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GLOBAL LYAPUNOV STABILITY AND EXPONENTIAL CONVERGENCE OF DIRECT ADAPTIVE CONTROL WITH RECURSIVE LEAST SQUARES IDENTIFICATION

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Abstract

Investigation of stability properties of deterministic direct adaptive control with recursive least squares identification is made in this paper. The approach is towards traditional Lyapunov theory of stability. Global Lyapunov stability and exponential convergence are shown for one algorithm. Some instability problems are demonstrated for two other adaptive algorithms.

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INTRODUCTION

The problem of stability of direct discrete-time adaptive control with recursive least squares identification is investigated in this report.

The problem of direct adaptive control was formulated in the original article by Åström and Wittenmark (1973). These authors left the the stability problem as an open problem. Some results were soon established for systems with positive real transfer functions and systems with time delay equal to one.

The difficulties to give proofs of stability for a general class of systems inspired work into new directions, see Goodwin et al (1980) and Egardt (1980). With this approach it was possible to show stability for a broad class of adaptive control systems.

The stability concept was however different from traditional concepts of stability. The new concept was more qualitative while the traditional concepts of control theory are quantitative.

Thus, using the new stability concept it was not longer possible to give bounds for convergence rates and error magnitudes. It was neither possible to explain impacts of modelling errors on the convergence properties.

A rigorous comparison between the traditional concepts and the new concept has not been made and it is therefore difficult to translate the "new" stability properties to well known properties of e.g. asymptotic stability. Some doubts on the stability properties have been presented despite of the stability proofs. Rohrs et al (1982) showed with some examples that some disturbances in presence of unmodeled dynamics resulted in unstably growing errors. As a result the scientific discussion changed into the direction of robustness rather than stability.

This work tries to follow a traditional way of stability analysis in control theory. A first attempt in this direction was made by Johansson (1983) where global Lyapunov stability and exponential convergence were shown for a discrete time pole placement adaptive control scheme with a simple gradient method for parameter estimation. The present contribution gives similar results for deterministic pole placement adaptive control with recursive least squares estimation.

The paper starts with a formulation of algorithm elements and presents then some adaptive algorithms to be investigated. Then follows state space formulations and definitions of Lyapunov function candidates. Finally, the global Lyapunov stability and global exponential convergence are shown for one algorithm and some instability problems are demonstrated for two other but related algorithms.

THE ALGORITHM ELEMENTS

Algorithms of direct adaptive control are given below. No derivation is given here. A general introduction to the problem and may be found in Egardt (1980) where it is shown how many discrete-time adaptive control schemes in the literature may be reformulated into a unified scheme of the form below.

The adaptive algorithms require choices of an algebraic structure of polynomials to describe the process model, regulator model, a reference model etc.. Let this be denoted by <a>. Further, it is needed to choose a parameter vector to be estimated <p1|p2|p3>, a recursive least-squares type of identification method <i> and a control law <c1,c2,c3>. A particular choice of adaptive algorithm will then be described by a quadruple e.g. <a,p1,i,c>. Several other identification methods and control laws would have been possible to use with good result. This presentation does however not pretend to cover all possible cases but is restricted to three different choices of parameter vectors. These specific algorithms are described in next section.

Consider the following algebraic structure for direct adaptive control suited for model reference control and explicit pole placement control.

Process model

$$A^*(q^{-1})y(t) = b_0 q^{-k} B^*(q^{-1})u(t)$$ <a>, (1)

Regulator model

$$R^*(q^{-1})u(t) = - S^*(q^{-1})y(t) + T^*(q^{-1})u_c(t)$$ <a>, (2)

Reference model

$$y_m(t) = q^{-k}\frac{B_M^*(q^{-1})}{A_M^*(q^{-1})}u_c(t) = \frac{q^{-k}}{T_1^* A_M^*}\, y_c(t)$$ <a>, (3)

Regulator equations

$$\underset{\uparrow}{R^*}A + \underset{\uparrow}{S^*}\left[b_0 q^{-k} B^*\right] = b_0 P^*$$ <a>, (4)

$$P^*(q^{-1}) = T_1^* A_M^* B^* = 1 + p_1 q^{-1} + \ldots\ldots + p_{n_P} q^{-n_P}$$ <a>, (5)

$$T^* = T_1^* B_M^* \qquad \text{<a>, (6)}$$

Direct parametric model

$$y_f(t) = R^*\left[q^{-k}u(t)\right] + S^*\left[q^{-k}y(t)\right]; \; y_f(t) = T_1^* A_M^* y(t) \qquad \text{<a>, (7)}$$

$$e_f(t) = R^*\left[q^{-k}u(t)\right] + S^*\left[q^{-k}y(t)\right] - y_c(t-k); \; e_f(t) = T_1 \mathring{A}_M^* \mathring{e}(t)$$

