e. to do a high gain proportional feedback $K$ around the system. Sometimes we then specialize to $K = kI$.

For pencils like

$$s \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} A & B \\ -C & 0 \end{pmatrix}$$

we also have the concept of generalized eigenvalues and eigenvectors, GEIG in Matlab.

### Suitable exercises

6.4 8, 11
6.5 1, 3, 12, 13, 14

### Homework problems

**8.26** Find a relation between controllability indices and a $B$ that is not of full column rank.

**8.27** Perform the searches by Scheme I and II of the crate for the following three examples from Friedland Control Systems Design, distillation column, double effect evaporator, and aircraft lateral motion. Start from suitable state space models to be found in the text copied. I suggest that you do the calculations in Matlab, but don't use the PACLIB general purpose Macro's. Use Matlab to help to evaluate the ranks as you go along in the crate and for documentation. Determine the controllability indices for the three problems. Give an example on how the length of the chains is indifferent to input ordering but not coupled to the specific inputs. Determine also for the three examples the important canonical controller form as deduced in example 6.4.-7 and written out on pp500-501.

**8.28** Find the Smith-McMillan form for the satellite and differentiator examples.

## Thirteenth Session

### Reading assignment

This week pages 502-525 and 532-540 of Kailath should be read. Notice how the controller form can be used for almost straightforward pole-placement just like in the SISO-case.

I believe that eq (6)-(8) of 7.1.1 are very important. The feedback chosen in eq (16) is just to prove the possibility of arbitrary poleplacement. The same type of method is used in 7.1.2. Remember that this is an existence proof, so you may very well first make one big block and then use $n-1$-impulse derivatives. Otherwise you could instead group your desired poles to the different blocks and use the corresponding inputs. The $k_i$-core block structure really supports the latter approach. In discrete time the core structure means dead-beat in $\max(k_i)$ steps. The "almost any" and "generically" on p504 should be used with greatest care. The existence is the important property. The Brunovsky form in 7.1.3 is really a nice and concise account of Scheme II and the controller form.

Now section 7.2 should be interpreted both as an MFD section and as another controller form section. The facts 1-4 on p508 are very important. The eigenvector formulas on p510-513 are nice and sometimes used but I consider them of much less importance. Further, the formula (26) should really be written using the inverse image concept, so we can handle the nonunique case, when $\mu$ is an eigenvalue of $A$.

The control structure theorem in 7.2.2 is intricate and in order to understand it read the example 7.2-1 as you go along in the proofs of the lemmas. We want to investigate to what extent we can place not only det $D(s)$ but also the invariant factors of $D(s)$.

Notice that the examples of invariant factors that could not be obtained are very special. They contain a lot of equal complex modes, and the modes should not be coupled. Notice also that exercise 7.2-1 says that for max $k_i - k_j \leq 1$ there is no restriction imposed by the structure theorem.

Repeat the proof of Lemma 7.2-2 using the controller form $sI - A_c$ instead of $D(s)$. First you work with each of the $m$ diagonal blocks of size $k_i$ and get ones in the diagonal and its characteristic polynomial $a_i(s)$ of degree $k_i$ in the upper left corner. In the off-diagonal blocks you get zeros except for in the columns and rows where the $a_i$'s are. There you get a polynomial of lower degree than the $a_i$ of that column. By row and column permutations you get two blocks, one unit matrix and one $m$ by $m$ matrix with column degrees $k_i$. Then you can mimic the degree proof for $D(s)$, but you can also start to see how different special structures of $A_c$ are reflected in the invariant polynomials. It is particularly simple to investigate a triangular such matrix, since then we have that the characteristic polynomial of the whole matrix is the product of the characteric polynomials of the blocks. The distinct eigenvalue case gives only one nonunity invariant polynomial as usual, and such matrices are called simple on p504. They never loose more than one in rank for any $s$. It is the structure of the degeneracy that is reflected in the invariant polynomials.

The proof of Lemma 7.2-3 is the difficult part of theorem 7.2-4. It gives a constructive method even if it is quite involved.

