Invariants Based on Areas and Volumes in Projective Spaces

Nielsen, Lars; Sparr, Gunnar

1987

CODEN: LUTFD2/(TFRT-7357)/1-37/(1987)

Invariants based on areas and volumes in projective spaces

Lars Nielsen
Gunnar Sparr

Department of Automatic Control
Lund Institute of Technology
April 1987

Department of Automatic Control
Lund Institute of Technology
P.O. Box 118
S-221 00 Lund Sweden

Document name

Date of issue
April 1987

Document Number
CODEN: LUTFD2/(TFRT-7357)/1-37/(1987)

Author(s)
Lars Nielsen
Gunnar Sparr

Supervisor

Sponsoring organisation

Title and subtitle
Invariants based on areas and volumes in projective spaces

Abstract

Areas and volumes in projective spaces are defined, and invariant relations between them are derived. The treatment is inspired by geometrical interpretations on the projective line and plane.

Key words
Areas, invariants, projective geometry.

Classification system and/or index terms (if any)

Supplementary bibliographical information

ISSN and key title

ISBN

Language
English

Number of pages
37

Security classification

Recipient's notes

The report may be ordered from the Department of Automatic Control or borrowed through the University Library 2, Box 1010, S-221 03 Lund, Sweden, Telex: 33248 lubbis lund.

Contents

1. Introduction

Plane projective geometry treats properties of geometric figures that are invariant under certain transformations, the "projectivities". The classical theorems often deal with incidences of points and lines. On the contrary, metrical concepts, such as distance and area, have less natural sites in projective geometry. One well known exception is the cross-ratio, relating distances between colinear points. In this paper similar relations will be derived for the areas of certain plane figures. Analogous results will also be obtained for intervals on the line and volumes in higher dimensional spaces.

The plan is as follows. In Chapter 2 the very concepts of area and invariance are discussed. We start in Chapter 3 with the projective line, to illustrate the geometrical ideas and the relation to the cross-ratio. The projective plane is treated in Chapter 4. Chapter 5 is devoted to the general n-dimensional case. In Chapter 6, finally, we consider area-invariants for conical sections in the plane. As our main results we consider the theorems of Chapters 4 and 6, dealing with the projective plane. (This is also the reason for choosing the word "area" in the title.)

The present work has its origin in image analysis. There the scene is projected through a camera lens to form an image. The problem is to recognize objects in the scene by measurements in the image. Our goal is to find certain classes of objects suited for recognition by computer vision. These objects can be used as sign posts or marking symbols in the applications. For reasons of error robustness and existing hardware, *area* measurements are preferable. The regions of interest must be identified in the image before the area measurements can be done. Different identification methods are available, but they are all more successful in images with high contrast. Especially well suited are marking symbols composed by regions with non-intersecting boundaries. (In particular such symbols may be colored by means of black and white only.)

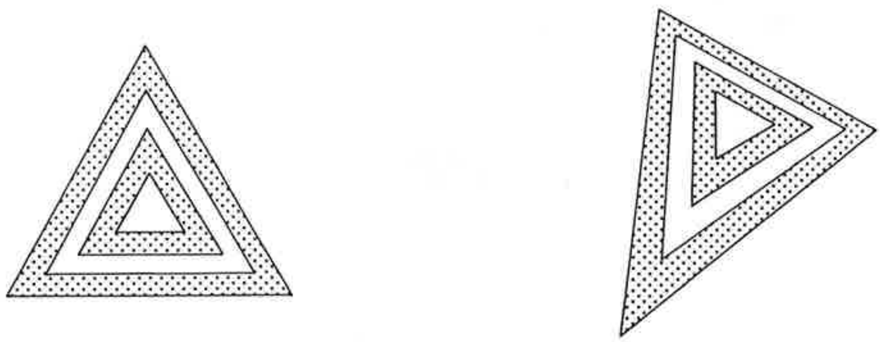

Figure 1.1 An example of a feasible object and a possible image of it

Figure 1.1 shows one feasible object and a possible image of it. This symbol, composed by inscribed triangles, and also symbols composed by inscribed rectangles, were studied first in Nielsen [N] and later in Nielsen-Sparr [N-S]. There the formulas in Lemma 4.1 (ii) and Theorem 4.2 were derived for perspectivities. A forthcoming article will concentrate on the application of the present results.

In the growth of this article a particular role has been played by the symbolic manipulation program Macsyma. It was especially useful in the first discovery of area-invariants in [N], and has also been used for experimental purposes thereafter.

2. Areas and Invariants

2.1 Preliminaries

The n-dimensional *projective space* $\mathbb{P}^n$ is defined as $(\mathbb{R}^{n+1} \setminus \{0\})/\sim$, where $x \sim y \Leftrightarrow x = \varrho y$, $0 \neq \varrho \in \mathbb{R}$. (In other words $\mathbb{P}^n$ consists of all non-zero $(n+1)$-tuples of real numbers, identifying proportional ones.) This will be the fundamental set for this work. The n-dimensional *affine space* A^n is obtained from $\mathbb{P}^n$ by deletion of one specific hyperplane $\Sigma_0^n \alpha_i x_i = 0$. What remains of $\mathbb{P}^n$ may then be represented by e.g. the hyperplane $\pi_\alpha : \Sigma_0^n \alpha_i x_i = 1$ in $\mathbb{R}^{n+1}$. This set thus serves as one possible model for A^n. On the contrary, an affine space $\pi_\alpha : \Sigma_0^n \alpha_i x_i = 1$ in $\mathbb{R}^{n+1}$, augmented with the lines through the origin in the hyperplane $\Sigma_0^n \alpha_i x_i = 0$, makes a model of $\mathbb{P}^n$. Here the latter hyperplane is called "the plane at infinity". Introducing in A^n a concept of orthogonality, one obtains the *Euclidean space* $\mathbb{E}^n$. Here it is possible to compare segments on different lines.

Given a set S and a group G of bijections on S, by a *geometry* is meant, loosely speaking, the collection of all properties involving elements of S which are preserved under all transformations of G. In *projective geometry* the set S is $\mathbb{P}^n$ while the group of transformations G is $PGL(n) = GL(n+1)/\sim$, where $A \sim B \Leftrightarrow A = \varrho B, 0 \neq \varrho \in \mathbb{R}$. (In other words, $PGL(n)$ consists of all square non-singular $(n+1)$-matrices, identifying proportional ones.) Elements of $PGL(n)$ are called *projectivities.* In *affine geometry* S is A^n while G is the subgroup of $PGL(n)$ consisting of transformations leaving the plane at infinity invariant. These are called *affinities.* In *Euclidean geometry* the set S is $\mathbb{E}^n$ while G is the subgroup of the affinities whose elements are distance-preserving. These are called *isometries.*

By definition, *distance* between points is thus a property dealt with in Euclidean geometry. On the contrary, in affine geometry it is not possible to compare distances between points, unless they all lie on the same line. In projective geometry even the concept of distance remains to be defined.

It is somewhat remarkable that the concept of *area* (the term area will also sometimes be used for volumes in higher dimensions) makes sense in affine geometry, despite the fact that in elementary geometry it is usually defined in terms of distances. This is so because, once coordinates are introduced, the definition of area is a purely analytical matter of measure and integration theory. In particular the frequent use of rectangular coordinates, and rectangular areas, in the definition of the integral is immaterial. Under affine transformations areas are changed by a factor equal to the determinant of the transformation matrix only. Hence it is a "relative invariant" in affine geometry.

For projective geometry the question arises to what extent the concept of area in an affine space can be transferred to a projective space, claiming the existence of area-relations that are invariant under projectivities. For $n = 1$ the cross-ratio makes an example. Analogous expressions were also studied for triangles and tetrahedrons in Möbius [M] "Der barycentrische Calcul" (1829).

Else, although a natural problem, it seems to have been little studied. An effort is made in this paper.

2.2 Areas

We start with an example.

EXAMPLE 2.1
Let A_0, A_1, A_2 be three points in a plane π in the Euclidean space $\mathbb{E}^3$. The points are represented by their coordinates. Suppose that the origin $O \notin \pi$. Then

$\det(A_0, A_1, A_2)$ = the volume, with signs depending on the orientation, of the parallelepiped spanned by the vectors $\overline{OA_0}, \overline{OA_1}, \overline{OA_2}$ =

6·(the volume of the tetrahedron with vertices in (O, A_0, A_1, A_2))=

3·(the area of the triangle with vertices (A_0, A_1, A_2)) · (the distance between O and π).

In other words, apart from a factor of proportionality, $\det(A_0, A_1, A_2)$ measures the area of the triangle in π, having vertices in A_0, A_1, A_2. □

The same argument in a general *Euclidean space* $\mathbb{E}^{n+1}$ shows that, given $n+1$ points $A_0, \ldots, A_n$ in π, then $\det(A_0, \ldots, A_n)$ measures the volume (with signs) of the polyhedron in π with vertices $A_0, \ldots, A_n$. A calculation of this volume by integration would of course lead to the same formula. However, as was pointed out above, the rules for integrals do not depend on the Euclidean structure, and thus will give the same result in *affine spaces*. This observation allows us to think of $\det(A_0, \ldots, A_n)$ as the signed volume of a polyhedron, even in the affine case.

Under affine transformations of π all volumes are changed by a common factor, the determinant of the transformation matrix. One thus arrives at the first invariancy result: For any two $(n+1)$-tuples $(A_0, \ldots, A_n)$ and $(B_0, \ldots, B_n)$ in π, the quotient

$$\det(A_0, \ldots, A_n) / \det(B_0, \ldots, B_n)$$

is invariant under affine transformations on π. An equivalent statement is that the quotient of the volumes of two given polyhedrons is independent of their coordinate representations.

We now turn to *projective spaces*, where the notion of area/volume has no a priori meaning. A definition will be made, inspired by the above affine considerations in the case $\pi : \Sigma_0^n x_i = 1$. This plane augmented with the plane at infinity $\Sigma_0^n x_i = 0$, will then serve as a model for $\mathbb{P}^n$.

For $X = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$, put

$$\sigma(X) = \sum_0^n x_k \tag{2.1}$$

Further, for $X_0, \ldots, X_n \in \mathbb{R}^{n+1}$, put

$$\delta(X_0, \ldots, X_n) = \begin{cases} \frac{\det(X_0, \ldots, X_n)}{\sigma(X_0) \ldots \sigma(X_n)} & \text{if } \sigma(X_0) \ldots \sigma(X_n) \neq 0 \\ 0 & \text{if } \det(X_0, \ldots, X_n) = 0 \\ \infty & \text{if } \det(X_0, \ldots, X_n) \neq 0, \\ & \quad \sigma(X_0) \ldots \sigma(X_n) = 0 \end{cases} \tag{2.2}$$

Here one notices that, by homogeneity, δ is in fact a function on $\mathbb{P}^n \times \ldots \times \mathbb{P}^n$ ($n+1$ times). This fact alone does not qualify it to be a meaningful object in projective geometry. For this also some sort of projectivity invariance is needed. Clearly δ standing for itself does not have such a property. However, there are equations involving several δ-expressions which have (cf. e.g. Theorems 2.1 and 5.1). Thus, when appearing together with others in such an equation, δ gets a projective meaning. Here it can in fact also be interpreted in terms of affine volumes. To see this, denote by $\tilde{A}_i$ the representative in π for $A_i \in \mathbb{P}^n$, not a point at infinity, $i = 0, \ldots, n$. Then δ has the following interpretation

$$\delta(A_0, \ldots, A_n) = \begin{cases} & \text{the signed volume of the polyhedron} \\ & \text{with vertices } \tilde{A}_0, \ldots, \tilde{A}_n \text{ if} \\ & \text{no } A_k \text{ is a point at infinity.} \\ 0 & \text{if the } (n+1)\text{-tuple is degenerate} \\ \infty & \text{if some } A_k \text{ is a point at infinity} \end{cases}$$

The particular choice of $\pi : \sum_0^n x_i = 1$ may seem somewhat arbitrary at first sight. However, the discussion above may also be formulated in terms of coordinate changes instead of mappings. (Both operations correspond to premultiplication by a matrix.) In the new coordinate system the role played by π will be played by another plane. (In particular, premultiplication by a diagonal matrix will correspond to a perspectivity between the two planes.) The invariance equations mentioned above will relate, in the old and new planes, the volumes associated to a given set of $(n+1)$-tuples. When dealing with invariancy, it is thus no restriction to consider the particular plane π only.

Compared to the affine case, there are some problems to describe the regions in $\mathbb{P}^n$ having the points $A_0, \ldots, A_n$ as vertices ("sense-classes", cf. [V-Y] vol II, Ch IX). To avoid these considerations, we will argue in terms of the points themselves. In concrete cases, for $n = 1, 2$, there will be no difficulty to make the region interpretation (cf. Remark 4 after Theorem 4.1).

We have now motivated the following two definitions. (Strictly speaking, the terminology of the second one will not be fully justified until the end of this paper. The postfix "ad" in the first one is borrowed from Veblen-Young.)

DEFINITION 2.1 By a *polyad* in $\mathbb{P}^n$ is meant a non-degenerate ordered $(n+1)$-tuple of points in $\mathbb{P}^n$

$$\mathcal{A} = (A_0, \ldots, A_n).$$

If n=1,2,3 also the terms *dyad*, *triad*, and *tetrad* will be used. □

DEFINITION 2.2 The *volume* of the polyad is defined by eqs. (2.1) and (2.2). For dyads and triads the terms *length* and *area* will also be used. □

Our first theorem on area-invariance will be formulated in terms of the following concept, cf. [V-Y] vol II p. 55 or [M] p. 266 ff.