Estimation models

$$y_f(t) = b_0\bar{u}(t) + \theta_1^T\bar{\varphi}_1(t) \triangleq b_0\bar{u}(t) + \theta^T\bar{\varphi}(t) \qquad \text{<p1>}$$

$$y_f(t) = b_0\bar{u}(t) + \theta_1^T\bar{\varphi}_1(t) \triangleq \theta^T\bar{\varphi}(t) \qquad \text{<p2>, (8)}$$

$$\bar{u}(t) = -\left(\frac{1}{b_0}\theta_1\right)^T\bar{\varphi}_1(t) + \left(\frac{1}{b_0}\right)y_f(t) \triangleq \theta^T\bar{\varphi}(t) \qquad \text{<p3>}$$

where φ contains components of delayed input-output data. The '—' means delay of k steps.

$$\bar{\varphi}(t) = \left[\; u(t-k) \;\; u(t-k-1) \;\; \ldots\ldots y(t-k) \ldots\ldots \right]^T \qquad (9)$$

in the case of <p2>. Similarly, the parameter vector θ_1 contains the coefficients of R^* and S^* of (2).

Recursive identification of θ

$$\zeta(t) = \theta^T\bar{\varphi}(t)$$

$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t)\bar{\varphi}(t)\varepsilon(t) \qquad \text{<i>}$$

$$P(t) = P(t-1) - \frac{P(t-1)\bar{\varphi}(t)\bar{\varphi}^T(t)P(t-1)}{1 + \bar{\varphi}^T(t)P(t-1)\bar{\varphi}(t)}$$

$$\varepsilon(t) = \zeta(t) - \hat{\theta}^T(t-1)\bar{\varphi}(t)$$

The control laws

$$u(t) = -\frac{1}{b_0}\left[\hat{\theta}_1^T(t)\varphi_1(t) - y_c(t)\right] \;;\; b_0 \text{ known a priori} \qquad \text{<c1>}$$

$$u(t) = -\frac{1}{\hat{b}_0}\left[\hat{\theta}_1^T(t)\varphi_1(t) - y_c(t)\right] \qquad \text{<c2>}$$

$$u(t) = -\left[\widehat{\frac{1}{b_0}\theta_1}\right]^T(t)\varphi_1(t) + \left[\widehat{\frac{1}{b_0}}\right]y_c(t) \qquad \text{<c3>}$$

where a correct control law would be

$$u(t) = -\frac{1}{b_0}\left[\theta^T\varphi(t) - y_c(t)\right] \qquad (10)$$

with

$$y_c(t) = T^*(q^{-1})u_c(t) \qquad (11)$$

THE ADAPTIVE ALGORITHMS

The following specific algorithms may be composed and will be investigated below.

<a,p1,i,c1> A prototype self-tuner where the gain b_0 is supposed to be known exactly a priori.

<a,p2,i,c2> An output-matching algorithm where the controller parameters including b_0 are estimated. The estimate of b_0 is used in the control law.

<a,p3,i,c3> An input-matching algorithm where the controller parameters are estimated along with the inverse $(1/b_0)$ of the gain.

It is in all cases supposed that the correct orders of polynomials are known and that the time delay k is known.

A STATE VECTOR FOR PARAMETER ESTIMATES

The parameter estimates are denoted by $\hat{\theta}$ and the error is $\tilde{\theta}=\hat{\theta}-\theta$. The least-squares criterion is denoted by $J(\theta)$. The one-step ahead convergence properties of recursive least-squares estimation is treated in textbooks on identification, see Ljung and Söderström (1983).

The parameter error vector $\tilde{\theta}$ is a natural state representation. It is however necessary to have k consecutive θ:s in order to have a full state description for the k step delay case. One state representation k steps ahead is given by

$$\Xi(t) = \left[\tilde{\theta}^T(t+k-1) \; \ldots . \tilde{\theta}^T(t)\right]^T \tag{12}$$

where the P-matrix generalizes to

$$\Pi(t) = \mathrm{diag}\left[P(t+k-1), \ldots ., P(t)\right]$$

The convergence properties of Ξ is described in detail in appendix 1 where it is also shown that the parameter error Lyapunov function candidate

$$V_\theta\left[\Xi(t)\right] = \sum_{i=t}^{t+k-1} J\left[\hat{\theta}(i)\right] \tag{13}$$

develops over time as a non-increasing function, see (A1.25).

