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LINEAR SAMPLED SYSTEM WITH  
TIME DELAY WHICH IS A  
FRACTION OF THE SAMPLING PERIOD

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# LINEAR SAMPLED SYSTEM WITH TIME DELAY WHICH IS A FRACTION OF THE SAMPLING PERIOD

K.J. Åström

Abstract

## 1. INTRODUCTION

## 2. ANALYSIS

Single Input Single Output Systems

An Example

## 3. REFERENCE

$$\frac{dx}{dt} = Ax(t) + Bu(t-\tau h)$$

Assume that the input signal is kept constant over sampling intervals of length  $h$ . The input  $u$  and the state  $x$  is illustrated in Fig. 1.

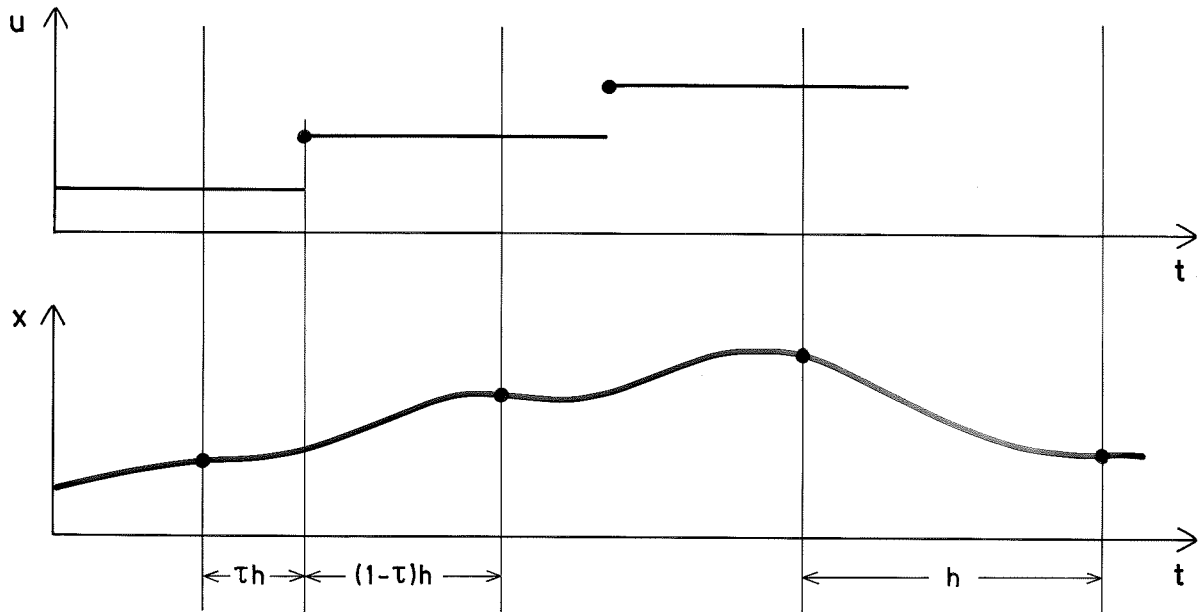


Fig. 1.

The value of the state vector at the sampling points is then given by

$$x(t+h) = \Phi x(t) + \Gamma_1 u(t) + \Gamma_2 u(t-h) \quad (1)$$

where

$$\Phi = e^{Ah} \quad (2)$$

$$\Gamma_1 = \int_0^{h(1-\tau)} e^{As} B ds \quad (3)$$

$$\Gamma_2 = \int_{h(1-\tau)}^h e^{As} B ds \quad (4)$$

Notice that

$$\Gamma_1 + \Gamma_2 = \int_0^h e^{As} B ds$$

The only difference compared to the usual case is thus that the term  $u(t-h)$  appears in (1).