DEFINITION 2.3 For given points X, Y and a polyad $\mathcal{A}$ in $\mathbb{P}^n$, the following collection of *cross-ratios* is formed for $i, j = 0, \ldots, n$, $i \neq j$,

$$k_{ij} = k_{ij}(X, Y; \mathcal{A}) = k_{ij}(X, Y; A_0, \ldots, A_n) =$$
$$= \frac{\delta(A_0, \ldots, \overset{i}{X}, \ldots, A_n)}{\delta(A_0, \ldots, \underset{j}{X}, \ldots, A_n)} \Big/ \frac{\delta(A_0, \ldots, \overset{i}{Y}, \ldots, A_n)}{\delta(A_0, \ldots, \underset{j}{Y}, \ldots, A_n)} \tag{2.3}$$

In particular, if $n = 1$, we write CR for k_{10} i.e.

$$CR(X,Y;A_0,A_1) = k_{10} = \frac{\delta(A_0,X)}{\delta(X,A_1)} \Big/ \frac{\delta(A_0,Y)}{\delta(Y,A_1)} \tag{2.4}$$

The cases when the values 0 or ∞ appear somewhere are treated by the natural limit conventions. □

Remark. If $n = 1$, by the discussion above, all the δ:s can be interpreted as lengths on a line. In particular, we recognize CR as the familiar concept of cross-ratio on $\mathbb{P}^1$. □

THEOREM 2.1 *The cross-ratios k_{ij} are invariant under projectivities.* □

Proof: To fix the ideas let $n = 2$ and consider k_{12}. Let $T \in PGL(2)$ and $X, Y, A_0, A_1, A_2 \in \mathbb{P}^2$. Suppose at first that no three of the points are colinear and that none of them or their images under T is a point at infinity. Fixing representatives of the points and the projectivity, one has

$$\begin{aligned} &k_{12}(TX,TY;TA_0,TA_1,TA_2) = \\ &= \frac{\delta(TA_0,TX,TA_2)}{\delta(TA_0,TA_1,TX)} \Big/ \frac{\delta(TA_0,TY,TA_2)}{\delta(TA_0,TA_1,TY)} \\ &= \frac{\frac{\det T \det(A_0,X,A_2)}{\sigma(TA_0)\sigma(TX)\sigma(TA_2)}}{\frac{\det T \det(A_0,A_1,X)}{\sigma(TA_0)\sigma(TA_1)\sigma(TX)}} \Big/ \frac{\frac{\det T \det(A_0,Y,A_2)}{\sigma(TA_0)\sigma(TY)\sigma(TA_2)}}{\frac{\det T \det(A_0,A_1,Y)}{\sigma(TA_0)\sigma(TA_1)\sigma(TY)}} \\ &= \frac{\det(A_0,X,A_2)}{\det(A_0,A_1,X)} \Big/ \frac{\det(A_0,Y,A_2)}{\det(A_0,A_1,Y)} \end{aligned}$$

The last expression also equals $k_{12}(X,Y;A_0,A_1,A_2)$, which is shown in the same way. This proves the theorem under the imposed extra assumptions. By limit considerations the result is proved for general projectivities. □

EXAMPLE 2.2
Let $n = 2$. Consider the two figures in Figure 2.1.

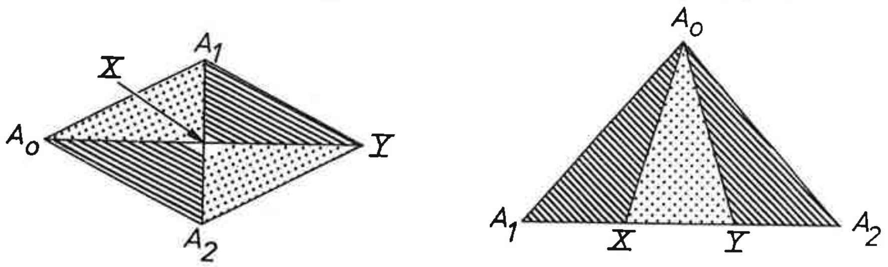

Figure 2.1 Objects that have an area-invariant k_{12}. However, the regions have intersecting boundaries

There we have

$$k_{12} = \frac{\delta(A_0,X,A_2)}{\delta(A_0,A_1,X)} \Big/ \frac{\delta(A_0,Y,A_2)}{\delta(A_0,A_1,Y)}$$

where each factor can be interpreted as sums of areas of shaded or dotted regions in the figure. The same value of k_{12} will be obtained after any projective, and in particular any perspective, mapping of the points. □

This could be utilized for our purposes, described in the introduction. However, a marking symbol relying on this example does not fulfil the particular claims we posed there. Instead we proceed by studying regions with nonintersecting boundaries (where it is possible to use two colours only).

2.3 Invariants and elimination theory

The word "invariant" has many meanings in mathematics. E.g. in classical algebraic invariant theory, given a subgroup G of $GL(n)$, the term is used for polynomials p such that

$$p(gx) = \kappa_g p(x), \quad x \in \mathbb{R}^n, g \in G.$$

Although we too deal with polynomial invariants under a group of transformations, our concept does *not* adhere to the one above. In this section we precise what we mean, and at the same time give an abstract setting for our construction of invariants.

To every point P and hyperplane p of $\mathbb{P}^n$ we assign a one-parameter subgroup of projectivities

$$\mathcal{H}_{P,p} = \left(H^t_{P,p}; 0 \neq t \in \mathbb{R}\right)$$

(homologies, cf. Definition 4.1 below). A group action on the set of all polyads is defined by

$$H^t_{P,p}\mathcal{A} = \mathcal{A}_t, \quad \text{where} \quad \mathcal{A} = (A_0, \dots, A_n), \\ \mathcal{A}_t = \left(H^t_{P,p}A_0, \dots, H^t_{P,p}A_n\right)$$

We will show (cf. Lemma 4.1) that the "orbits" $\mathcal{A}_t$, $t \in \mathbb{R}$, are constrained by certain polynomial equations

$$R(\lambda; t; 1/\delta(\mathcal{A}_t)) = 0 \tag{2.5}$$

where

$$\lambda \in \mathbb{R}^{n+1}, t = (t_0, \dots, t_m) \in \mathbb{R}^{m+1} \\ 1/\delta(\mathcal{A}_t) = (1/\delta(\mathcal{A}_{t_0}), \dots, 1/\delta(\mathcal{A}_{t_m})) \in \mathbb{R}^{m+1}$$

Here λ, the "configuration coefficients", depend on $P, p, \mathcal{A}$. In forming δ, the choice of affine basis in π may depend on $\mathcal{A}$.

Now fix P, p and $\mathcal{A}$. Let $T \in PGL(n)$ with $TP = \tilde{P}$, $Tp = \tilde{p}$, $T\mathcal{A} = \tilde{\mathcal{A}}$, and let $\tilde{\lambda}$ be the configuration coefficients associated to $\tilde{P}, \tilde{p}, \tilde{\mathcal{A}}$. Along with (2.5) holds

$$R\left(\tilde{\lambda}; t; 1/\tilde{\delta}(\tilde{\mathcal{A}}_t)\right) = 0 \tag{2.6}$$

where $\tilde{\delta}$ is assigned to $\tilde{\mathcal{A}}$. Since δ and $\tilde{\delta}$ measure volumes in two different affine bases for π, they are proportional. It will turn out that R is homogeneous in $1/\delta$. Hence $\tilde{\delta}$ may be replaced by δ in (2.6).

Suppose that $\mathcal{H}_{P,p}$ and $\mathcal{H}_{\tilde{P},\tilde{p}}$ are conjugate and that, more precisely,

$$TH^t_{P,p}T^{-1} = H^t_{\tilde{P},\tilde{p}}$$

(cf. Lemma 4.3 below). Then

$$\tilde{\mathcal{A}}_t = H^t_{\tilde{P},\tilde{p}}T\mathcal{A} = TH^t_{P,p}\mathcal{A} = T\mathcal{A}_t$$

Suppose further that $\tilde{\lambda} = \lambda$. Then (2.6) says that

$$R(\lambda; t; 1/\delta(T\mathcal{A}_t)) = 0 \tag{2.7}$$

This then holds independently of T.

To sum up, given $P, p, \mathcal{A}$ and t, i.e. keeping fixed the first two groups of variables in (2.5), we have a polynomial equation in the remaining ones $1/\delta(\mathcal{A}_t)$. By eq (2.7) this equation (2.5) is unaltered by projectivities. In other words we have constructed an algebraic manifold to which $1/\delta(T\mathcal{A}_t)$ will belong for every $T \in PGL(n)$. This manifold will be our *invariant*.

For particular values of λ, the polynomial R will in fact be linear in $1/\delta$. The number $m+1$ of t-variables will equal $n+2$ in the situation above. Replacing $\mathcal{A}$ by other configurations it is sometimes possible to carry out the same construction. For $n = 2$, this will be done below in two cases, cf. Section 4.3 and Chapter 6.

We may also describe, in general terms, the procedure for finding R. First, for $t \in \mathbb{R}$ we express $1/\delta(TA_t)$ as a polynomial in some parameters describing T. Combining a set of such equations, corresponding to different t:s, one gets a system of homogeneous polynomial equations in the T-parameters. The fact that this system has a nontrivial solution imposes conditions on the coefficients of the polynomials. These conditions will serve as the invariants searched for. To determine them, we will trust on a tool from classical algebra, namely "elimination theory", cf. van der Waerden [vdW] Ch XI or Hogde-Pedoe [H-P] Ch IV. (For the first reference, this chapter is only to be found in the three first editions, before 1955.) For the sake of reference, we formulate

Elimination lemma: Consider a general system of n homogeneous polynomial equations

$$f_1(x) = 0, \ldots, f_n(x) = 0 \quad (*)$$

where $x = (x_1, \ldots, x_n)$. *Then there exists an irreducible polynomial* R *with integer coefficients and the coefficients of* $f_1, \ldots, f_n$ *as variables such that*

$$(*) \text{ has a non-trivial solution} \Leftrightarrow R(\text{coeff. of}(*)) = 0$$

If $\deg(f_i) = n$, $i = 1, \ldots, n$, *then* R *is homogeneous of degree* $n(n-1)$ *in the coefficients of* f_i, $i = 1, \ldots, n$. R *is called the resultant of* $(*)$. □

If all f_i:s are linear, then the resultant coincides with the determinant of the system. If $n = 2$ then R may always be expressed as a certain determinant in the coefficients (cf. the proof of Lemma 3.1 below). The computation of R for $n > 2$ is in general a formidable task, cf. the remark after Lemma 4.1. A possible procedure, built upon the elimination of one variable at a time, is outlined in [H-P] p 159 ff.

3. The Projective Line

In this chapter we derive certain projectivity invariant relations between the lengths of certain intervals on the line. Relying on a geometric picture, we treat perspectivities by themselves. Parts of our results, namely those for translated intervals, will in fact be valid in that case only.

3.1 Perspectivities

Let ℓ, ℓ' be two lines and $O \notin \ell \cup \ell'$ a point in the affine plane, cf. Figure 3.1. Augmented with points at infinity, the lines may be thought of as models for $\mathbb{P}^1$. Let P', A', B', with $A' \neq B'$, be three points on ℓ'. P' may coincide with A' or B'. Under the perspectivity with center O the points P',A',B' are mapped on P,A,B respectively. Let p be the point on ℓ corresponding to the point at infinity on ℓ', i.e. such that Op is parallel to ℓ'.

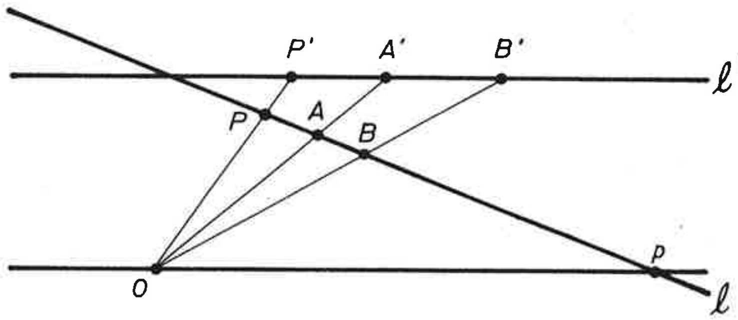

Figure 3.1

Our construction of invariants will be carried out by means of two types of *homothetic* transformations on ℓ', dilations (the main case) and translations. ([C] and [H] serve as general references on these and other transformations appearing later, where [H] emphasizes the group theoretic point of view.)

Dilations

By a *dilation* on ℓ' with center P' and scale t is meant a mapping

$$H'_{P'} : X' \to X'_t \quad \text{where} \quad \overline{P'X'_t} = t\overline{P'X'} \tag{3.1}$$

This can also be expressed as $\overline{OX'_t} = (1-t)\overline{OP'} + t\overline{OX'}$. In particular

$$\overline{OA'_t} = (1-t)\overline{OP'} + t\overline{OA'}$$
$$\overline{OB'_t} = (1-t)\overline{OP'} + t\overline{OB'}$$

Then e.g. $A'_0 = P'$, $A'_1 = A'$. Let A_t, B_t be the corresponding points on ℓ. Our aim is to derive, for a set of values of t, certain relations between the lengths of the dyads (A_t, B_t), cf. Figure 3.2. These relations shall be valid for *any* perspective image of ℓ'.