A STATE SPACE MODEL

The growth rate of a state vector x(t) representing the states of the control object and the controller will now be investigated.

The state vector of the control object (1) is easy to represent via a reformulation to a fraction form

$$\begin{aligned} A^*(q^{-1})\xi(t) &= u(t) \\ y(t) &= b_0 B^*(q^{-1})\xi(t) \end{aligned} \tag{14}$$

where $\xi(t)$ is referred to as the partial state. A state vector for the control object is

$$\left[\xi(t-1)\ \xi(t-2)\ldots\ldots\xi(t-n_A)\right]^T \qquad (15)$$

where n_A is the appropriate order. The states of the regulator (2) may be represented in the same way. However, the regulator makes use of old input--output data. It is then natural to express also the regulator states of the vector φ in terms of ξ.

$$x(t) = \left[\xi(t-1)\ \xi(t-2)\ldots\ldots\xi(t-n_x)\right]^T \qquad (16)$$

The order of x is some number n_x that is chosen large enough to enable us to write

$$\varphi_1(t) = M_\varphi x(t) \qquad (18)$$

for a matrix M_φ which contains the parameters of the A^*- and B^*-polynomials.

$$\varphi_1(t)=\begin{bmatrix} u(t-1) \\ \vdots \\ u(t-k+1) \\ y(t) \\ y(t-1) \\ \vdots \end{bmatrix} = \begin{bmatrix} 1 & a_1 & a_2 & \ldots a_n & 0\ldots0 \\ 0 & \ldots & 1 & \ldots\ldots a_n & \\ 0 & \ldots\ldots 0 & b_0 & b_1 & \ldots \\ 0 & \ldots\ldots & & & \\ \vdots & & & & \end{bmatrix} \begin{bmatrix} \xi(t-1) \\ \vdots \\ \xi(t-n_x) \end{bmatrix} = M_\varphi x(t)$$

A special form of the state-space equation sufficient to cover (1) and (2) will now be used. Define via (5)

$$\nu(t) \triangleq P^*(q^{-1})\xi(t) = u(t) + \frac{1}{b_0}\cdot\theta_1^T\varphi(t) \qquad (19)$$

Then it follows that a state space description is given by

$$x(t+1) = \Phi x(t) + \Gamma\nu(t)$$

The Φ-matrix and the Γ-vector are then

$$\Phi=\begin{bmatrix} -p_1 & -p_2 & -p_3\cdot\ldots\ldots & -p_n \\ 1 & 0\ldots\ldots\ldots\ldots & & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & \\ 0\ldots\ldots\vdots & .0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} p^T & \\ \hline I_{(n-1)x(n-1)} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \end{bmatrix};\quad \Gamma=\begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \qquad (20)$$

The components of the vector p are the coefficients p_i of the polynomial P^*. Notice that the polynomial P^* represents all poles of the closed-loop system - also those poles which cancel the zeros of the B^*-polynomial.

THE GROWTH RATE OF $\xi(t)$

We will now investigate the growth rate of the state vector. It is then possible to introduce a norm or a Lyapunov function candidate which grows when the norm of x grows. Introduce therefore the function

$$V_x\left[x(t)\right] = \ln\left[1 + \mu x^T(t)\Lambda x(t)\right] \tag{22}$$

with a constant $\mu>0$ and with Λ chosen as a positive definite matrix. The matrix Λ will be chosen as a solution to the Lyapunov equation

$$\Phi^T\Lambda\Phi - \Lambda = - Q - pp^T \tag{23}$$

where we also require that

$$\Lambda\Gamma = \Gamma \tag{24}$$

The vector Γ is then an eigenvector of Λ. Firstly, we need to show existence of solutions to (23-24) in order to proceed. This is done in the following lemma.

Lemma:

There are solutions to the equations (23-24) if

$$\mathrm{Re}\left\{(z+1)\Gamma^T(I+\Phi)\left[zI-\Phi\right]^{-1}\Gamma\right\} > 0 \quad ; \qquad z = e^{i\omega} \quad ; \; -\pi \le \omega \le \pi \tag{25}$$

Proof: See appendix 2.

A necessary condition for (23) to hold for positive definite Λ and Q is that Φ has all its eigenvalues within the unit circle. For example, all stable first order polynomials satisfy (25). The theorem is valid for closed-loop pole locations in a restricted part of the stable region.