The section 7.2.3 is of less importance. View it as a collection of examples in the use of polynomial matrices.

The section 7.3 parallels the SISO section. To obtain (4) from (3) we introduce a state transformation to block triangularize the system, ie sustitute $\hat{x}$ by $\tilde{x} = x - \hat{x}$, as in session 4. For the reduced order observer extend the derivation of session 5 to the multivariable case. It is useful to do at least exerc 7.3-1 as well. For the estimation of linear combinations of the states that are of reduced dimension read p524, but the underlying algebraic problem is very complicated. Anders Rantzer has given some results for the even more difficult problem to estimate a linear state feedback, that can be used to stabilize the plant, with an observer of least order.

The MFD approach of 7.5 to get an observer+controller compensator in MFD-form is limitted by (2), ie the $N(s)$ is unchanged. However this is exactly what state-feedback $u(t) = v(t) - Kx(t)$ would give as discussed on p508. It is interesting to see how the right MFD is used to find the partial-state feedback $M(s)$ giving the closed loop denominator $P(s)$, while it is advantagous to use the corresponding left MFD to get an observer for $\xi(s)$ with the dynamics $\Delta(s)$. Notice also how polynomial division is used to get a proper, stable, two-input-vector formulation just like in the observer-controller formulation. A feedforward precompensator can of course be used to change $N(s)$ at least to some extent. Lars Pernebo gave in his thesis a more complete theory. Vidyasagar treats the problem in his design book, and Gunnar Bengtsson worked in this area as well as in the pure state space formulation as we shall see in section 7.6. Words that appear in this context are: feedforward design - feedback implementation, the interactor matrix, the Pernebo structure matrix, parametrization of stabilizing feedback compensators, etc.

**Suitable exercises**

7.1–7.5 and 7.7–7.9 (BoB).

**Homework problems**

**8.29** Regard the distillation column example. Try to make state feedback so that you get the invariant polynomials $\psi_1 = \psi_2 = s^2 + 2\zeta\omega s + \omega^2$. What about $\psi_1 = 1, \psi_2 = (s^2 + 2\zeta\omega s + \omega^2)^2$? Use $\zeta = 0.7$ and $\omega = 1$.

**8.30** Find the Kronecker form and necessary transformation matrices for the differentiator example, ie go from

$$(sE + A)y = BV_{in}$$

to

$$\begin{aligned}(sI - F)z_1 &= B_1 V_{in}\\ (sJ - I)z_2 &= B_2 V_{in}\\ y &= C_1 z_1 + C_2 z_2\end{aligned}$$

Show that $J = 0$ for unless $1/K = 0$ in which case $J^2 = 0$, so that

$$z_2(t) = -\left(I + J\frac{d}{dt}\right) B_2 V_{in}(t)$$

Notice that the pencil is regular in both cases.

**8.31** The following system is used in a paper (IEEE AC-27 1241-1243, 1982) to discuss multivariable root loci.

$$A = \begin{pmatrix} -4 & 7 & -1 & 13 \\ 0 & 3 & 0 & 2 \\ 4 & 7 & -4 & 8 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 0 \\ -2 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & -5 & 2 & -2 \\ 8 & -14 & 0 & 2 \end{pmatrix}$$

Determine the poles, zeros, and zero directions, ie $x_0$ and $u_0$ at p449. Derive what happens with the closed loop poles for output feedback $u = -kIy$ as $k \to \infty$. What happens for the feedback $u_1 = u_2 = -k(y1 + y2)$?

# Fourteenth Session

## Reading assignment

This week pages 540–557 of Kailath should be read.

For the comments on p538 on Pernebo you could consult his thesis, Vidyasagars book Control System Synthesis, or better the short report by Michael Lundh. This report contains also a number of Macsyma macros, (handout). BoB has a preliminary report on the geometric theory with examples and macros, (handout). You could also scan the report by Gunnar Bengtsson, Lecture notes on geometric theory, and a recent survey on multivariable zeros, (handout).