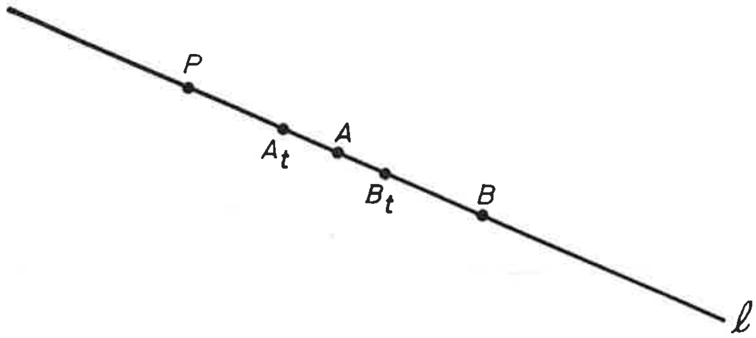

Figure 3.2

On ℓ', let $\overline{OP'} = \lambda\overline{OA'} + \mu\overline{OB'}$, $\lambda + \mu = 1$. Then

$$\overline{OA'_t} = ((1-t)\lambda + t)\overline{OA'} + (1-t)\mu\overline{OB'}$$
$$\overline{OB'_t} = (1-t)\lambda\overline{OA'} + ((1-t)\mu + t)\overline{OB'}$$

Now fix a coordinate system O, $\overline{OA} = \frac{1}{\alpha}\overline{OA'}$, $\overline{OB} = \frac{1}{\beta}\overline{OB'}$, in the plane, cf. Figure 3.1. The line ℓ then has the equation $x + y = 1$. In the corresponding homogeneous coordinates holds

$$A_t = ((1-t)\lambda\alpha + t\alpha, (1-t)\mu\beta)$$
$$B_t = ((1-t)\lambda\alpha, (1-t)\mu\beta + t\beta)$$

Application of δ (Definition 2.2) yields

$$\delta(A_t, B_t) = \frac{1}{((1-t)(\lambda\alpha + \mu\beta) + t\alpha)((1-t)(\lambda\alpha + \mu\beta) + t\beta)} \cdot$$
$$\cdot \begin{vmatrix} (1-t)\lambda\alpha + t\alpha & (1-t)\lambda\alpha \\ (1-t)\mu\beta & (1-t)\mu\beta + t\beta \end{vmatrix} \delta(A_1, B_1)$$

Here the unit $\delta(A_1, B_1)$ is included for homogeneity reasons. To achieve homogeneity in t also, we replace A_1, B_1 with A_{t_0}, B_{t_0} and substitute t/t_0 for t. An algebraic computation gives

$$((t_0 - t)^2\tilde{S}_1^2 + t(t_0 - t)S_1\tilde{S}_1 + t^2 S_2) \cdot \delta(A_t, B_t) = S_2 t t_0 \cdot \delta(A_{t_0}, B_{t_0}) \qquad (3.2)$$

where

$$S_1 = \alpha + \beta, \quad S_2 = \alpha\beta, \quad \tilde{S}_1 = \lambda\alpha + \mu\beta \qquad (3.3)$$

Equivalently we may write

$$(a_t\alpha^2 + b_t\alpha\beta + c_t\beta^2) \cdot \delta(A_t, B_t) = t t_0 \alpha\beta \cdot \delta(A_{t_0}, B_{t_0}) \qquad (3.4)$$

where

$$\begin{aligned} a_t &= \lambda(t_0 - t)((t_0 - t)\lambda + t) \\ b_t &= 2(t_0 - t)^2\lambda\mu + t t_0 \\ c_t &= \mu(t_0 - t)((t_0 - t)\mu + t) \end{aligned} \qquad (3.5)$$

LEMMA 3.1 *With the notation introduced above, for any perspective image of ℓ' hold the relations:*

(i) If $\lambda = 1$, $\mu = 0$ (i.e. $P = A = A_t$, $t \in \mathbb{R}$), then

$$\frac{t_0(t_1 - t_2)}{\delta(A, B_{t_0})} + \frac{t_1(t_2 - t_0)}{\delta(A, B_{t_1})} + \frac{t_2(t_0 - t_1)}{\delta(A, B_{t_2})} = 0 \tag{3.6}$$

(ii) If $\lambda = \mu = 1/2$, then

$$\frac{t_0(t_1^2 - t_2^2)}{\delta(A_{t_0}, B_{t_0})} + \frac{t_1(t_2^2 - t_0^2)}{\delta(A_{t_1}, B_{t_1})} + \frac{t_2(t_0^2 - t_1^2)}{\delta(A_{t_2}, B_{t_2})} = 0 \tag{3.7}$$

(iii) For general λ, μ holds, using the notation $\delta_{t_i} = \delta(A_{t_i}, B_{t_i})$,

$$\begin{vmatrix} a_{t_1} & b_{t_1}/\delta_{t_0} - t_0 t_1/\delta_{t_1} \\ a_{t_2} & b_{t_2}/\delta_{t_0} - t_0 t_2/\delta_{t_2} \end{vmatrix} \begin{vmatrix} b_{t_1}/\delta_{t_0} - t_0 t_1/\delta_{t_1} & c_{t_1} \\ b_{t_2}/\delta_{t_0} - t_0 t_2/\delta_{t_2} & c_{t_2} \end{vmatrix} - \frac{1}{\delta_{t_0}^2} \begin{vmatrix} a_{t_1} & c_{t_1} \\ a_{t_2} & c_{t_2} \end{vmatrix}^2 = 0 \tag{3.8}$$

□

Proof: The proof relies on (3.2) or equivalently (3.4). We follow the path outlined in Section 2.3. Since the perspectivity is uniquely determined by the non-zero numbers α and β, "invariants under perspectives" must be independent of α, β.

Writing

$$m_t = \delta(A_t, B_t)/\delta(A_{t_0}, B_{t_0}) \tag{3.9}$$

the basic formula (3.2) becomes

$$(t_0 - t)^2 \tilde{S}_1^2 + t(t_0 - t) S_1 \tilde{S}_1 + t(t - \frac{t_0}{m_t}) S_2 = 0 \tag{3.10}$$

Although (i) and (ii) are special cases of (iii), we prefer to treat them separately.

(i) If $\lambda = 1, \mu = 0$, then (3.10) simplifies into

$$(t_0 - t)\alpha + t(1 - \frac{1}{m_t})\beta = 0 \tag{3.11}$$

Putting together two such equations, corresponding to $t = t_1$ and $t = t_2$, one gets a homogeneous system of linear equations in the unknowns α and β. This system is known to have a nontrivial solution, determined by the geometrical construction above. Hence the determinant of the system is zero, i.e.

$$\begin{vmatrix} t_0 - t_1 & t_1(1 - \frac{1}{m_{t_1}}) \\ t_0 - t_2 & t_2(1 - \frac{1}{m_{t_2}}) \end{vmatrix} = 0$$

Expansion of the determinant directly gives (3.6).

(ii) If $\lambda = \mu = 1/2$ then $\tilde{S}_1 = S_1/2$ which simplifies (3.10) into

$$(t_0^2 - t^2) S_1^2 + 4t(t - \frac{t_0}{m_t}) S_2 = 0 \tag{3.12}$$

By the same argument as above, combination of two such equations yields

$$\begin{vmatrix} t_0^2 - t_1^2 & 4t_1(t_1 - \frac{t_0}{m_{t_1}}) \\ t_0^2 - t_2^2 & 4t_2(t_2 - \frac{t_0}{m_{t_2}}) \end{vmatrix} = 0$$

Expansion of the determinant gives (3.7).
(iii) Combination of two equations (3.4), corresponding to t_1 and t_2, gives a system of two homogeneous polynomial equations of second order in α, β. This system is known to have a non-trivial solution. By the Elimination lemma (cf. Section 2.3) this happens if and only if the resultant of the system vanishes. But here, in the case of two variables, the resultant can be written down explicitely as (cf. [vdW] Ch XI)

$$\begin{vmatrix} a_{t_1} & b_{t_1} - t_0 t_1 \delta_{t_0}/\delta_{t_1} & c_{t_1} & 0 \\ 0 & a_{t_1} & b_{t_1} - t_0 t_1 \delta_{t_0}/\delta_{t_1} & c_{t_1} \\ a_{t_2} & b_{t_2} - t_0 t_2 \delta_{t_0}/\delta_{t_2} & c_{t_2} & 0 \\ 0 & a_{t_2} & b_{t_2} - t_0 t_2 \delta_{t_0}/\delta_{t_2} & c_{t_2} \end{vmatrix} = 0 \tag{3.13}$$

This determinant is easily rewritten as (3.8). □

Remark 1. The case (i) is in fact the ordinary *cross-ratio* relation for CR (cf. Definition 2.3 and Theorem 2.1). To see this, note that the sum of the nominators in (3.5) is zero. Hence

$$t_1(t_2 - t_0)\left(\frac{1}{\delta(P, B_{t_1})} - \frac{1}{\delta(P, B_{t_0})}\right) + t_2(t_0 - t_1)\left(\frac{1}{\delta(P, B_{t_2})} - \frac{1}{\delta(P, B_{t_0})}\right) = 0$$

or, equivalently (cf. Figure 3.3)

$$\frac{\delta(P, B_{t_0}) - \delta(P, B_{t_2})}{\delta(P, B_{t_2})} / \frac{\delta(P, B_{t_0}) - \delta(P, B_{t_1})}{\delta(P, B_{t_1})} = \frac{t_0 - t_2}{t_2} / \frac{t_0 - t_1}{t_1} \tag{3.14}$$

But here $\delta(P, B_{t_0}) - \delta(P, B_{t_1}) = \delta(B_{t_1}, B_{t_0})$, $\delta(P, B_{t_0}) - \delta(P, B_{t_2}) = \delta(B_{t_2}, B_{t_0})$ by the geometric interpretation of δ. The left hand side in (3.14) is thus the cross-ratio (2.4), and we have reproved that it is invariant under perspectivities. (The possibility of expressing the invariance of cross-ratios by a formula like (3.6) was known already by Möbius [M], "Von der metrischen Relationen im Gebiete der Lineal-Geometrie" (1829).) □

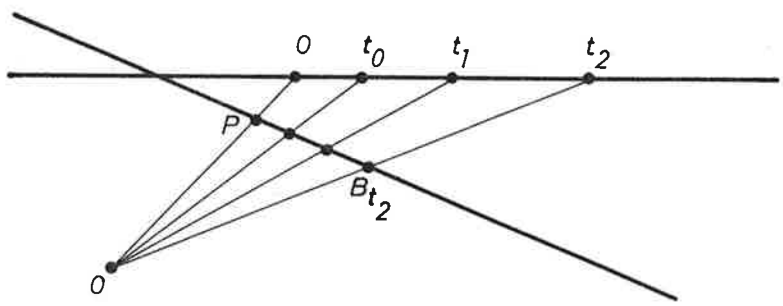

Figure 3.3

Remark 2. The case (ii) relates the lengths of perspective images of dyads with a common center. For reasons that will be apparent in the next chapter (cf. the remark after Theorem 4.1), we will call this the *polar* case.

The polar case (ii) and the cross-ratio case (i) are not as independent as they may seem. In fact (i) can be derived from (ii) as a limit case $P \to A$, $p \to A$. Likewise (ii) can be derived from (i).