An Example

Assume that the desired closed-loop pole locations are given by the polynomial

$$P^*(q^{-1}) = 1 \tag{26}$$

This means that $p_1,\dots,p_n$ of (27) are all zero. Let ϱ be some constant in the open interval $]0,1[$. Then define the matrix

$$\Lambda_\varrho = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & \varrho^2 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & \varrho^{2n} \end{bmatrix} \qquad (27)$$

It is then easy to see that Λ_ϱ and Q_ϱ below satisfies equations (23-24).

$$Q_\varrho = \begin{bmatrix} 1-\varrho^2 & 0 & \cdots & \cdots & 0 \\ 0 & \varrho^2-\varrho^4 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & \cdots & \cdots & 0 & \varrho^{2n} \end{bmatrix} > \left[1-\varrho^2\right]\Lambda_\varrho \qquad (28)$$

It is now possible to give a statement about the growth rate of function (21) and the state magnitude.

Result:

The adaptive control algorithm <a,p1,i,c1>, <a,p2,i,c2>, <a,p3,i,c3> may at most result in a growth rate of the state vector such that

$$V_x\left[x(t+1)\right] - V_x\left[x(t)\right] \le \qquad (29)$$

$$\le -\mu \frac{x^T(t)Qx(t)}{1+\mu x^T(t)\Lambda x(t)} - \frac{\left[p^T x(t) - \nu(t)\right]^2}{1+\mu x^T(t)\Lambda x(t)} + 2\mu \frac{\nu^2(t)}{1+\mu x^T(t)\Lambda x(t)}$$

Proof: See appendix 3.

Remark:

The fact that the condition for existence of solutions to equation (25) does <u>not</u> mean that the proof only covers the class of systems usually thought of as "positive real transfer functions", cf. Landau (1979) and Ljung (1977).

Notice also that the condition enters in such away that there are only restrictions on the closed-loop pole locations. There are e.g. no restrictions on the number of time delays.

THE P-MATRIX

In order to represent the states it is finally necessary to consider the P-matrix. A scalar function to represent the Π-matrix is e.g.

$$V_P\left[\Pi(t)\right] = \sum_{i=t}^{t+k-1} \operatorname{tr} P(i) \tag{30}$$

It is trivial to show that each tr(P) decreases at each recursion. Since the P-matrix is a positive definite matrix with positive eigenvalues, it follows that (30) is decreasing and positive.

GLOBAL LYAPUNOV STABILITY

We will now give a Lyapunov function for the considered adaptive systems for the reference value $y_c=0$. Start to consider the parameter convergence properties. Applying (A1.25) to <p1>, <p2> and <p3> when the control laws are <c1>, <c2> and <c3>, respectively, gives via (A1.21) and (29) that

$$V_\theta\left[\Xi(t+1)\right] - V_\theta\left[\Xi(t)\right] \le - \frac{\upsilon^2(t)}{1 + \bar{\varphi}^T(t+k)P(t)\bar{\varphi}(t+k)} \tag{A1.25}$$

with

$$\upsilon^2(t) = \begin{cases} b_0^2\nu^2(t) & \langle a, p1, i, c1\rangle \\ b_0^2\nu^2(t) & \langle a, p2, i, c2\rangle \\ b_0^2\nu^2(t)\left[\dfrac{\hat{i}}{b_0}\right]^2(t) & \langle a, p3, i, c3\rangle \end{cases} \tag{32}$$

Assume that the initial value of the P-matrix (A1.4) for <p1> is $P_0=P_0^T>0$. Let the initial value of the P-matrix of <p2> and <p3> be $\operatorname{diag}(1,P_0)$. This a special choice for <p2> and <p3> but all arguments are the same in the general case. Choose the constant μ of (22) so that

$$\mu\cdot\Lambda > M_\varphi^T P_0 M_\varphi \tag{33}$$

where M_φ is defined in (18). With this choice of μ, it holds for <p1> that

$$\frac{-\,\upsilon^2(t)}{1+\varphi^T(t)P(t)\varphi(t)} \le \frac{-\,\upsilon^2(t)}{1+\varphi_1^T(t)P_0\varphi_1(t)} \le \frac{-\,\upsilon^2(t)}{1+\mu x^T(t)\Lambda x(t)} \qquad (34)$$

The function

$$V\left[X(t)\right] = V_\theta\left[\Xi(t)\right] + KV_x\left[x(t)\right] + V_P\left[\Pi(t)\right] \qquad (35)$$

composed of (13), (22) and (30) decreases with time at least as

$$V\left[X(t+1)\right] - V\left[X(t)\right] \le -\,\frac{1}{2}b_0^2\cdot\frac{\mu x^T(t)Qx(t)}{1+\mu x^T(t)\Lambda x(t)} \qquad (36)$$

for

$$K > \frac{1}{2\mu}\cdot b_0^2 \qquad (37)$$

This is shown via (32), (34), (A1.25) and (A3.6). The function V of (35) is thus shown to be a Lyapunov function. The right hand side is non-positive but not necessarily negative for a non-zero parameter error. This is sufficient for global Lyapunov stability but not sufficient to claim asymptotic stability.