The section 7.6 is one of my favourites. It combines the geometric theory by Wonham with the polynomial matrix formalism or at least the pencil formalism. The presentation is very condensed. The theorem 7.6-2 really needs a thorough penetration. Notice how a coordinate free representation in terms of subspaces, maps of subspaces, and restrictions of maps to subspaces compactifies the notation. On the other hand I need the insight provided by the special choice of basis as in the theorem. Notice also the input transformation necessary to obtain (4). Perfect observability means no finite zeros and no zero-directions. The $\epsilon_i$'s are all equal to one. This part is represented by $A_{11}$. The multivariable zeros are found in the $A_{22}$-part, while the strange modes that can be made unobservable with arbitrary poles, are represented by $\bar{A}_{33}$. In the proof of (16), the Smith-form of $P(z)$, only constant transformation matrices are used up to (13), while unimodular polynomial matrices are needed for (14) and (15). Thus (13) contains the structure at infinity as well.

Be sure to understand the different invariant subspaces: $A$-invariant, maximal $A$-invariant in $\mathcal{N}(C)$ , $(A,B)$-invariant, $\mathcal{V}^*$, ie the maximal $(A,B)$-invariant subspace in $\mathcal{N}(C)$, and $\mathcal{R}^*$, ie the maximal controllability subspace in $\mathcal{N}(C)$. $\mathcal{R}^*$ contains the strange modes.

We characterize the zeros of a system using the concept restriction of a map to an invariant subspace. Let $L$ be the feedback map connected with $\mathcal{V}^*$, ie $\mathcal{V}^*$ is $A+BL$-invariant in $\mathcal{N}(C)$. Then the zeros of the system are the eigenvalues of the restriction of $A+BL$ to $\mathcal{V}^*$ except those that are eigenvalues of the restriction to $\mathcal{R}^*$. If there is any $\mathcal{R}^*$ then the system is not invertible, while the true zeros are the poles of the inverse.

The algorithms for the different subspaces require an intuitive feeling for intersections of subspaces that are the images of other subspaces.

Section 8 is important and really the aim of the whole book. When trying to give natural physical models we often find that we want to start from something more general than a state space formulation. The left MFD's were natural in some cases, when most variables were measured, while the right MFD's were natural, when the inputs entered like forces directly into the dynamical equations. Pencils were a different special class, that I liked quite much, but we really need expressions in terms of at least second order polynomials. Thus we generalize all the four matrices $A, B, C, D$ to polynomial matrices $\{P(s), Q(s), R(s), W(s)\}$ and call it a PMD.

Kailath goes through all operations on the four matrices, that do not 'change the system'. This concept is introduced step by step and the line of reasoning should be possible to follow. If there is something you don't understand, mark it, and go back to it later. It is important to catch the way the Rosenbrock *system matrix*, $P(s)$, is used together with unimodular transformations to represent the basic operations on p552. Remember that MFD's are special PMD's, and go back to the previously strange p409 again. Although the system matrix is mainly introduced to represent nice operations easily, don't forget that (21) is a way to write (5). The need for rectangular $M$'s is not clear to me from Ex 8.1-1. Instead I had great help of p409, where it also follows that it would be nice with (18) instead of (7). This increasing the dimension step is really the most difficult one in the whole PMD formalism. It is no longer sufficient with unimodular $M$, and then new requirements on the transformations have to be deduced. System equivalence is therefore introduced by requiring that state realizations corresponding to the transformed systems are equivalent.

### Suitable exercises

Problems 1-7 of BoB's geometric report. plus the enclosed two examples on 8.1.

### Homework problems

**8.32** Choose a state feedback for the aircraft lateral dynamics for suitable pole-placement. A suitable choice would be to use the rudder to decouple the sideslip equation so that $T\dot{\beta} = -\beta$ with $T = 0.2$ for instance. Notice that this means that the rudder signal will depend on the aileron signal to be determined, and that the resulting lower order system will have modified both $A$ and $B$ matrices. Then the aileron can be used to place the remaining poles at for instance $s = -1, -1 \pm 3i$. Determine a reduced order observer.