We prove the latter statement using a process that also works in certain cases in higher dimensions, cf. Section 4.3. On ℓ' in Figure 3.4, change the notation B', B'_t into B'^+, B'^+_t, and introduce corresponding points B'^-, B'^-_t symmetrically spaced around A'. With obvious notations on ℓ, the equation (3.11) may be written (it suffices to consider the case $t_0 = 1$)

$$\frac{\delta(A, B_t^+)}{\delta(A, B^+)} = \frac{\alpha\beta^+ t}{(1-t)\alpha + t\beta^+}$$
$$\frac{\delta(A, B_t^-)}{\delta(A, B^-)} = \frac{\alpha\beta^- t}{(1-t)\alpha + t\beta^-}$$

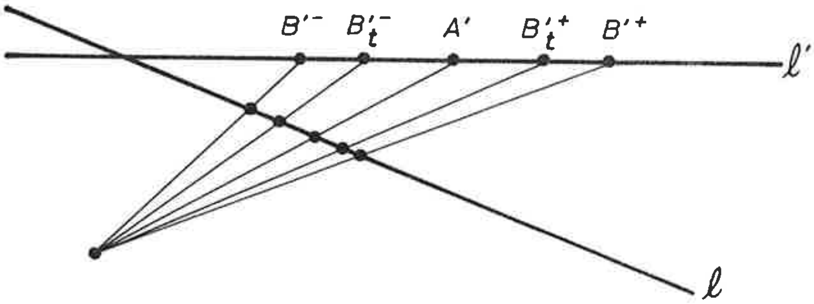

Figure 3.4

The fact that $\delta(A', B'^+) = -\delta(A', B'^-)$ yields

$$\alpha\beta^+\delta(A, B^+) = -\alpha\beta^-\delta(A, B^-)$$
$$\beta^+ + \beta^- = 2\alpha$$

Denoting the common value of the members of the first equation by c we get

$$\begin{aligned}\delta(B_t^-, B_t^+) &= \delta(A, B_t^+) - \delta(A, B_t^-) = \\ &= ct\left(\frac{1}{(1-t)\alpha + t\beta^+} + \frac{1}{(1-t)\alpha + t\beta^-}\right) = \\ &= c\frac{4S_1 t}{(1-t^2)S_1^2 + 4t^2 S_2}\end{aligned}$$

with $S_1 = \beta^+ + \beta^-$, $S_2 = \beta^+\beta^-$. Insertion of $t = 1$ gives $c = S_2\delta(B^-, B^+)/S_1$, which in turn gives eq (3.12). □

Remark 3. The cases (i) and (ii) are the only situations where (iii) reduces to a linear relation in $1/\delta_i$. This happens if and only if completion of squares in (3.8), as a quadratic form in $1/\delta_i$, gives only two quadratic expressions of different signs. This in turn happens if and only if either the first term in (3.8) is a square in itself of the second cancels. In both cases the condition is that

$$\begin{vmatrix} a_{t_1} & c_{t_1} \\ a_{t_2} & c_{t_2} \end{vmatrix} = 0$$

for all t_1, t_2. We obtain $\lambda = 0$ or $\mu = 0$ or $\lambda = \mu$. □

Remark 4. Two points A, B divide the projective line into two "intervals". One of these, the one that not contains the point at infinity, may be called the "finite" one. Under a perspectivity a finite interval may be mapped onto a non-finite one. This situation is reflected by a change of signs in δ, but does not alter the validity of the lemma. It is in order to avoid such considerations, irrelevant for the invariants, that we talk about dyads instead of intervals. These aspects are still more accentuated in higher dimensions. □

Translations

Now consider dyads on ℓ' obtained from each others by *translations.* Thus let the points A'^τ, B'^τ be defined by

$$\overline{A'A'^\tau} = \tau\overline{A'B'}, \quad \overline{B'B'^\tau} = \tau\overline{A'B'}$$

(The vector $\overline{A'B'}$ thus serves as unit.) Let A^τ, B^τ be the corresponding points on ℓ. The relation between $\delta(A^\tau, B^\tau)$ and $\delta(A, B)$ can be derived either directly or as a limit case of Lemma 3.1. We do it in the latter way. From $\overline{OP'} = \lambda\overline{OA'} + \mu\overline{OB'}$, $\lambda + \mu = 1$, follows that $\overline{A'P'} = \mu\overline{A'B'}$. Consider dilations with center P'. Choose t so that $A'_t = A'^\tau$, i.e.

$$t = \overline{P'A'^\tau}/\overline{P'A'} = (\overline{A'A'^\tau} + \overline{P'A'})/\overline{P'A'} = 1 - \tau/\mu.$$

By (3.2) holds, with $t_0 = 1$,

$$\begin{aligned}((\tau/\mu)^2\tilde{S}_1^2 + (1-\tau/\mu)(\tau/\mu)S_1\tilde{S}_1 + (1-\tau/\mu)^2S_2)\cdot\delta(A_t, B_t) \\ = S_2(1-\tau/\mu)\cdot\delta(A, B)\end{aligned} \tag{3.15}$$

Now let $\mu \to \infty$. Then the τ-translation is obtained as a limit of the dilations above. In particular $A_t = A^\tau, B_t \to B^\tau$. Moreover

$$\tilde{S}_1/\mu = ((1-\mu)\alpha + \mu\beta)/\mu \quad \to \quad \beta - \alpha.$$

Equation (3.15) yields, in the limit,

$$\tau^2(\beta-\alpha)^2 + \tau(\beta^2-\alpha^2) + \alpha\beta(1 - \delta(A, B)/\delta(A^\tau, B^\tau)) = 0$$

or equivalently

$$\alpha^2(\tau^2 - \tau) + \beta^2(\tau^2 + \tau) + \alpha\beta(1 - 2\tau^2 - \delta(A, B)/\delta(A^\tau, B^\tau)) = 0. \tag{3.16}$$

Put

$$a^\tau = \tau^2 - \tau, \quad b^\tau = 1 - 2\tau^2, \quad c^\tau = \tau^2 + \tau$$

$$\delta^\tau = \delta(A^\tau, B^\tau), \quad \delta = \delta(A, B)$$

Combining two expressions (3.16), for τ_1 and τ_2, exactly as in Lemma 3.1, case (iii), we obtain

LEMMA 3.2 *With the notations introduced above, for any perspective images of translated intervals on ℓ' hold the relation*

$$\begin{aligned}\begin{vmatrix} a^{\tau_1} & b^{\tau_1}/\delta - 1/\delta^{\tau_1} \\ a^{\tau_2} & b^{\tau_2}/\delta - 1/\delta^{\tau_2}\end{vmatrix}\begin{vmatrix} b^{\tau_1}/\delta - 1/\delta^{\tau_1} & c^{\tau_1} \\ b^{\tau_2}/\delta - 1/\delta^{\tau_2} & c^{\tau_2}\end{vmatrix} \\ -\frac{1}{\delta^2}\begin{vmatrix} a^{\tau_1} & c^{\tau_1} \\ a^{\tau_2} & c^{\tau_2}\end{vmatrix}^2 = 0\end{aligned} \tag{3.17}$$

□

3.2 Projectivities. Homologies

We are now going to study projectivities on ℓ, considered as a model for $\mathbb{P}^1$. We will prove that the formulas of Lemma 3.1 remain true when a projectivity is applied to all appearing points A_t, B_t. (Concerning Lemma 3.2, cf. the remark ending this section.) In formulating such a result, no reference must be made to ℓ' or the specific perspectivity used.

In Lemma 3.1, as we have seen in Section 2.2, the δ:s describe intrinsic properties of the line ℓ itself. (Note that a change of reference points on ℓ will change the values of all δ by a common factor, which not affects the validity of the formulas.) However, the other ingredients, λ, μ, t_i of the lemma refer to ℓ' and the perspectivity of ℓ' onto ℓ. To get rid of this dependence, note that

$$t = CR(A'_t, A'; P', \infty') = CR(B'_t, B'; P', \infty')$$

where ∞' denotes the point at infinity on ℓ'. Taking into account the invariance of the cross-ratio under the perspectivity $\ell' \to \ell$, the following concept seems adequate to describe the situation on ℓ.

DEFINITION 3.1 By a *homology* with center P, axis p, scale t, is meant the mapping

$$H^t_{P,p} : \begin{cases} X \to X_t & \text{if } X \neq P \text{ and } X \neq p \\ P \to P \\ p \to p \end{cases}$$

where X_t is the unique point determined by

$$CR(X_t, X; P, p) = t$$

By convention, let $X_t = p$ correspond to $t = \infty$. For a given dyad (A, B), the set of all dyads (A_t, B_t) is called the *homological range* of (A, B) and is denoted $\mathcal{H}_{P,p}(A, B)$. □

Holding P, p, t fixed, $H^t_{P,p}$ is in fact a projectivity on ℓ. This follows from the fact that it is the composite of a perspectivity $\ell' \to \ell$ and a dilation (i.e. a projectivity) on ℓ'. By definition $H^t_{P,p}$ has two fixed points P, p. Hence it is a *hyperbolic projectivity*. However, to obtain uniformity with the next chapter, we have preferred the terminology "homology" and its associates "center,axis".

On ℓ', the fact that $P' = (\lambda, \mu)$ with respect to the points of reference A', B', may also be expressed as

$$\lambda = CR(A', P'; \infty', B'), \quad \mu = CR(P', B'; A', \infty')$$

The invariance of the cross-ratio under perspectivities leads to

DEFINITION 3.2 By the *configuration coefficients* of $(P, p; A, B)$ are meant

$$\lambda = CR(A, P; p, B), \quad \mu = CR(P, B; A, p)$$

□

The following lemma is a direct consequence of the invariance of the cross-ratio under projectivities (Theorem 2.1).

LEMMA 3.3 *Let $T : \ell \to \ell$ be a projectivity, $\tilde{P} = TP$, $\tilde{p} = Tp$. Then the following diagram is commutative*

$$\begin{array}{ccc} X & \xrightarrow{T} & \tilde{X} \\ \downarrow H^t_{P,p} & & \downarrow H^t_{\tilde{P},\tilde{p}} \\ X_t & \xrightarrow{T} & \tilde{X}_t \end{array}$$

If $TA = \tilde{A}$, $TB = \tilde{B}$, it follows that $(P, p; A, B)$ and $(\tilde{P}, \tilde{p}; \tilde{A}, \tilde{B})$ have the same configuration coefficients and that

$$T : \mathcal{H}_{P,p}(A, B) \to \mathcal{H}_{\tilde{P},\tilde{p}}(\tilde{A}, \tilde{B}), \quad T : (A_t, B_t) \to (\tilde{A}_t, \tilde{B}_t) \tag{3.18}$$

□

Summing up, Lemma 3.1 gives relations, homogeneous in $1/\delta$, between the lengths of dyads belonging to a particular homological range $\mathcal{H}_{P,p}(A, B)$ on ℓ, with configuration coefficients λ, μ. Lemma 3.3 says that projectivities on ℓ transfer homological ranges onto homological ranges, without altering t and λ, μ. We have thus arrived at the situation described in Section 2.3, and conclude

THEOREM 3.1 *Let $(A_{t_i}, B_{t_i}) \in \mathcal{H}_{P,p}(A, B)$, $i = 0, 1, 2$, and let λ, μ be the configuration coefficients of $(P, p; A, B)$. Then the cases (i), (ii), and (iii) of Lemma 3.1 describe invariants under projectivities (i.e. the equations remain valid when applying a projectivity to all points involved).* □

Remark. In Section 3.1 we also considered *translations*, corresponding to $P' = \infty$. By a perspectivity $\ell' \to \ell$, every translation on ℓ' is transferred to a projectivity on ℓ. This projectivity has a single fixed point and is thus a *parabolic projectivity.* It is also associated to the *elations* in the plane case. By means of a suitable limit process every such elation may be parameterized by the same τ as was used on ℓ'. However, since the analogue of Lemma 3.3 does not hold for the full group of projectivities (but only for a group of elations), we do not develop this case any further. □

4. The Projective Plane

This chapter is devoted to projectivity invariant relations between the areas of certain triangles in the plane. In Section 4.3 we also consider other polygons. As in Chapter 3, we start with perspectivities.

4.1 Perspectivities

Let π, π' be two distinct planes and $O \notin \pi \cup \pi'$ a point in the three-dimensional affine space, cf. Figure 4.1. Augmented with lines at infinity, the planes may be thought of as models for $\mathbb{P}^2$. Let P', A', B', C', be four points in π' with A', B', C' non-colinear. Under the perspectivity from π' to π with center O, the points P', A', B', C' are mapped on P, A, B, C respectively. Let p denote the line in π such that Op is parallel to π'. It thus corresponds to the line at infinity in π'.

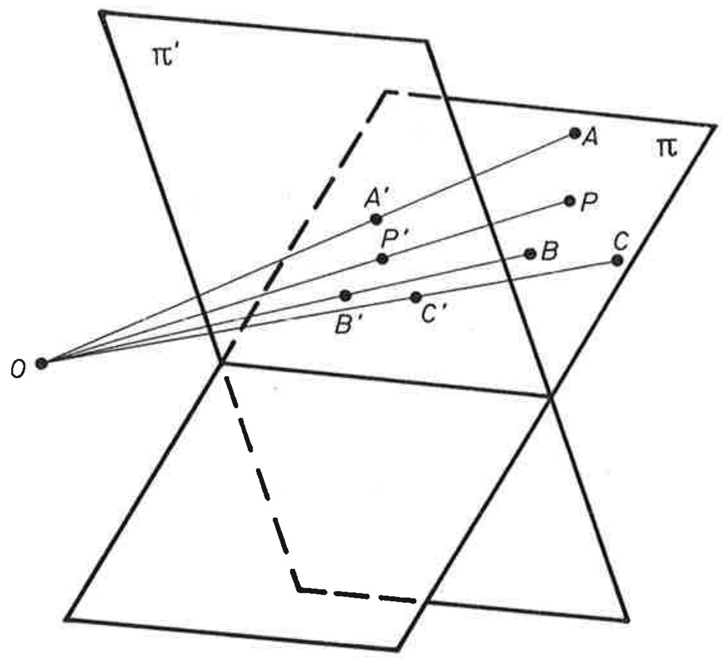

Figure 4.1

Our construction of invariants will be based on the same ideas as in Chapter 3. In particular we use the two types of *homothetic* transformations on π', dilations (the main case) and translations.