STABILITY PROBLEMS OF FULL OUTPUT- AND INPUT-MATCHING

Calculations similar to (34) for the full output- and input-matching algorithms gives respectively

$$\frac{-\,\upsilon^2(t)}{1+\varphi^T(t)P(t)\varphi(t)} \le \begin{cases} \dfrac{-\,b_0^2\nu^2(t)}{1+u^2(t)+\mu x^T(t)\Lambda x(t)} \\[2ex] \dfrac{-\,b_0^2\nu^2(t)}{1+b_0^2\nu^2(t)+\mu x^T(t)\Lambda x(t)}\left[\dfrac{1}{\hat{b}_0}\right]^2 \end{cases} \qquad (38a\text{-}b)$$

It is seen that no K is sufficient to make a Lyapunov function for neither the output-matching nor the input-matching algorithms. The problem for the output-matching algorithm is associated with the denominator (38a) which depends on u. For any K it is possible to find a (large) u which results in growth of V.

For the input matching algorithm there are two problems. Firstly, for any K and when $|\nu|$ is large enough it follows that V grows. Secondly, when the estimate of the inverse gain is small, it follows that (38b) is small in magnitude and V may grow.

However, in harmony with the arguments in stability proofs ad modum Goodwin et al, it follows that a large u or ν will bound the right hand sides of (38) away from zero and will eventually result in parameter convergence. There is then however no longer any obvious relation between the growth of V_x and the descent of V_θ.

EXPONENTIAL CONVERGENCE

Exponential convergence to an arbitrarily small neighbourhood around the origin may be shown. Consider e.g. the unit sphere of the state space x. Let q_{min} and λ_{max} be extremal eigenvalues of Q and Λ, respectively. For $\|x\|\geq 1$ it follows from (36) that

$$V\left[X(t+1)\right] - V\left[X(t)\right] \leq \tag{39}$$

$$\leq -\frac{1}{2}b_0^2 \cdot \frac{x^2(t)q_{min}}{1+\mu x^2(t)\lambda_{max}} \leq -\frac{1}{2}b_0^2 \cdot \frac{q_{min}}{1+\mu\lambda_{max}} \triangleq -\alpha$$

Then it follows from the properties of the Lyapunov function and that the maximally attainable value of

$$V_x\left[x(t)\right] \leq V\left[X(t)\right] \leq V\left[X(0)\right] - \alpha t \tag{40}$$

Since V_x is logarithmic it follows that

$$x^T(t)\Lambda x(t) \tag{41}$$

is bounded from above by an exponentially decreasing function with time constant α of (39). Hence, there is global exponential convergence to the neighbourhood of the origin.

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APPENDIX 1

The parameters θ of (A1.1) will be estimated given $(\zeta,\bar{\varphi})$.

$$\zeta(t) = \theta^T\bar{\varphi}(t) \tag{A1.1}$$

Least-squares criterion with respect to N pairs of $(\zeta,\bar{\varphi})$.

$$J(\hat{\theta}) = \sum_{i=1}^{N} \left[\zeta(i) - \hat{\theta}^T\bar{\varphi}(i)\right] \tag{A1.2}$$

Recursive equation

$$\hat{\theta}(t) = \hat{\theta}(t-1) + P(t)\bar{\varphi}(t)\varepsilon(t) \tag{A1.3}$$

$$P(t) = P(t-1) - \frac{P(t-1)\bar{\varphi}(t)\bar{\varphi}^T(t)P(t-1)}{1 + \bar{\varphi}^T(t)P(t-1)\bar{\varphi}(t)} \tag{A1.4}$$

$$\varepsilon(t) = \zeta(t) - \hat{\theta}^T(t-1)\bar{\varphi}(t) \tag{A1.5}$$

Cf. (Ljung & Söderström,1983; (2.21)).

