Linear Sampled System with Time Delay Which is a Fraction of the Sampling Period

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$$\frac{dx}{dt} = Ax(t) + Bu(t-\tau h)$$

Assume that the input signal is kept constant over sampling intervals of length h. The input u and the state x is illustrated in Fig. 1.

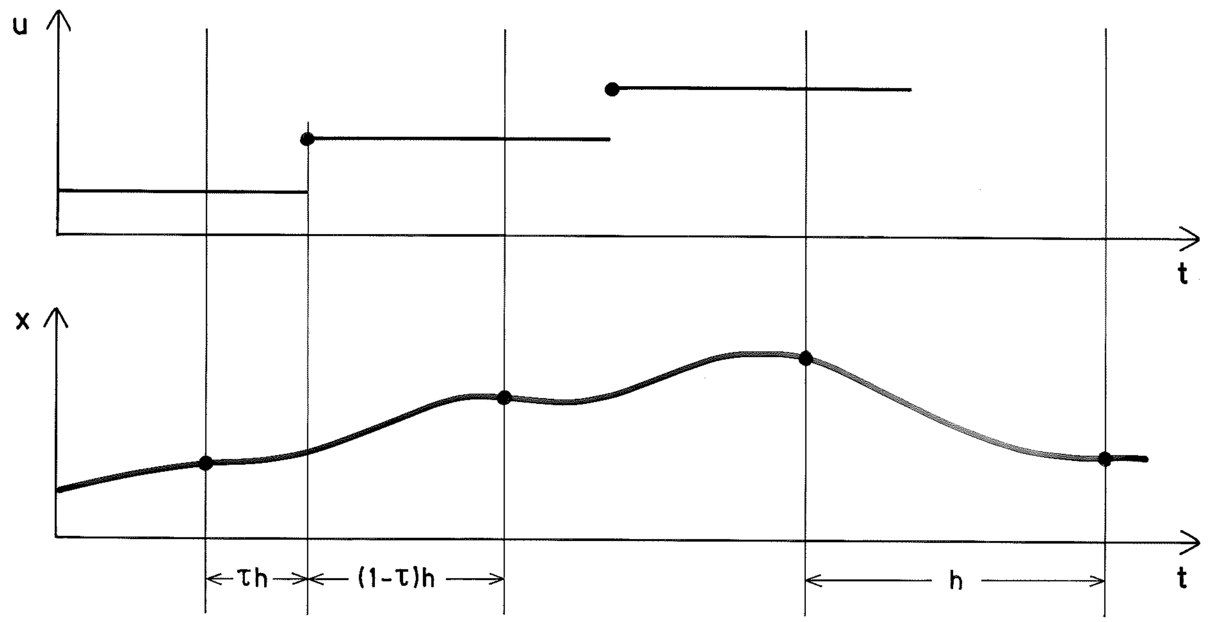

Fig. 1.

The value of the state vector at the sampling points is then given by

$$x(t+h) = \Phi x(t) + \Gamma_1 u(t) + \Gamma_2 u(t-h) \tag{1}$$

where

$$\Phi = e^{Ah} \tag{2}$$

$$\Gamma_1 = \int_0^{h(1-\tau)} e^{As} \, Bds \tag{3}$$

$$\Gamma_2 = \int_{h(1-\tau)}^{h} e^{As} \, Bds \tag{4}$$

Notice that

$$\Gamma_1 + \Gamma_2 = \int_0^h e^{As} \, B ds$$

The only difference compared to the usual case is thus that the term u(t-h) appears in (1).

The puls transfer function relating the state to the input is thus given by

$$H(z) = \left[zI - \Phi\right]^{-1} \left[\Gamma_1 + z^{-1}\Gamma_2\right] \tag{6}$$

<u>Single Input Single Output Systems</u>

In the single-input single-output case we choose the coordinates in the state space representation so that the matrix Φ is a companion matrix i.e.

$$\Phi = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix} \tag{7}$$

Furthermore denote

$$\Gamma_1 = \begin{bmatrix} \gamma_1^1 \\ \gamma_2^2 \\ \vdots \\ \gamma_n^1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} \gamma_1^2 \\ \gamma_2^2 \\ \vdots \\ \gamma_n^2 \end{bmatrix} \tag{8}$$

If the output y is chosen as the first component of the state-vector i.e. $y=x_1$ we thus find that the input output relation is given by

$$y(t)+a_1y(t-1)+\dots+a_ny(t-n) = \gamma_1^1u(t-1)+(\gamma_2^1+\gamma_1^2)u(t-2)+$$
$$\dots(\gamma_n^1+\gamma_{n-1}^2)u(t-n)+\gamma_n^2\,u(t-n-1) \tag{9}$$

which is identical to the standard form

$$A(q)y(t) = B(q)u(t-1) \tag{10}$$

where the polynomials A and B are both of degree n. Notice that in the case when the timedelay is an integer multiple of the sampling interval the polynomial B in (10) is of degree n-1.

A consequence of importance for system identification is thus that it is reasonable to consider models where the polynomials A and B are of the same degree as the standard case. It is also clear that the model (9) can be used as a basis for adaptive algorithms that can handle variable time delays.

An Example

As an illustration we will consider the first order system

$$\dot{x}(t) = -x(t) + u(t-\tau h)$$

It is assumed that the input u is kept constant over sampling intervals of length h, we thus find that the values of the state variable at the sampling intervals are given by (1) where

$$\begin{aligned} \phi &= e^{-h} &&= a \\ \Gamma_1 &= 1-e^{-h(1-\tau)} &&= b_1 \\ \Gamma_2 &= e^{-h(1-\tau)}-e^{-h} &&= b_2 \end{aligned}$$

The transfer function of the system is thus given by

$$H(z) = \frac{b_1+b_2z^{-1}}{z+a}$$

Notice that the pulse transfer function has a zero

$$z = -\frac{b_2}{b_1} = - \frac{1 - e^{-h(1-\tau)}}{e^{-h(1-\tau)}-e^{-h}}$$

which is outside the unit circle if

$$\tau > 1 + \frac{1}{n} \log \frac{1+e^{-h}}{2} = f(h)$$

A graph of the function f is shown in Fig 2.

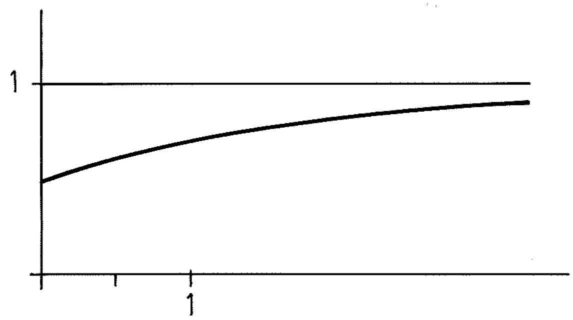

Fig. 2
Graph of $f(x) = 1 + \frac{1}{x} \log \frac{1+e^{-x}}{2}$

The sampled system will thus be nonminimum phase if the delay τ is sufficiently large. The critical value of τ depends on the sampling interval. Notice that if $\tau < 0.5$ the zero of the pulse transfer function is always inside the unit circle.

3. REFERENCE

[1] Bolam, F. "Papermaking Systems and their Control" Trans of the Symposium held in Oxford Sept 1969. The British Paper and Board Maker's Association, London 1970.