Dilations

A *dilation* on π' with center P' and scale t is defined by the same formula as in Section 3.1:

$$H'_{P'} : X' \to X'_t \quad \text{where} \quad \overline{P'X'_t} = t\overline{P'X'} \tag{4.1}$$

It follows that, cf. Figure 4.2,

$$\overline{OA'_t} = (1-t)\overline{OP'} + t\overline{OA'}$$
$$\overline{OB'_t} = (1-t)\overline{OP'} + t\overline{OB'}$$
$$\overline{OC'_t} = (1-t)\overline{OP'} + t\overline{OC'}$$

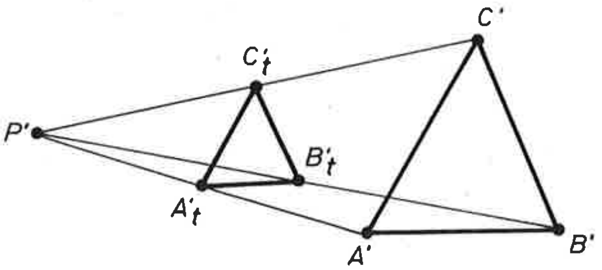

Figure 4.2

On π', there exist λ, μ, ν (barycentric coordinates) such that

$$\overline{OP'} = \lambda\overline{OA'} + \mu\overline{OB'} + \nu\overline{OC'}, \qquad \lambda + \mu + \nu = 1 \tag{4.2}$$

Then

$$\overline{OA'_t} = ((1-t)\lambda + t)\,\overline{OA'} + (1-t)\mu\overline{OB'} + (1-t)\nu\overline{OC'}$$
$$\overline{OB'_t} = (1-t)\lambda\overline{OA'} + ((1-t)\mu + t)\,\overline{OB'} + (1-t)\nu\overline{OC'}$$
$$\overline{OC'_t} = (1-t)\lambda\overline{OA'} + (1-t)\mu\overline{OB'} + ((1-t)\nu + t)\,\overline{OC'}$$

Let A_t, B_t, C_t be the points in π corresponding to A'_t, B'_t, C'_t in π'. Fix the coordinate system

$$O, \quad \overline{OA} = \frac{1}{\alpha}\overline{OA'}, \quad \overline{OB} = \frac{1}{\beta}\overline{OB'}, \quad \overline{OC} = \frac{1}{\gamma}\overline{OC'}$$

for the space. The plane π then has the equation $x + y + z = 1$. In the corresponding homogeneous coordinates holds

$$A_t = (((1-t)\lambda + t)\,\alpha, (1-t)\mu\beta, (1-t)\nu\gamma)$$
$$B_t = ((1-t)\lambda\alpha, ((1-t)\mu + t)\,\beta, (1-t)\nu\gamma)$$
$$C_t = ((1-t)\lambda\alpha, (1-t)\mu\beta, ((1-t)\nu + t)\,\gamma)$$

Application of δ and introduction of t_0 and $\delta(A_{t_0}, B_{t_0}, C_{t_0})$ yields, in the same way as in Section 3.1,

$$((t_0-t)^3\tilde{S}_1^3 + t(t_0-t)^2\tilde{S}_1^2 S_1 + t^2(t_0-t)\tilde{S}_1 S_2 + t^3 S_3)\cdot\delta(A_t, B_t, C_t) = t^2 t_0 S_3 \delta(A_{t_0}, B_{t_0}, C_{t_0}) \tag{4.3}$$

where

$$S_1 = \alpha + \beta + \gamma, \tilde{S}_1 = \lambda\alpha + \mu\beta + \nu\gamma, S_2 = \alpha\beta + \beta\gamma + \gamma\alpha, S_3 = \alpha\beta\gamma$$

In the following lemma a number of special cases for λ, μ, ν, single out naturally. Introduce first the notation

$$h_3(t_1, t_2, t_3) = t_1^2 t_2^3 - t_1^3 t_2^2 + t_2^2 t_3^3 - t_2^3 t_3^2 + t_3^2 t_1^3 - t_3^3 t_1^2$$
$$= (t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 t_2 + t_2 t_3 + t_3 t_1)$$

(Here the subscript 3 refers to triads, cf. 4-points in Section 4.3.)

LEMMA 4.1 · *With the notation introduced above, for any perspective image of the configuration in π' hold the relations (cf. Figure 4.3):*

(i) If $\lambda = 1$, $\mu = \nu = 0$ (i.e. $P = A = A_t$, $t \in \mathbb{R}$), then

$$\frac{t_0^2(t_1-t_2)(t_2-t_3)(t_3-t_1)}{\delta(A,B_{t_0},C_{t_0})} - \frac{t_1^2(t_2-t_3)(t_3-t_0)(t_0-t_2)}{\delta(A,B_{t_1},C_{t_1})} + \frac{t_2^2(t_3-t_0)(t_0-t_1)(t_1-t_3)}{\delta(A,B_{t_2},C_{t_2})} - \frac{t_3^2(t_0-t_1)(t_1-t_2)(t_2-t_0)}{\delta(A,B_{t_3},C_{t_3})} = 0 \tag{4.4}$$

(ii) If $\lambda = \mu = \nu = 1/3$ (i.e. P' = the center of A', B', C') then

$$\frac{t_0^2 h_3(t_1,t_2,t_3)}{\delta(A_{t_0},B_{t_0},C_{t_0})} - \frac{t_1^2 h_3(t_2,t_3,t_0)}{\delta(A_{t_1},B_{t_1},C_{t_1})} + \frac{t_2^2 h_3(t_3,t_0,t_1)}{\delta(A_{t_2},B_{t_2},C_{t_2})} - \frac{t_3^2 h_3(t_0,t_1,t_2)}{\delta(A_{t_3},B_{t_3},C_{t_3})} = 0 \tag{4.5}$$

(iii) For general λ, μ, ν holds

$$R(\overline{\lambda}, \overline{t}, 1/\overline{\delta}) = 0 \tag{4.6}$$

with

$$\overline{\lambda} = (\lambda,\mu,\nu), \quad \overline{t} = (t_0,t_1,t_2,t_3), \quad 1/\overline{\delta} = (1/\delta_{t_0}, 1/\delta_{t_1}, 1/\delta_{t_2}, 1/\delta_{t_3})$$

where R is the resultant of the system of three equations (4.3) corresponding to t_1, t_2, t_3. R is homogeneous of degree 6 in $1/\overline{\delta}$. □

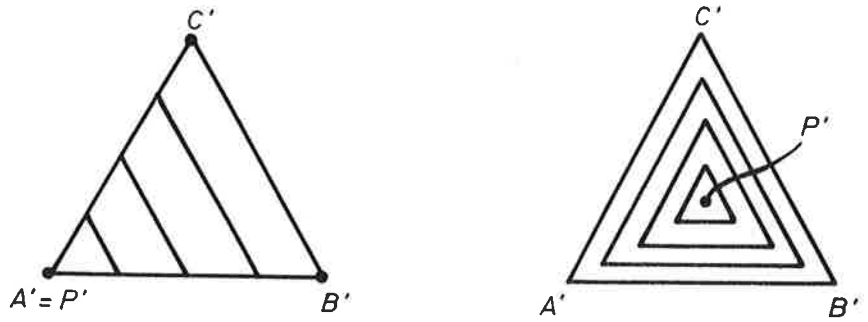

Figure 4.3

Proof: Put

$$m_t = \delta(A_t,B_t,C_t)/\delta(A_{t_0},B_{t_0},C_{t_0}) \tag{4.7}$$

Introduced in (4.3) it yields

$$((t_0-t)^3\tilde{S}_1^3 + t(t_0-t)^2\tilde{S}_1^2 S_1 + t^2(t_0-t)\tilde{S}_1 S_2 + t^2(t - \frac{t_0}{m_t})S_3 = 0 \tag{4.8}$$

For the general case (iii), cf. the Elimination lemma of Chapter 2. The proofs of the special cases (i) and (ii) are similar to the proof of Lemma 3.1, and will only be indicated here.

(i) If $\lambda = 1, \mu = \nu = 0$, equation (4.8) becomes

$$(t_0-t)^2\alpha^2 + t(t_0-t)\alpha(\beta+\gamma) + t^2(1-\frac{1}{m_t})\beta\gamma = 0$$

Three such equations, corresponding to $t = t_1, t_2, t_3$, give

$$\begin{vmatrix} (t_0 - t_1)^2 & t_1(t_0 - t_1) & t_1^2(1 - \frac{1}{m_{t_1}}) \\ (t_0 - t_2)^2 & t_2(t_0 - t_2) & t_2^2(1 - \frac{1}{m_{t_2}}) \\ (t_0 - t_3)^2 & t_3(t_0 - t_3) & t_3^2(1 - \frac{1}{m_{t_3}}) \end{vmatrix} = 0$$

Expansion of the determinant gives (4.4).

(ii) If $\lambda = \mu = \nu = 1/3$ then $\tilde{S}_1 = S_1/3$. This simplifies (4.8) into

$$(t_0 - t)^2(t_0 + 2t)S_1^3 + 9t^2(t_0 - t)S_1S_2 + 27t^2(t - \frac{t_0}{m_t})S_3 = 0$$

By the same argument as above, combination of three such equations yields

$$\begin{vmatrix} (t_0 - t_1)^2(t_0 + 2t_1) & t_1^2(t_0 - t_1) & t_1^2(t_1 - \frac{t_0}{m_{t_1}}) \\ (t_0 - t_2)^2(t_0 + 2t_2) & t_2^2(t_0 - t_2) & t_2^2(t_2 - \frac{t_0}{m_{t_2}}) \\ (t_0 - t_3)^2(t_0 + 2t_3) & t_3^2(t_0 - t_3) & t_3^2(t_3 - \frac{t_0}{m_{t_3}}) \end{vmatrix} = 0$$

Expansion of the determinant gives (4.5). □

Remark. One alternative to obtain a more explicit expression for the resultant condition $R(\overline{\lambda}, \overline{t}, 1/\overline{\delta}) = 0$ is the following.

Consider

$$S_1 = \alpha + \beta + \gamma \qquad (1)$$
$$S_2 = \alpha\beta + \beta\gamma + \gamma\alpha \qquad (2)$$
$$S_3 = \alpha\beta\gamma \qquad (3)$$
$$\tilde{S}_1 = \lambda\alpha + \mu\beta + \nu\gamma \qquad (4)$$

Here the relation between roots and coefficients gives

$$\alpha^3 - S_1\alpha^2 + S_2\alpha - S_3 = 0$$

Suppose we are not in case (ii) and that (without restriction) $\mu \neq \nu$. Using (1) and (4) to solve for β, γ, and then substituting into (2) one obtains:

$$\alpha^2(1 - 3\sigma) + \alpha(S_1(3\sigma - 1 + \lambda^2 - \mu\nu) + \tilde{S}_1(1 - 3\lambda)) + S_2(\mu - \nu)^2 + (\mu S_1 - \tilde{S}_1)(\nu S_1 - \tilde{S}_1) = 0$$

where we have used $\lambda + \mu + \nu = 1$ and the notation $\sigma = \lambda\mu + \mu\nu + \nu\lambda$. The resultant between these two polynomials in α is the invariant searched for. It can be written as a fifth order determinant, having the same structure as (3.13). (However, for the explicit computation of resultants, especially of higher orders, there exist better methods than the evaluation of such determinants, cf. [H-P] Ch IV.)

A system of three equations (4.8) corresponding to t_1, t_2, t_3 can be used to solve for S_1, S_2, and S_3 expressed in $\tilde{S}_1, t_0, t_1, t_2, t_3$. When S_1, S_2, and S_3 are introduced in the resultant then $\tilde{S}_1$ will (by homogeneity) become a common factor that can be omitted. (These calculations have been carried out by means of computer algebra. The result fills too much space to be included here.) □

Translations

Now consider the images of triads obtained from each other by a *translation* on π'. By this is meant a mapping

$$X' \to X'^{\tau} \quad \text{where} \quad \overline{X'X'^{\tau}} = \tau(\lambda_0\overline{C'A'} + \mu_0\overline{C'B'})$$

(The vector $\lambda_0\overline{C'A'} + \mu_0\overline{C'B'}$ thus serves as unit). As usual, let $X \in \pi$ be the perspective image of X'.

As in Section 3.2, translations are treated as limits of dilations. Thus let P' be defined by $\overline{C'P'} = x(\lambda_0\overline{C'A'} + \mu_0\overline{C'B'})$. Choose t so that the images of C under the translation above and a dilation with center P' coincide, i.e. so that $C'_t = C'^{\tau}$. This means that

$$t = \overline{P'C'^{\tau}}/\overline{P'C'} = (\overline{P'C'} + \overline{C'C'^{\tau}})/\overline{P'C'} = 1 - \tau/x$$

Put $\nu_0 = -\lambda_0 - \mu_0$. By (4.2) holds, with $t_0 = 1$,

$$\begin{aligned}((\tau/x)^3\tilde{S}_1^3 + (1-\tau/x)(\tau/x)^2\tilde{S}_1^2 S_1 + (1-\tau/x)^2(\tau/x)\tilde{S}_1 S_2 + \\ +(1-\tau/x)^3 S_3)\delta(A_t, B_t, C_t) = (1-\tau/x)^2 S_3\delta(A,B,C)\end{aligned} \tag{4.9}$$

Now let $x \to \infty$. Then the dilation approaches a translation and, in particular,

$$A'_t \to A'^{\tau}, \quad B'_t \to B'^{\tau}, \quad C'_t = C'^{\tau}$$

Moreover

$$\tilde{S}_1/x = (x\lambda_0\alpha + x\mu_0\beta + (1 - x\nu_0)\gamma)/x \to \lambda_0\alpha + \mu_0\beta + \nu_0\gamma = \overline{S_1}$$

where the last equality defines $\bar{S}_1$. In the limit (4.9) becomes

$$\left(\tau^3\bar{S}_1^3 + \tau^2\bar{S}_1^2 S_1 + \tau\bar{S}_1 S_2 + S_3\right)\delta\left(A^{\tau}, B^{\tau}, C^{\tau}\right) = S_3\delta(A,B,C) \tag{4.10}$$

Elimination of α, β, γ within three such equations gives, exactly as in Lemma 4.1 (iii) (cf. also the remark above):

LEMMA 4.2 *With the notations introduced above, for any perspective images of translated triangles in π' holds the relation*

$$R(\overline{\lambda}, \overline{\tau}, 1/\overline{\delta}) = 0$$

with

$$\overline{\lambda} = (\lambda_0, \mu_0, \nu_0), \quad \overline{\tau} = (0, \tau_1, \tau_2, \tau_3), \quad 1/\overline{\delta} = (1/\delta_0, 1/\delta_{\tau_1}, 1/\delta_{\tau_2}, 1/\delta_{\tau_3})$$

Here R is the resultant of the system of three equations (4.3) corresponding to τ_1, τ_2, τ_3. R is homogeneous of degree 6 in $1/\overline{\delta}$. □

4.2 Projectivities. Homologies

About the projective plane an analogous remark to the one in the beginning of Section 3.2 can be made. In this case we note that the δ:s in Lemma 4.1 represent dignities (certain areas) of the plane π itself, inherited from its affine structure, without reference to the particular perspectivity used in the proof.