$$\hat{\theta}(t_1) = P(t_1) \left[P^{-1}(t_0)\hat{\theta}(t_0) + \sum_{i=t_0+1}^{t_1} \bar{\varphi}(i)\zeta(i)\right] \tag{A1.6}$$

$$\hat{\theta}(t_1) = \left[P^{-1}(t_0) + \sum_{i=1}^{t_1} \bar{\varphi}(i)\bar{\varphi}^T(i)\right]^{-1} \left[P^{-1}(t_0)\hat{\theta}(t_0) + \sum_{i=1}^{t_1} \bar{\varphi}(i)\zeta(i)\right] \tag{A1.7}$$

Reformulate the cost criterion J of (A1.2) with the parameter error $\tilde{\theta}=\hat{\theta}-\theta$.

$$J\left[\hat{\theta}(t)\right] = \tilde{\theta}^T(t)P^{-1}(t)\tilde{\theta}(t) \tag{A1.8}$$

Any pair $(\zeta,\bar{\varphi})$ which satisfies (A1.1) and which is used in the recursion of (A1.3-5) results in a decreasing cost criterion J. In each step it holds that

$$J\left[\hat{\theta}(t)\right] - J\left[\hat{\theta}(t-1)\right] = - \frac{\varepsilon^2(t)}{1 + \bar{\varphi}^T(t)P(t-1)\bar{\varphi}(t)} \tag{A1.9}$$

Now consider the parameter error state vector (12)

$$\Xi(t) = \left[\tilde{\theta}^T(t+k-1) \;.....\; \tilde{\theta}^T(t)\right]^T \qquad (A1.10)$$

and the corresponding P-matrices

$$\Pi^{-1}(t) = \text{diag}\left[P^{-1}(t+k-1),.....,P^{-1}(t)\right] \qquad (A1.11)$$

A Lyapunov function candidate with respect to the parameter errors that covers all the k steps is defined by

$$V_\theta\left[\Xi(t)\right] = \sum_{i=t}^{t+k-1} J\left[\hat{\theta}(i)\right] \qquad (13),\;(A1.12)$$

It is shown via (A1.8) and (A1.12) that

$$V_\theta\left[\Xi(t)\right] = \Xi^T(t)\Pi^{-1}(t)\Xi(t) = \sum_{i=t}^{t+k-1} \tilde{\theta}^T(i)P^{-1}(i)\tilde{\theta}(i) \qquad (A1.13)$$

The Lyapunov function candidate (A1.12) develops one step ahead as

$$V_\theta\left[\Xi(t+1)\right] - V_\theta\left[\Xi(t)\right] = J\left[\hat{\theta}(t+k)\right] - J\left[\hat{\theta}(t)\right] \qquad (A1.14)$$

The $\bar{\varphi}$-vector to be used in the identification at time (t+k) is already determined by the φ-vector used in the control law at time k. Also, the output y at time (t+k) is determined by the inputs up to time t. Thus, the updatings of θ at time (t+1),...,(t+k) are fully determined already at time t although they are not performed by the recursive algorithm until k steps later. The Ξ-vector belongs therefore to the state-vector of the system.

Reconsider now (A1.7) with respect to the updatings to come in the next k recursions. Formally, we have

$$\hat{\theta}(t+k) = \left[P^{-1}(t) + \sum_{i=t+1}^{t+k} \bar{\varphi}(i)\bar{\varphi}^T(i)\right]^{-1}\left[P^{-1}(t)\hat{\theta}(t) + \sum_{i=t+1}^{t+k} \bar{\varphi}(i)\zeta(i)\right]$$

$$P^{-1}(t+k) = P^{-1}(t) + \sum_{i=t+1}^{t+k} \bar{\varphi}(i)\bar{\varphi}^T(i) \qquad (A1.14-15)$$

The computation of (A1.14) is implemented by the recursive equations (A1.3-5) operating on the following time sequence of φ-vectors

$$\bar{\varphi}(t+1) \rightarrow \bar{\varphi}(t+2) \rightarrow ..... \rightarrow \bar{\varphi}(t+k) \qquad (A1.16)$$

The result is the parameter vector sequence

$$\hat{\theta}(t) \rightarrow \hat{\theta}(t+1) \rightarrow \hat{\theta}(t+2) \rightarrow \ldots\ldots \rightarrow \hat{\theta}(t+k) \tag{A1.17}$$

which gives the least-squares estimate based on the accumulated data collected at each time instant.