**8.33** Gunnar Bengtsson and Sture Lindahl introduced a standard multivariable example based on a boiler model by Karl Eklundh:

$x_1$= drum pressure (bar)
$x_2$= drum liquid level (m)
$x_3$= drum liquid temperature ($^o$ C)
$x_4$= riser wall temperature ($^o$ C)
$x_5$= steam quality (%)

$$A = \begin{pmatrix} -0.129 & 0 & 0.396*10^{-1} & 0.250*10^{-1} & 0.191*10^{-1} \\ 0.329*10^{-2} & 0 & -0.779*10^{-4} & 0.122*10^{-3} & -0.621 \\ 0.718*10^{-1} & 0 & -0.100 & 0.887*10^{-3} & -0.385*10^{1} \\ 0.411*10^{-1} & 0 & 0 & -0.822*10^{-1} & 0 \\ 0.361*10^{-3} & 0 & 0.350*10^{-4} & 0.426*10^{-4} & -0.743*10^{-1} \end{pmatrix}$$

$$B = \begin{pmatrix} 0 & 0.139*10^{-2} \\ 0 & 0.359*10^{-4} \\ 0 & -0.989*10^{-2} \\ 0.249*10^{-4} & 0 \\ 0 & -0.534*10^{-5} \end{pmatrix}$$

Calculate the zeros when $u_1$ is input and $x_1$ output. Calculate the zeros when $u_1$ is input and $x_2$ output. What can be said about controllability, observability and minimum phase? Now consider both inputs and two outputs $x_1$ and $x_2$. Find $\mathcal{V}^*$ and perform the transformation to the form of Kailath theorem 7.6-2. Which are the zeros?

**8.34** Calculate the zeros for the system $S(A, B, C)$:

$$A = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & -1 & -2 \\ 1 & 0 & -1 \end{pmatrix}$$

Find the transfer function and show that the system

$$\dot{w} = -w + \begin{pmatrix} -1 & 1 \end{pmatrix} y$$
$$u = -4w + \begin{pmatrix} -3 & 3 \end{pmatrix} y + \begin{pmatrix} 0 & 1 \end{pmatrix} \frac{dy}{dt}$$

is a left inverse.

**8.35** Regard the system

$$\begin{aligned} (s+1)x_1 &= u_1 \\ (s+2)x_2 &= u_2 \\ (s+3)x_3 &= x_1 + u_1 + u_2 \\ y_1 &= x_3 \\ y_2 &= x_1 + x_2 \end{aligned}$$

Determine $\mathcal{R}^*$, $\mathcal{V}^*$, zeros and zero-directions. What happens with the poles of the system in case of the feedback $u = -ky$ when $k \to \infty$?

## Fifteenth Session

### Reading assignment

This week pages 557-640 of Kailath should be read. The chapters 9 and 10 are only to be scan through.

Kailath's standard scheme for realization consists of four parts:

1. Make $P(s)$ row reduced (3a)
2. Divide to get a strictly proper input part (4b)
3. Use its naturally associated observer realization (5)
4. Perform a new division (9) to determine the $J(s)$

Really this scheme gives a unique associated realization apart from some arbitrariness in the first step. This is also the only step that might change the properties at infinity. There is a substantial relationship with the introduction of the then difficult controllability form on p418-420. The footnote on p559 contains bad notation. What is really meant is

$$C = R(A_0)C_0 = R_0 C_0 A_0^n + R_1 C_0 A_0^{n-1} \dots$$

The compact descriptions (11) and (15) are now extensions of p409, while (20) is the final generalization. The example 8.2-1 is a comment to example 8.1-1, and shows that $(s+1)$ could be moved from the output side to the input side only if it is not a common mode with $P(s)$. The page 561 uses generalized Bezout identities and is thus complicated. Left and right should be interchanged in (16). I guess it is time to look back at the proofs of Lemma 6.4-2 and 6.3-9.