It remains to characterize also the parameters t and λ, μ, ν appearing in the nominators by means of intrinsic properties of π only.

We recapitulate some notions from plane projective geometry. As a model for $\mathbb{P}^2$ we use the augmented plane π in the three-dimensional affine space. A subclass of the projectivities will play a particular role in the sequel. Thus, let P be a point and p a line in π. By a *perspective collineation* with *center* P and *axis* p is meant a projectivity leaving fixed every point on p and every line on (=through) P. In particular, if $P \notin p$ the collineation is called a *homology*, and if $P \in p$ an *elation.*

It is well-known that a homology is uniquely determined by its center P and axis p, together with one point Q and its image $\tilde{Q}$, cf. [C] p 53. For later reference we repeat the proof, beginning with the uniqueness. Here and in the sequel we denote the intersection of the lines a and b by $a \cdot b$.

For any Y, by the invariance of the lines on P, the image $\tilde{Y}$ lies on the line PY, cf. Figure 4.4. On the other hand, if $Y \notin Q\tilde{Q}$ then by the invariance of $Y_p = QY \cdot p$, the line $QY = Y_pY$ is mapped onto the line $Y_p\tilde{Q}$. It follows that $\tilde{Y} = PY \cdot Y_p\tilde{Q}$, uniquely. The case $Y \in Q\tilde{Q}$ is treated by repeated use of this argument, first using the known property $Q \to \tilde{Q}$ to construct a pair $Z \to \tilde{Z}$ with $Z \notin Q\tilde{Q}$, then using the property $Z \to \tilde{Z}$ to construct $Y \to \tilde{Y}$.

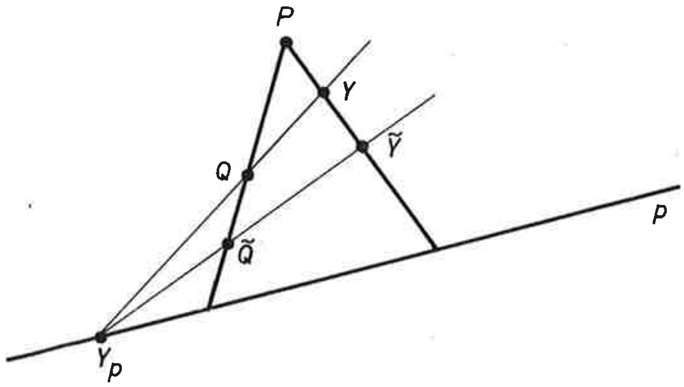

Figure 4.4

The existence of such a homology, and a bit more, can be proved by means of a perspectivity $\pi \to \pi'$. Choose O and π' so that p corresponds to the line at infinity in π'. Let Q and $\tilde{Q}$ in π correspond to Q' and $\tilde{Q}'$ in π'. Then it is possible to find a value of t such that, cf. (4.1),

$$H^t_{P'} : Q' \to \tilde{Q}'$$

Letting "persp" stand for "perspectivity with center O", a mapping $X \to X_t$ in π is defined by the diagram

$$\begin{array}{ccc} \pi' & \xrightarrow{H^t_{P'}} & \pi' \\ \uparrow \text{persp} & & \downarrow \text{persp} \\ \pi & \longrightarrow & \pi \end{array}$$

Since the dilation $H^t_{P'}$ on π' may be described as a homology with center P' and with the line at infinity as axis, the composite map in the diagram is a homology with center P and axis p. Moreover it maps Q onto $\tilde{Q}$. Hence the existence of a homology with the stated properties is established.

The proof also indicates the possibility to parameterize the set of homologies with a given center and axis. In fact, letting ∞' denote the line at infinity in π', $H^t_{P'}$ is characterized by

$$CR(X'_t, X'; P', \infty' \cdot P'X') = t$$

The invariance of cross-ratios under perspectives then legitimates the following alternate definition of homologies.

DEFINITION 4.1 By the *homology* in π with *center* P, *axis* p, *scale* t, is meant the mapping

$$H^t_{P,p} : X \to X_t$$

where X_t is the unique point on PX determined by

$$CR(X_t, X; P, p \cdot PX) = t$$

For a given triad (A, B, C) the set of all triads (A_t, B_t, C_t) is called the *homological range* of (A, B, C) and is denoted by $\mathcal{H}_{P,p}(A, B, C)$. □

Given A', B', C' in π', the barycentric coordinates of P' are obtained by solving (4.2) for λ, μ, ν. Cramer's rule gives

$$\lambda = \frac{det(P', B', C')}{det(A', B', C')}$$

$$\mu = \frac{det(A', P', C')}{det(A', B', C')}$$

$$\nu = \frac{det(A', B', P')}{det(A', B', C')}$$

In terms of the cross-ratios of Definition 2.1 and by the conventions for treating points at infinity, one checks that

$$\lambda = k_{01}(P', A'; A'P' \cdot \infty', B', C')$$

$$\mu = k_{12}(P', B'; A', B'P' \cdot \infty', C')$$

$$\nu = k_{20}(P', C'; A', B', C'P' \cdot \infty')$$

(cf. the proof of Theorem 2.1). The invariance of cross-ratios under perspectivities leads to

DEFINITION 4.2 By the *configuration coefficients* of $(P, p; A, B, C)$ are meant

$$\lambda = k_{01}(P, A; AP \cdot p, B, C)$$

$$\mu = k_{12}(P, B; A, BP \cdot p, c)$$

$$\nu = k_{20}(P, C; A, B, CP \cdot p)$$

□

(Note that, contrary to π', in π the configuration coefficients have no interpretation as barycentric coordinates.) By means of the uniqueness of homologies (stated above) and another reference to Theorem 2.1, we obtain

LEMMA 4.3 *Let T be a projectivity on π with $\tilde{P} = TP$, $\tilde{p} = Tp$. Then the following diagram is commutative*

$$\begin{array}{ccc} X & \xrightarrow{T} & \tilde{X} \\ \downarrow {\scriptstyle H^t_{P,p}} & & \downarrow {\scriptstyle H^t_{\tilde{P},\tilde{p}}} \\ X_t & \xrightarrow{T} & \tilde{X}_t \end{array}$$

If $TA = \tilde{A}, TB = \tilde{B}, TC = \tilde{C}$ it follows that $(P, p; A, B, C)$ and $(\tilde{P}, \tilde{p}; \tilde{A}, \tilde{B}, \tilde{C})$ have the same configuration coefficients and that

$$T : \mathcal{H}_{P,p}(A, B, C) \to \mathcal{H}_{\tilde{P},\tilde{p}}(\tilde{A}, \tilde{B}, \tilde{C}), \quad T : (A_t, B_t, C_t) \to (\tilde{A}_t, \tilde{B}_t, \tilde{C}_t)$$

□

Summing up, Lemma 3.1 gives homogeneous relations between the areas of triads belonging to a particular homological range $\mathcal{H}_{P,p}(A, B, C)$ on π, with configuration coefficients λ, μ, ν. Lemma 4.3 says that projectivities on π transfer homological ranges onto homological ranges, without altering t and λ, μ, ν. We have thus proved (cf. Section 2.3)

THEOREM 4.1 *Let $(A_{t_i}, B_{t_i}, C_{t_i}) \in \mathcal{H}_{P,p}(A, B, C)$, $i = 0, 1, 2, 3$, and let λ, μ, ν be the configuration coefficients of $(P, p; A, B, C)$. Then the cases (i), (ii), and (iii) of Lemma 4.1 describe invariants under projectivities.* □

Remark. The case (ii) $\lambda = \mu = \nu = 1/3$ has some special features. Let P be a point and (A, B, C) a triangle in a plane π. A new triangle (A_1, B_1, C_1) is defined by $A_1 = PA \cdot BC$, $B_1 = PB \cdot CA$, $C_1 = PC \cdot AB$. The triangles (A, B, C) and (A_1, B_1, C_1) are then perspective from P. By Desargue's theorem this happens if and only if they also are perspective from a line p. (This means that the points of intersection $AB \cdot A_1B_1$, $BC \cdot B_1C_1$ and $CA \cdot C_1A_1$ all lie on p.) The situation is described by saying that P and p are *pole* and *polar* with respect to (A, B, C), cf. Figure 4.5 and [C] p 29.

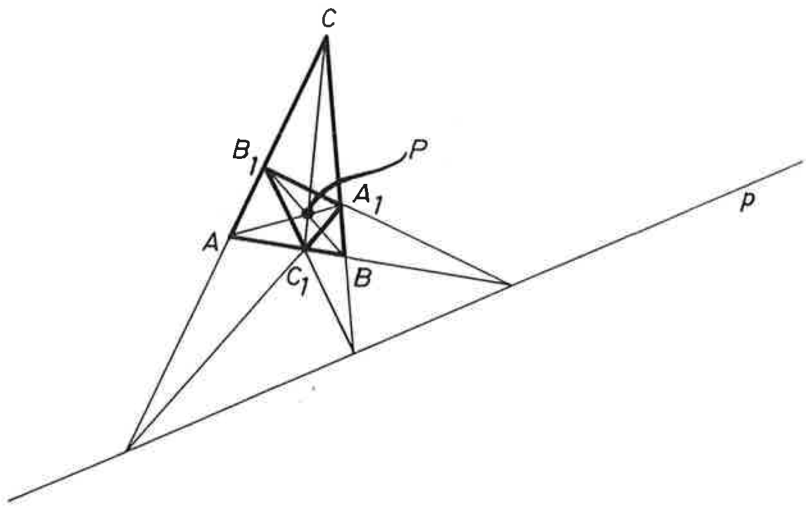

Figure 4.5

The corresponding λ, μ, ν are found by means of the invariance of the pole/polar property under perspectivities. The perspectivity $\pi \to \pi'$ maps the polar p onto the line at infinity, $P \to P'$, $(A, B, C) \to (A', B', C')$, and

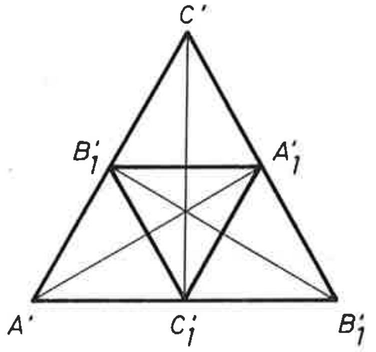

Figure 4.6

$(A_1, B_1, C_1) \rightarrow (A'_1, B'_1, C'_1)$, (cf. Figure 4.6). Hence P' is pole and the line at infinity is polar with respect to the triangle (A', B', C'). By means of similar triangles and medians one finds that P' is the center of the triangle (A, B, C) i.e. $OP' = (OA' + OB' + OC')/3$. This shows that $\lambda = \mu = \nu = 1/3$. For this reason we refer to (ii) as the *polar case.* □

4.3 Simple k-points

Generally speaking, by a *simple k-point* in $\mathbb{P}^n$ is meant an ordered k-tuple of points in $\mathbb{P}^n$. If $k = n+1$ it is a polyad, and if $k < n+1$ it may be considered as a polyad in a k-dimensional projective subspace of $\mathbb{P}^n$. Since polyads are treated in the main line of this work, only the case $k > n+1$ remains to be studied. We restrict ourselves to the case $n = 2$. By means of Definition 2.2, in a natural way one associates an "area" to every simple k-point in $\mathbb{P}^2$ by

$$\Delta(X_1, \ldots, X_k) = \delta(X_1, X_2, X_3) + \delta(X_1, X_3, X_4) + \ldots + \delta(X_1, X_{k-1}, X_k)$$

(cf. [V-Y] vol II, p 104 for the affine case). For polyads $k = 3$ we know from Theorem 4.1 that there exist area-invariants. The natural question arises whether this is true for $k > 3$.

We will consider the case $k = 4$ in a particular situation, reminding of the polar case (ii) in Theorem 4.1. Starting as usual in an affine plane π', let A', B', C', D' be a parallelogram and let P' be the intersection of its diagonals (cf. Figure 4.7).