Notice that an off-line least-squares estimate of the parameters at time (t+k) would give the same result. Any other iteration order of the k recursion steps would also give the same end result of (A1.14-15) and hence (A1.13). In order to evaluate $J(\hat{\theta}(t+k))$ it is thus not necessary to respect the natural iteration order as long as no significance is given to the intermediate estimates.

A recursion order which is suitable for evaluation of the cost function and which results in the same parameter estimates $\hat{\theta}$ and the same value of J after k steps is e.g. the reverse order:

$$\bar{\varphi}(t+k) \rightarrow \bar{\varphi}(t+k-1) \rightarrow \ldots\ldots \rightarrow \bar{\varphi}(t+1) \tag{A1.19}$$

This order of recursion would give the following sequence of parameter estimates

$$\hat{\theta}(t) \rightarrow \hat{\theta}'(t+1) \rightarrow \ldots\ldots \rightarrow \hat{\theta}'(t+k-1) \rightarrow \hat{\theta}(t+k) \tag{A1.20}$$

where $\hat{\theta}'$ denotes the intermediate parameter estimates. Let $\tilde{\theta}'$ denote the corresponding parameter error vector. The first "reverse order" updating with $\bar{\varphi}(t+k)$ then gives rise to the prediction error $\varepsilon'(t+1)$

$$\begin{aligned}\varepsilon'(t+1) &= \zeta(t+k) - \hat{\zeta}(t+k|t) = \\ &= \zeta(t+k) - \hat{\theta}(t)\bar{\varphi}(t+k) = -\tilde{\theta}^T(t)\bar{\varphi}(t+k) \triangleq \upsilon(t)\end{aligned} \tag{A1.21}$$

where the last expression is easy to relate to the output error at time (t+k).

$$J\left[\hat{\theta}'(t+1)\right] - J\left[\hat{\theta}(t)\right] = -\frac{\upsilon^2(t)}{1 + \bar{\varphi}^T(t+k)P(t)\bar{\varphi}(t+k)} \tag{A1.22}$$

Since all recursive updatings decrease J, we have

$$J\left[\hat{\theta}(t+k)\right] \le J\left[\hat{\theta}'(t+k-1)\right] \le \ldots \le J\left[\hat{\theta}'(t+1)\right] \tag{A1.23}$$

The cost function J of (A1.8) then develops over k recursions at least as

$$J\left[\hat{\theta}(t+k)\right] - J\left[\hat{\theta}(t)\right] \le -\frac{\upsilon^2(t)}{1 + \bar{\varphi}^T(t+k)P(t)\bar{\varphi}(t+k)} \tag{A1.24}$$

It is then possible to establish that V_θ of (A1.13) develops at least as

$$V_\theta\left[\Xi(t+1)\right] - V_\theta\left[\Xi(t)\right] \le -\frac{\upsilon^2(t)}{1 + \varphi^T(t)P(t)\varphi(t)} \tag{A1.25}$$

APPENDIX 2

Solutions to the Lyapunov equation

The existence of solutions to (23-24) may be brought back to Kalman-Yakubovich lemma as is shown in this appendix.

Consider the equations (23-24)

$$\Phi^T\Lambda\Phi - \Lambda = - Q - pp^T \tag{23}$$

$$\Lambda\Gamma = \Gamma \tag{24}$$

Define now the mapping

$$F = \left(\Phi - I\right)\left(\Phi + I\right)^{-1} \quad \text{or} \quad \Phi = \left(I + F\right)\left(I - F\right)^{-1} \tag{A2.1}$$

According to Kalman-Yakubovich lemma (see Narendra and Taylor (1973) p.48ff.) there exists a positive definite matrix Λ solving

$$\Lambda F + F^T\Lambda = - Q_0 - qq^T \quad ; \quad \Lambda g = h + \sqrt{\gamma}\, q \tag{A2.2}$$

for vectors g, h, q and $Q_0=Q_0^T>0$ if

$$\operatorname{Re}\left\{h^T\left(sI - F\right)^{-1}g\right\}_{s=i\omega} \geq - \frac{1}{2}\gamma \tag{A2.3}$$

Taking $g=\Gamma$ it is seen that all F must obey

$$\operatorname{Re}\left\{h^T\left(sI - F\right)^{-1}\Gamma\right\}_{s=i\omega} \geq - \frac{1}{2}\gamma \quad ; \quad h = \Lambda g - \sqrt{\gamma}\, q \tag{A2.4}$$

Now take

$$Q = 2\left(I - F\right)^{-T}Q_0\left(I - F\right)^{-1} \quad ; \quad q = \frac{1}{\sqrt{2}}\left(I - F\right)^T p \tag{A2.5}$$