The theorem 8.2-1 is for me clear up to the proof of $T$ invertible. So far we don't even know that $T$ is quadratic. $T$ and $sI - A_2$ left coprime means that $(A_2, T)$ is a controllable pair, but $A_2 T = T A_1$ means that no more than $T$ is spanned by the controllability matrix. Thus $T$ has to span the whole space for controllability and rank $T = \dim A_2$. Similarly we show that rank $T = \dim A_1$ by $T$ and $sI - A_1$ being right coprime, i.e. observability. Therefore $\dim A_1 = \dim A_2$ and $T$ is square and invertible.

For theorem 8.2-2 I prefer to go the detour by equivalence with the natural state representations and then use 8.2-1 to show that PMD equivalence is the same as MFD equivalence. Don't forget that equivalence means the same transfer function and no change in uncontrollable and unobservable parts.

In the p565 motivation for expansion of the system matrix by a unity block we use the generalized Bezout once again. Here it seems really worthwhile. The zero in the (3,1)-element of unimodular matrix to the left must however be $-R_2$. Notice that it does not matter if you extend too much. The last matrix is unimodular because the (1,1)-block in (27) gives the (1,2)-block in (29).

Theorem 8.2-3 is really a compact formulation of what is preserved by equivalence. For the Rosenbrock extended system matrix it can actually be shown that it is always sufficient to extend the dimension to the determinental degree of $P$, as formulated in (31b), although not very clearly and certainly not proven. Thus for a given pair of PMD's we extend to equal dimensions and try to find transformations that fulfil (31c,d). Condition (31b) just says that if we really tried hard we would find one for the dimension $q$ if we can find one at all. The Morf system equivalence is more esoteric and could be skipped.

In section 8.3 the beginning on minimality really clears thing up. I guess we should add that (3a) means controllability and (3b) means observability. Don't forget that MFD's are special PMD's so their properties better be preserved as well. Unfortunately reducible PMD's (and MFD's) have some problems with their zeros and $\epsilon(s)$. Rosenbrock gave these new strange zeros the name decoupling zeros. Remember also the discussion on p449 on the transmission blocking effect of a zero, and how we have three characteristics, the zero, the input direction and the corresponding initial vector. As a really disturbing fact we note that an uncontrollable or unobservable pole i.e. a decoupling zero may be hidden by a zero $\epsilon(s)$, ie singularity of $\mathbf{P}(s)$ and even worse by it being nonsquare, as discussed on p584.

Controllability and observability of interconnected systems can be determined to some extent. Since I suggest building systems as interconnected simpler systems, the available results deserves much emphasis.

### Suitable exercises

8.3–8.9 (BoB)

### Homework problems

**8.36** Show that FSE of the two left and right MFD's

$$\{D_L(s), N_L(s)\}, \qquad \{D_R(s), N_R(s)\}$$

means that both MFD's are minimal.

**8.37** Apply the Kailath 8.2 standard PMD realization algorithm to the PMD

$$P(s) = \begin{pmatrix} s^2 & 0 \\ 0 & s \end{pmatrix}, \qquad Q(s) = \begin{pmatrix} s^2 & 1 \\ 1 & 1 \end{pmatrix},$$
$$R(s) = \begin{pmatrix} 0 & s^2 \\ 0 & 1 \end{pmatrix}, \qquad W(s) = \begin{pmatrix} -1 & 1 \\ 0 & 0 \end{pmatrix}$$

**8.38** Regard the PMD

$$P(s) = \begin{pmatrix} s^3 & 0 \\ s^2 & s \end{pmatrix}, \qquad Q(s) = I,$$
$$R(s) = \begin{pmatrix} s & 1 \\ s^2 & 0 \end{pmatrix}, \qquad W(s) = 0$$

If we consider $\{P(s), R(s)\}$ as a right MFD, we have a naturally associated controller form. Find a state similarity transformation giving the realization obtained by the standard PMD algorithm.