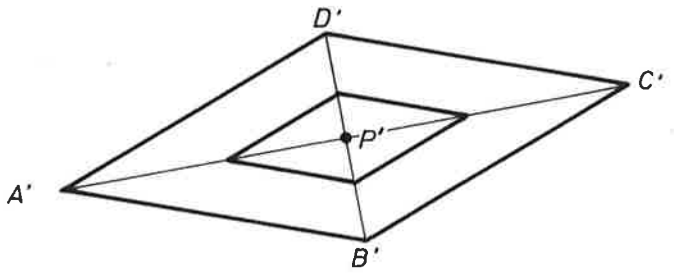

Figure 4.7

By a dilation with center P', scale t, the points A'_t, B'_t, C'_t, D'_t are constructed. After a perspectivity $\pi' \rightarrow \pi$ one obtains the situation of Figure

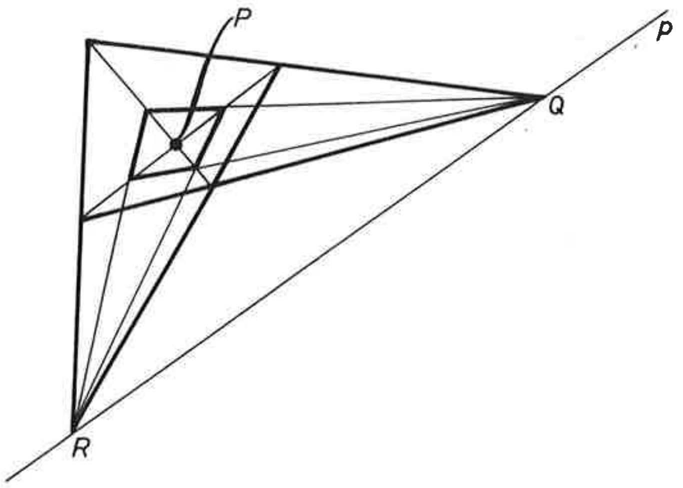

Figure 4.8

4.8. Let A_t, B_t, C_t, D_t be the images of A'_t, B'_t, C'_t, D'_t, and let the line p in π correspond to the line at infinity in π'.

By considering the homological ranges $\mathcal{H}_{P,p}(A,B,C)$ and $\mathcal{H}_{P,p}(A,D,C)$ separately, the whole 4-point may be treated. In both cases the configuration coefficients are $(1/2, 0, 1/2)$. Let

$$\overline{OA} = \frac{1}{\alpha}\overline{OA'}, \quad \overline{OB} = \frac{1}{\beta^+}\overline{OB'}, \quad \overline{OC} = \frac{1}{\gamma}\overline{OC'}, \quad \overline{OD} = \frac{1}{\beta^-}\overline{OD'}$$

Put

$$S_3^+ = \alpha\beta^+\gamma, \quad S_3^- = \alpha\beta^-\gamma$$

Then

$$\Delta(A_t, B_t, C_t, D_t) = \delta(A_t, B_t, C_t) - \delta(A_t, D_t, C_t)$$

Computation of δ, as in (4.3), gives

$$\Delta(A_t, B_t, C_t, D_t) = \frac{t^2 S_3^+ \delta(A,B,C)}{\sigma(A_t)\sigma(B_t)\sigma(C_t)} - \frac{t^2 S_3^- \delta(A,D,C)}{\sigma(A_t)\sigma(D_t)\sigma(C_t)}$$

The facts that $\delta(A', B', C') = -\delta(A', D', C')$ and that P' is the midpoint of $A'C'$ and $B'D'$ yield

$$S_3^+\delta(A,B,C) = -S_3^-\delta(A,D,C)$$
$$\alpha + \gamma = \beta^+ + \beta^-$$

Denote by c the common value of the members of the first equation. Let s_1 be the common value of the second. Put $s_2 = \alpha\gamma + \beta^+\beta^-$, $s_4 = \alpha\gamma\beta^+\beta^-$. Then

$$\begin{aligned}\Delta(A_t, B_t, C_t, D_t) &= ct^2\left(\frac{1}{\sigma(A_t)\sigma(B_t)\sigma(C_t)} - \frac{1}{\sigma(A_t)\sigma(D_t)\sigma(C_t)}\right) \\ &= \frac{ct^2 s_1}{\sigma(A_t)\sigma(B_t)\sigma(C_t)\sigma(D_t)}\end{aligned}$$

Insertion of $t = 1$ gives

$$cs_1 = s_4\Delta(A,B,C,D)$$

After a straightforward calculation we obtain the analogue of (4.3):

$$((1-t^2)^2 s_1^4 + 4t^2(1-t^2)s_1^2 s_2 + 16t^4 s_4)\Delta(A_t, B_t, C_t, D_t) =$$
$$= 16t^2 s_4 \Delta(A, B, C, D)$$

In analogy with h_3 above we define

$$h_4(t_1, t_2, t_3) = t_1^2 t_2^4 - t_1^4 t_2^2 + t_2^2 t_3^4 - t_2^4 t_3^2 + t_3^2 t_1^4 - t_3^4 t_1^2$$
$$= (t_1^2 - t_2^2)(t_2^2 - t_3^2)(t_3^2 - t_1^2)$$

By a now familiar argument we obtain

LEMMA 4.4 *For any perspective image of the configuration in π' holds the relation*

$$\begin{aligned} &\frac{t_0^2 h_4(t_1, t_2, t_3)}{\Delta_{t_0}} - \frac{t_1^2 h_4(t_2, t_3, t_0)}{\Delta_{t_1}} + \\ &\frac{t_2^2 h_4(t_3, t_0, t_1)}{\Delta_{t_2}} - \frac{t_3^2 h_4(t_0, t_1, t_2)}{\Delta_{t_3}} = 0 \end{aligned} \tag{4.11}$$

□

This is in fact an invariant under general projectivities. To prove this, one needs some invariant configuration property, replacing the configuration coefficients in Lemma 4.3. To this end one notices that P is a vertex and p the opposite side of the diagonal triangle P, Q, R of the complete quadrangle defined by A, B, C, D (cf. [C] Ch 2). Let us in this case say that $(P, p; A, B, C, D)$ is a *diagonal configuration.* This property is preserved under projectivities. Defining in a natural way the homological range $\mathcal{H}_{P,p}(A, B, C, D)$, an analogue of Lemma 4.3 holds true in this particular case. We obtain

THEOREM 4.2 *Suppose that $(P, p; A, B, C, D)$ is a diagonal configuration. Let $(A_{t_i}, B_{t_i}, C_{t_i}, D_{t_i}) \in \mathcal{H}_{P,p}(A, B, C, D)$, $i = 0, 1, 2, 3$. Then the equation (4.11) is invariant uder projectivities.* □

Remark. Comparing (4.5) and (4.11), where in both cases the center of the figure was used as the center of the homology, one notes at least two common features. First, the number of figures needed were in both cases four, and second, the coefficients h_3 and h_4 in the invariant formula have the same structure. The problem arises whether this can be generalized to general k-points. The answer is *no*, at least in the sense that the number of figures needed depends on k. This number is highly dependent on the symmetry properties of the figure. Calculations with a symbolic manipulation program have showed that for regular pentagons, $k = 5$, one needs nine and for regular hexagons, $k = 6$, six t-values (i.e. homological images of the reference k-point).

□

5. The Projective n-Space

Above we have treated the projective line and the projective plane, emphasizing the geometric point of view. Here we will derive corresponding results for general projective spaces $\mathbb{P}^n$, using algebraic arguments mainly. As will be seen, these work as well for complex spaces. With the background given in the preceding chapters, we permit ourselves to be somewhat brief.

Let $\mathcal{A} = (A_0, \ldots, A_n)$ be a polyad in $\mathbb{P}^n$. Let P be a point and p an n-dimensional hyperplane in $\mathbb{P}^n$. The definitions of *homology* $H^t_{P,p}$ with center P, axis p, scale t as well as of *homological range* $H_{P,p}(\mathcal{A})$, are taken over word by word from Definition 4.1. In analogy with Definition 4.2 we define the *configuration coefficients* of $(P, p; \mathcal{A})$ by

$$\lambda_i = k_{i,i+1}(P, A_i; A_0, \ldots, A_i P \cdot p, \ldots, A_n), \qquad i = 0, 1, \ldots, n$$

where, by convention, $k_{n,n+1} = k_{n,0}$.

THEOREM 5.1 *Let $\mathcal{A}_t \in H_{P,p}(\mathcal{A})$, $i = 0, 1, \ldots, n$, and let λ be the configuration coefficients of $(P, p, \mathcal{A})$. Then the following formulas are invariant under projectivities.*

(i) If $\lambda_i = 1/(n+1)$, $i = 0, 1, \ldots, n$, then

$$\begin{vmatrix} \varrho_1(t_0,t_1) & \varrho_2(t_0,t_1) & \ldots & \varrho_{n+1}(t_0,t_1)/\delta_{t_0} - t_0 t_1^n/\delta_{t_1} \\ \varrho_1(t_0,t_2) & & & \\ \vdots & & & \vdots \\ \varrho_1(t_0,t_{n+1}) & & \ldots & \varrho_{n+1}(t_0,t_{n+1})/\delta_{t_0} - t_0 t_{n+1}^n/\delta_{t_{n+1}} \end{vmatrix} = 0 \tag{5.1}$$

where

$$\begin{aligned} \delta_t &= \delta(\mathcal{A}_t) \\ \varrho_1(t_0,t) &= (t_0 + nt)(t_0 - t)^n \\ \varrho_k(t_0,t) &= t^k (t_0 - t)^{n+1-k} \qquad k = 2, \ldots, n+1 \end{aligned} \tag{5.2}$$

(ii) Generally there exists a polynomial R such that

$$R(\lambda; t; 1/\delta(\mathcal{A}_t)) = 0$$

where

$$\lambda \in \mathbb{R}^{n+1}, \quad t \in \mathbb{R}^{n+2}$$

$$1/\delta(\mathcal{A}_t) = \big(1/\delta(\mathcal{A}_{t0}), \ldots, 1/\delta(\mathcal{A}_{t_{n+1}})\big) \in \mathbb{R}^{n+2}$$

R is homogeneous of degree $n(n+1)$ in $1/\delta(\mathcal{A}_t)$. □

Proof: By the fundamental theorem of projective geometry there exists a (unique) projectivity on $\mathbb{P}^n$, mapping p onto the hyperplane $\sum_0^n x_i = 0$, and $A_0, A_1, \ldots, A_n$ onto $(1,0,\ldots,0), (0,1,\ldots,0), \ldots, (0,0,\ldots,1)$ respectively. It thus suffices to prove the theorem for those particular choices of $A_i, i = 0, \ldots, n$, and p. As usual we choose the augmented hyperplane $\pi : \sum_0^n x_i = 1$ as a model for $\mathbb{P}^n$.

Now let $T = (T_0, \ldots, T_n)$ be a representative of a projectivity, where T_i stands for the i:th column, $i = 0, \ldots, n$. First suppose that $\sigma(T_i) = \alpha_i \neq 0, i = 0, \ldots, n$. Then we have the factorization

$$T = (T_0/\alpha_0 \ldots T_n/\alpha_n)\mathrm{diag}(\alpha_0, \ldots, \alpha_n)$$

Here all column sums in the first matrix are equal to 1. Hence it defines an affinity on the affine plane π. It changes all δ:s by a common factor only, while the homology scale t and configuration coefficients are left unchanged. An equation (i) or (ii) valid before application of this matrix is thus valid afterwards too. Hence, to verify the theorem for projectivities with non-zero column sums, it suffices to consider the second factor in the representation above. After this is done below, by a limit argument one sees that also the case of general projectivities is settled.

Thus let $T = \mathrm{diag}(\alpha)$ with $\alpha_i \neq 0$, $i = 0, \ldots, n$. Since $p : \sum_0^n x_i = 0$ the homologies $H_{P,p}^t$ are in fact dilations on $\pi : \sum_0^n x_i = 1$. If P has the barycentric coordinates $\lambda = (\lambda_0, \ldots, \lambda_n)$ with respect to $A_0, \ldots, A_n$, then the images under $H_{P,p}^t$ of these points are

$$A_{0t} = ((1-t)\lambda_0 + t, (1-t)\lambda_1, \ldots, (1-t)\lambda_n)$$
$$A_{1t} = ((1-t)\lambda_0, (1-t)\lambda_1 + t, \ldots, (1-t)\lambda_n)$$

$$A_{nt} = ((1-t)\lambda_0, (1-t)\lambda_1, \ldots, (1-t)\lambda_n + t)$$

Put

$$TA_{it} = \tilde{A}_{it}, \quad \mathcal{A}_t = (A_{0t}, \ldots, A_{nt}), \quad \tilde{\mathcal{A}}_t = (\tilde{A}_{0t}, \ldots, \tilde{A}_{nt})$$

Here the $\tilde{A}_{it}$ are obtained by insertion of factors $\alpha_0, \ldots, \alpha_n$ in the coordinates of A_{it}. Application of the definition of δ to $\tilde{\mathcal{A}}_t$ gives, after introduction of t_0 as in Chapters 3 and 4

$$p(\alpha; \lambda; t_0 t)\delta(\tilde{\mathcal{A}}_t) = \alpha_0 \ldots \cdot \alpha_n t_0 t^n \delta(\tilde{\mathcal{A}}_{t_0})$$

where

$$p(\alpha; \lambda; t_0, t) = \Pi_{i=0}^n \sigma(\tilde{A}_{it}) \tag{5.3}$$

Here p is homogeneous of degree $n+1$ in the variables α. Using $n+1$ such equations, for $t = t_1, \ldots, t_{n+1}$, the Elimination lemma yields (ii).