The puls transfer function relating the state to the input is thus given by

$$H(z) = [zI - \Phi]^{-1} [\Gamma_1 + z^{-1}\Gamma_2] \quad (6)$$

### Single Input Single Output Systems

In the single-input single-output case we choose the coordinates in the state space representation so that the matrix  $\Phi$  is a companion matrix i.e.

$$\Phi = \begin{bmatrix} -a_1 & 1 & 0 & \dots & 0 \\ -a_2 & 0 & 1 & \dots & 0 \\ \vdots & & & & \\ -a_{n-1} & 0 & 0 & \dots & 1 \\ -a_n & 0 & 0 & \dots & 0 \end{bmatrix} \quad (7)$$

Furthermore denote

$$\Gamma_1 = \begin{bmatrix} 1 \\ \gamma_1^1 \\ \gamma_2^2 \\ \vdots \\ \gamma_n^1 \end{bmatrix}, \quad \Gamma_2 = \begin{bmatrix} \gamma_1^2 \\ \gamma_2^2 \\ \vdots \\ \gamma_n^2 \end{bmatrix} \quad (8)$$

If the output  $y$  is chosen as the first component of the state-vector i.e.  $y=x_1$  we thus find that the input output relation is given by

$$\begin{aligned} y(t) + a_1 y(t-1) + \dots + a_n y(t-n) &= \gamma_1^1 u(t-1) + (\gamma_2^1 + \gamma_1^2) u(t-2) + \\ &\dots + (\gamma_n^1 + \gamma_{n-1}^2) u(t-n) + \gamma_n^2 u(t-n-1) \end{aligned} \quad (9)$$

which is identical to the standard form

$$A(q)y(t) = B(q)u(t-1) \quad (10)$$

where the polynomials A and B are both of degree n. Notice that in the case when the timedelay is an integer multiple of the sampling interval the polynomial B in (10) is of degree n-1.

A consequence of importance for system identification is thus that it is reasonable to consider models where the polynomials A and B are of the same degree as the standard case. It is also clear that the model (9) can be used as a basis for adaptive algorithms that can handle variable time delays.

### An Example

As an illustration we will consider the first order system

$$\dot{x}(t) = -x(t) + u(t-\tau h)$$

It is assumed that the input u is kept constant over sampling intervals of length h, we thus find that the values of the state variable at the sampling intervals are given by (1) where

$$\begin{aligned} \phi &= e^{-h} &= a \\ \Gamma_1 &= 1 - e^{-h(1-\tau)} &= b_1 \\ \Gamma_2 &= e^{-h(1-\tau)} - e^{-h} &= b_2 \end{aligned}$$

The transfer function of the system is thus given by

$$H(z) = \frac{b_1 + b_2 z^{-1}}{z + a}$$

Notice that the pulse transfer function has a zero

$$z = -\frac{b_2}{b_1} = -\frac{1 - e^{-h(1-\tau)}}{e^{-h(1-\tau)} - e^{-h}}$$

which is outside the unit circle if

$$\tau > 1 + \frac{1}{n} \log \frac{1+e^{-h}}{2} = f(h)$$

A graph of the function  $f$  is shown in Fig 2.

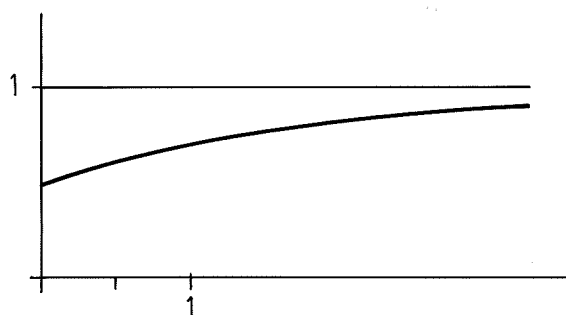


Fig. 2

Graph of  $f(x) = 1 + \frac{1}{x} \log \frac{1+e^{-x}}{2}$

The sampled system will thus be nonminimum phase if the delay  $\tau$  is sufficiently large. The critical value of  $\tau$  depends on the sampling interval. Notice that if  $\tau < 0.5$  the zero of the pulse transfer function is always inside the unit circle.

### 3. REFERENCE

- [1] Bolam, F. "Papermaking Systems and their Control" Trans of the Symposium held in Oxford Sept 1969. The British Paper and Board Maker's Association, London 1970.