It is then straightforward to show from (A2.1-5) that Λ and Q solves

$$\Phi^T\Lambda\Phi - \Lambda = - Q - pp^T \quad ; \quad \Lambda\Gamma = \Gamma \tag{A2.6}$$

when F satisfies

$$\operatorname{Re}\left\{\Gamma^T\left(sI - F\right)^{-1}\Gamma\right\}_{s=i\omega} \geq - \frac{1}{2}\gamma + \sqrt{\gamma}\operatorname{Re}\left\{q^T\left(sI - F\right)^{-1}\Gamma\right\}_{s=i\omega} \tag{A2.7}$$

for some $\gamma \geq 0$. In particular, when $\gamma=0$

$$\Phi^T\Lambda\Phi - \Lambda = - Q - pp^T \quad ; \quad \Lambda\Gamma = \Gamma \tag{A2.8}$$

if

$$\mathrm{Re}\left\{\Gamma^T\left(sI - F\right)^{-1}\Gamma\right\}_{s=i\omega} \geq 0 \tag{A2.9}$$

where (A2.9) is recognized as a positive real condition on the transfer function

$$G(s) = \Gamma^T\left(sI - F\right)^{-1}\Gamma \tag{A2.10}$$

This condition is transformed to a condition for the discrete-time description via (A2.1) and

$$s=\frac{z-1}{z+1} \tag{A2.11}$$

It follows that

$$sI - F = \frac{2}{z+1}\left(zI - \Phi\right)\left(\Phi + I\right)^{-1} \tag{A2.12}$$

and

$$G(s) = \frac{z+1}{2}\Gamma^T\left(\Phi + I\right)\left(zI - \Phi\right)^{-1}\Gamma \triangleq H(z) \tag{A2.13}$$

The condition (A2.9) now becomes

$$\mathrm{Re}\ H(z)\Big|_{z=e^{i\omega}} = \mathrm{Re}\left\{\frac{z+1}{2}\Gamma^T\left(\Phi + I\right)\left(zI - \Phi\right)^{-1}\Gamma\right\}\Bigg|_{z=e^{i\omega}} \geq 0 \tag{A2.14}$$

for real ω in the interval $-\pi \leq \omega \leq \pi$.

APPENDIX 3

$$V_x[x(t+1)] - V_x[x(t)] =$$

$$= \ln\left[\frac{1 + \mu x^T(t+1)\Lambda x(t+1)}{1 + \mu x^T(t)\Lambda x(t)}\right] \tag{A3.1}$$

$$= \ln\left[1 + \mu\frac{x^T(t+1)\Lambda x(t+1) - x^T(t)\Lambda x(t)}{1 + \mu x^T(t)\Lambda x(t)}\right] = \tag{A3.2}$$

Now apply (23) to (A3.2) to obtain

$$= \ln\left[1 + \mu\frac{x^T(t)\left[\Phi^T\Lambda\Phi - \Lambda\right]x(t) + 2\nu(t)\Gamma^T\Lambda\Phi x(t) + \nu^2(t)\Gamma^T\Lambda\Gamma}{1 + \mu x^T(t)\Lambda x(t)}\right] = \tag{A3.3}$$

Simplification of (A3.3) with (23) and (24) gives

$$= \ln\left[1 + \mu\frac{x^T(t)\left[- Q - pp^T\right]x(t) + 2p^Tx(t)\nu(t) + \nu^2(t)}{1 + \mu x^T(t)\Lambda x(t)}\right] = \tag{A3.4}$$

Completing the squares gives

$$= \ln\left[1 + \mu\frac{- x^T(t)Qx(t) - \left[p^Tx(t) - \nu(t)\right]^2 + 2\nu^2(t)}{1 + \mu x^T(t)\Lambda x(t)}\right] \le \tag{A3.5}$$

Since $\ln(1+z) \le z$ for all real $z>-1$, it follows that (A3.5) is bounded by

$$\le - \mu\frac{x^T(t)Qx(t)}{1 + \mu x^T(t)\Lambda x(t)} - \mu\frac{\left[p^Tx(t)-\nu(t)\right]^2}{1 + \mu x^T(t)\Lambda x(t)} + 2\mu\frac{\nu^2(t)}{1 + \mu x^T(t)\Lambda x(t)}$$

Notice that there is only one term responsible for the growth of V_x namely

$$2\mu\frac{\nu^2(t)}{1 + \mu x^T(t)\Lambda x(t)} \tag{A3.7}$$