As usual it is possible to be more explicit in the case $\lambda_i = 1/(n+1), i = 0, \ldots, n$. Introduce the symmetric functions

$$S_1 = \alpha_0 + \ldots + \alpha_n$$
$$S_2 = \sum_{i \neq j} \alpha_i \alpha_j$$
$$\ldots$$
$$S_n = \alpha_0 \cdot \ldots \cdot \alpha_n$$

Then

$$\sigma(\tilde{A}_{it}) = (t_0 - t)S_1/(n+1) + t\alpha_i, \quad i = 0, \ldots, n$$

A calculation of the polynomial p gives

$$p(\alpha; t_0, t) = \sum_{k=0}^{n+1} (t_0 - t)^{n+1-k} t^k S_1^{n+1-k} S_k/(n+1)^{n+1-k}$$

Hence $n+1$ equations (5.3), corresponding to $t = t_1, \ldots, t_{n+1}$, may be considered as a system of linear equations in the unknowns S_1^{n+1}, $S_1^n S_2, \ldots, S_{n+1}$. This system is known to have a non-trivial solution. With the notation (5.2), after simplification, the determinant criterion gives (5.1). □

6. Conical Area-invariants

In this chapter we derive two-dimensional area-relations for regions enclosed by ellipses and, after suitable interpretation, general conic sections. We restrict ourselves to the analogue of the "polar case" in Section 4.2, i.e. when the center and axis of the homological range are pole and polar of the configurations considered.

6.1 The Fundamental Form

We consider quadratic functions

$$q(x) = \frac{1}{2}x^T Qx + a^T x + b, \quad x \in \mathbb{R}^2,\ Q \text{ symmetric}$$

If Q is non-singular, let x^* be defined as the solution of

$$Qx^* + a = 0$$

Then

$$\begin{aligned} q(x) &= \frac{1}{2}(x - x^*)^T Q(x - x^*) - \frac{1}{2}x^{*T}Qx^* + b \\ &= \frac{1}{2}(x - x^*)^T Q(x - x^*) - \frac{1}{2}a^T Q^{-1}a + b \end{aligned}$$

Suppose for a moment that Q is positive definite with eigenvalues λ_1, λ_2. Diagonalization of the quadratic form $(x - x^*)^T Q(x - x^*)$ then yields that the area of the region $q(x) \leq 0$ in the Euclidean plane is

$$\pi \frac{a^T Q^{-1}a - 2b}{\sqrt{\lambda_1 \lambda_2}}$$

provided that the nominator is positive. (Otherwise the region is empty). Taking into account that $\lambda_1\lambda_2 = \det Q$, we are led to the following definition.

DEFINITION 6.1 By the *fundamental form* of q is meant

$$\alpha(q) = \begin{cases} \frac{a^T Q^{-1}a - 2b}{\sqrt{\det Q}} & \text{if } Q \text{ is nonsingular} \\ \infty & \text{otherwise} \end{cases}$$

□

(The terminology stems from the theory of the space of spheres, cf. [B] vol 5 p. 129, where a similar expression appears.)

Remark. Above q was expressed in non-homogeneous coordinates $x \in \mathbb{R}^2$. Using instead homogeneous coordinates $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ (where the non-homogeneous ones are obtained by letting $x_3 = 1$), then q may be expressed as $x^T Ax$, with

$$A = \frac{1}{2}\begin{pmatrix} Q & a \\ a^T & 2b \end{pmatrix}$$

An alternate formula for the fundamental form is then

$$\alpha(q) = -4\frac{\det A}{(\det Q)^{3/2}}$$

□

We have shown how $\alpha(q)$, apart from a factor π, can be interpreted as an area if Q is positive definite. This also holds in the affine plane, cf. the discussion in the beginning of Section 2.2. For Q indefinite, $\alpha(q)$ is purely imaginary. (Since in the sequel it always appears squared, this will cause no unambiguity). If one wishes, in the theorem below it can be given the following interpretation:

> The area assigned to a hyperbola with half-axes $a, b =$
> $\frac{1}{i}$ (the area assigned to an ellipse with half-axes a, b).

About $\alpha(q)$ a similar remark as the one for δ, preceding Definition 2.2 must be made. Thus α itself changes in an irregular way under projectivities. However, when grouping together a number of α:s in a particular equation, we will see that each of them allows a projectively meaningful interpretation as an area.

6.2 Perspectivities

Now consider a non-degenerate cone in the three-dimensional space. Let O be its vertex and let π, π' be two planes with $O \notin \pi \cup \pi'$, $\pi \neq \pi'$. Let $\ell = \pi \cap \pi'$. Two conics C and C' are defined by the intersections of the cone with π and π' respectively. Suppose that C' is an ellipse.

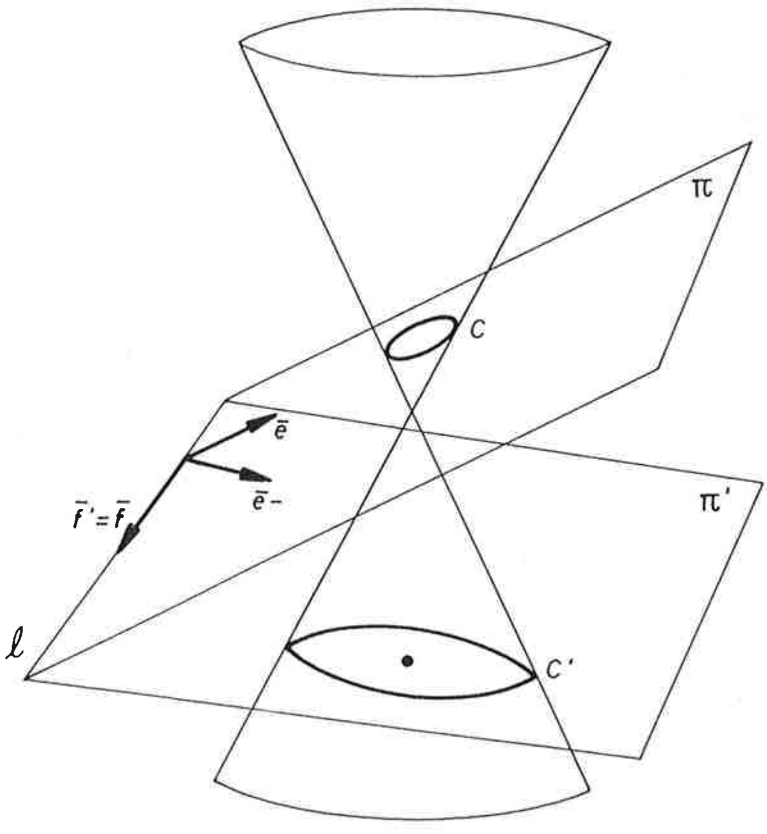

Figure 6.1

Referring to Figure 4.1, let P' be the center of C'. Let $\bar{e}'$ be the conjugate direction of ℓ with respect to C' (i.e. the direction determined by the locus of all midpoints of chords of C' parallel to ℓ). Choose the length of $\bar{e}'$ so that it can be represented by a directed segment connecting ℓ and P'. Then by classical theory of conics it is possible to choose $\bar{f}'//\ell$ so that, for some t,

$$C' : x'^2 + y'^2 = t^2 \tag{6.1}$$

Choosing the third axis as $\bar{g}' = \overline{OP'}$ and using O as origin, the equation of π is $z' = 1$ and the equation of the cone is

$$x'^2 + y'^2 - t^2 z'^2 = 0$$

To treat π a new basis

$$\bar{e} = \bar{e}' - \varrho \bar{g}', \quad \bar{f} = \bar{f}', \quad \bar{g} = \bar{g}'$$

is introduced, where ϱ is so chosen that $\bar{e}//\pi$. The change of coordinates $x' = Sx$ is then described by the matrix

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\varrho & 0 & 1 \end{pmatrix}$$

In the coordinate system $O, \bar{e}, \bar{f}, \bar{g}$, the equation of π is $z = 1 - \varrho$. The equation of the cone is

$$x^2 + y^2 - t^2(-\varrho x + z)^2 = 0$$

Now consider C, i.e. the intersection of π with the cone. In the coordinate system $P, \bar{e}, \bar{f}$ of π, its equation is

$$\begin{aligned} q_t(x,y) &= x^2 + y^2 - t^2(-\varrho x + 1 - \varrho)^2 = 0 \quad \Leftrightarrow \\ q_t(x,y) &= (1 - t^2\varrho^2)x^2 + y^2 + 2t^2\varrho(1-\varrho)x - t^2(1-\varrho)^2 = 0 \end{aligned} \tag{6.2}$$

Computation of the fundamental form gives

$$\alpha(q_t) = \frac{t^2(1-\varrho)^2}{(1 - t^2\varrho^2)^{3/2}}$$

Here ϱ is a parameter, describing the perspectivity. Putting together two such expressions, corresponding to t_0 and t_1 respectively, one obtains

$$\frac{\alpha(q_1)}{\alpha(q_0)} = \frac{t_1^2}{t_0^2}\left(\frac{1 - t_0^2\varrho^2}{1 - t_1^2\varrho^2}\right)^{3/2}$$

or equivalently

$$\varrho^2 = \frac{(\frac{\alpha(q_0)}{t_0^2})^{2/3} - (\frac{\alpha(q_1)}{t_1^2})^{2/3}}{t_0^2(\frac{\alpha(q_0)}{t_0^2})^{2/3} - t_1^2(\frac{\alpha(q_1)}{t_1^2})^{2/3}}$$

By means of two expressions, for t_0, t_1 and t_0, t_2 respectively, ϱ may be eliminated. Simplification yields

LEMMA 6.1 *With the notation introduced above, for any perspectivity holds*

$$\frac{t_1^2 - t_2^2}{(\frac{\alpha(q_0)}{t_0^2})^{2/3}} + \frac{t_2^2 - t_0^2}{(\frac{\alpha(q_1)}{t_1^2})^{2/3}} + \frac{t_0^2 - t_1^2}{(\frac{\alpha(q_2)}{t_2^2})^{2/3}} = 0 \tag{6.3}$$

Remark. At first sight this lemma only gives an analytical relation between the fundamental forms of three particular quadratic functions $q_i, i = 0, 1, 2$. However, since $\alpha(q_t)$ has an interpretation as the area connected with C_t : $q_t(x) = 0$, it changes only by a proportionality factor under affine coordinate transformations on π. Because of the homogeneity of (6.3), one may thus choose for q_i the polynomials defining C_{t_i} in *any* affine basis for π. □

6.3 Projectivities

Projectivities map conics onto conics. If C has the equation $q(x) = 0$, then the image under T has the equation $(Tq)(x) = 0$, where

$$(Tq)(x) = q(T^{-1}x)$$

Here it is preferable to work with homogeneous coordinates, since then the calculation of Tq can be done by means of matrix operations.

Equation (6.1) describes a family of conical sections, obtained from each other by dilations with center P'. After a perspectivity, the situation in π is described by homologies $H^t_{P,p}$, where P, p are the images of P' and the line at infinity, respectively. The *homological range* $\mathcal{H}_{P,p}(C)$ of conical sections is defined as for polyads.

The concepts of *pole* and *polar* are central in projective geometry, cf. e.g. [C] Ch. 8. In π' the center P' of C' and the line at infinity are pole and polar with respect to C'. These properties are preserved under perspectivities, i.e. P and p are pole and polar with respect to C. The situation is unaltered after any projectivity on π. For this particular pole-polar configuration it is thus possible to formulate an analogue of Lemma 4.3. Together with Lemma 6.1, cf. also the remark above, it yields:

THEOREM 6.1 *Let $C_{t_i} \in \mathcal{H}_{P,p}(C)$, $i = 0, 1, 2$, where P and p are pole and polar with respect to C. Let C_{t_i} have the equation $q_i(x) = 0$. Then the formula in Lemma 6.1 is an invariant under projectivities (i.e. when replacing q_i by Tq_i).* □

Remark. It is noteworthy that the number of terms in (6.3) is three, while it earlier in the plane has been at least four (cf. the remark after Theorem 4.2). □

References

[B] BERGER, M., *Géométrie.* Cedic/Fernand Nathan, Paris (1977).

[C] COXETER, H.S.M., *Projective Geometry.* University of Toronto Press, Toronto (1974).

[H] HARTSHORNE, R., *Foundations of Projective Geometry.* Benjamin, New York (1967).

[H-P] HODGE, W.V.D and PEDOE, D., *Methods of Algebraic Geometry.* Cambridge University Press, Cambridge (1947).

[M] MÖBIUS, A.F., *Gesammelte Werke*, Erster Band. Hirzel, Leipzig (1885).

[N] NIELSEN, L., "Simplifications in visual servoing," Ph.D. thesis CODEN: LUTFD2/TFRT-1027, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden (1985).

[N-S] NIELSEN, L. and SPARR, G., "Perspective area-invariants," Report CODEN: LUTFD2/TFRT-7313, Department of Automatic Control, Lund Institute of Technology, Lund, Sweden (1985).

[V-Y] VEBLEN, O. and YOUNG, J.W., *Projective Geometry.* Ginn and Company, Boston (1910).

[vdW] VAN DER WAERDEN, B.L., *Algebra*, 3rd edition. Springer-Verlag, Berlin-Heidelberg-New York (1955).