# Characteristics of a stable platform

Åström, Karl Johan

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# CHARACTERISTICS OF A STABLE PLATFORM

by

Karl-Johan Åström

TTN-GROUP

DIVISION OF APPLIED HYDROMECHANICS

ROYAL INSTITUTE OF TECHNOLOGY

STOCKHOLM, SWEDEN

# CHARACTERISTICS OF A STABLE PLATFORM.

by

Karl-Johan Åström

Summary.

In this report a three-axis stable platform system is discussed. The platform system consists of a stable element to which three single-axis gyros are mounted. The stable element is provided with (some kind of) suspension, e.g. a gimbal system, arranged in such a way that it is possible to apply a torque to the stable element.[*] The system is described in section 1. Possible ways of arranging the gyroscopes on the stable element are discussed in sections 2 and 3. The equations of motion are derived for arbitrary orientation of the gyros in sections 4 and 5. The equations of motion are linearized in section 6, where it is assumed that the input axes of the gyros are mutually orthogonal. In section 7 are given some physical interpretations of the results, so far obtained. In section 8 the conditions for stability are established. It is found that the stability of the system is greatly affected by the arrangement of the gyros. The main result of section 8 is that, for a system with orthogonal input axes, the gyroscopes should be arranged with the output axes in the same plane and with a total angular momentum equal to or greater than $\sqrt{3}$ H, where H is the angular momentum of one gyroscope. The stability conditions for systems designed on a single axis basis are also discussed in section 8. It is found that the output axis sensitivity of the gyroscopes gives interaction between the three channels. The effect of this interaction is also extensively treated. Systems with a characteristic equation of a low degree are analysed with algebraical methods. For systems with a

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[*] For an analysis of the dynamics of a gimbal system we refer to the work reported in references 5 and 6.

characteristic equation (of a high degree) the graphical method of Evans is adopted. In section 9 are analysed the angular deviation of the stable element, caused by stochastic disturbances. It is assumed that the disturbances are stationary processes with zero averages. It is found that the angular deviation caused by disturbing torques acting on the gyrofloats has a random walk character, with a variance increasing linearly with time. A relationship is given between the variance of the angular displacement and the autocorrelation function of the disturbing torque. A quality figure for the random drift of a gyroscope is suggested. In section 10 the synthesis problem is briefly delt with. A method is given for the synthesis of an inertial stabilized platform system. In the appendix the synthesis method due to Truxal is adopted to the synthesis of a single axis inertial stabilized platform system.

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# CHARACTERISTICS OF A STABLE PLATFORM

## Contents.

1. A short description of the system.

The principal function of the stable platform is to maintain a physical reference system (for navigation, fire-control, or other purposes). The (complex) of all components necessary to establish the reference is called the platform system. The component of the platform system which mechanizes the reference is called the stable element. Here it is assumed that the basic component of the platform system is the single-axis floated gyro. The features of such a gyro are shown in figure 1.1.



TORQUE GENERATOR STATOR
TORQUE GENERATOR ROTOR    FLOAT    SIGNAL GENERATOR ROTOR
STATOR
OUTPUT AXIS
SPIN AXIS
GIMBAL BEARING
CASE
BALANCE NUT
GYRO ROTOR
SPIN REFERENCE AXIS
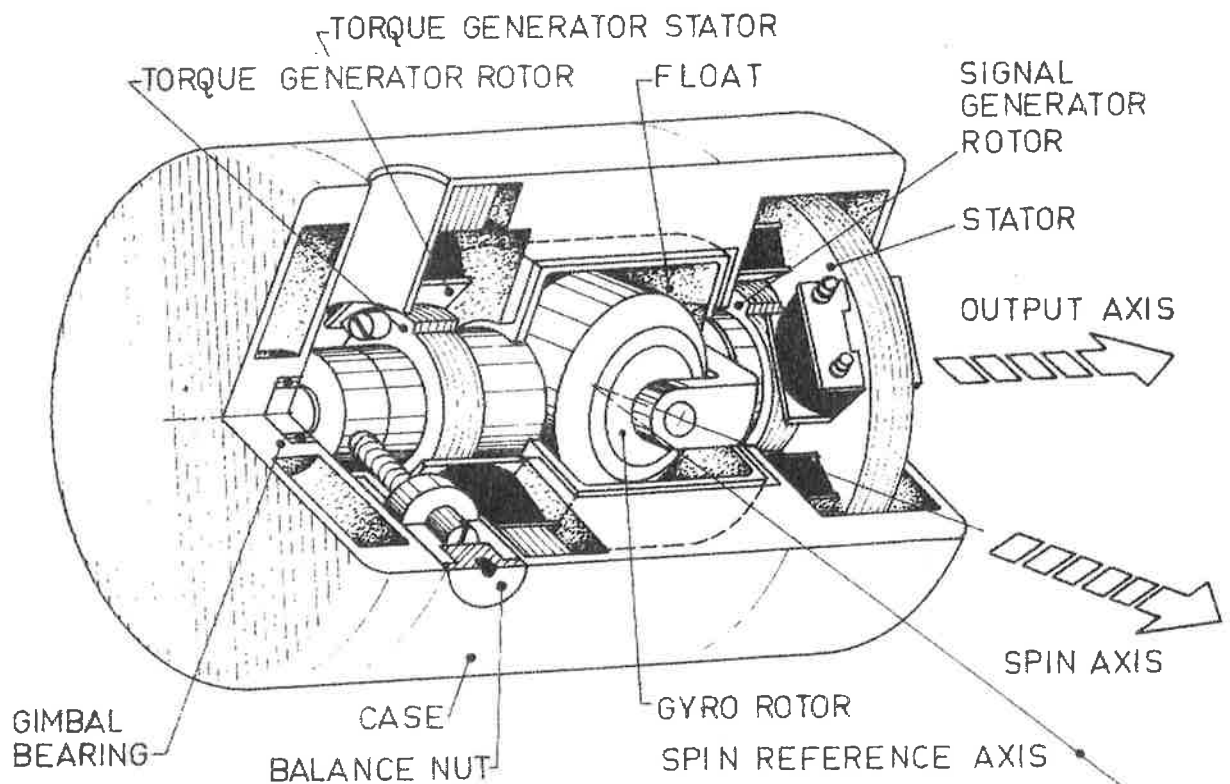
FIGURE  1.1

It consists of a gyrorotor supported by ball bearings in a cylindrical chamber, called the float, which is supported in the case by jewel bearings. The space between the cylinder and the case is filled with a high-density fluid. This fluid serves to float the cylinder in order to decrease the friction torque in the bearings. In some applications the fluid is also used in order to introduce damping between the float and the case. The pivot axis of the float is the output axis of the gyro, and the angle between the float and the case is the output signal of the gyro. The output signal is measured by a microsyn, the signal generator. A gyro is often provided with another microsyn, the torque generator, which makes it possible to

apply a torque to the float for control purposes. The spin axis of the gyro is coincident with the spin vector of the rotor. The axis coincident with the spin axis when no output signal is obtained from the gyro is called the spin reference axis. The input axis of the gyro is orthogonal to the spin reference axis and to the output axis. In pictorial diagrams the gyro is represented by the following simplified drawing.



SPIN REFERENCE AXIS

OUTPUT AXIS
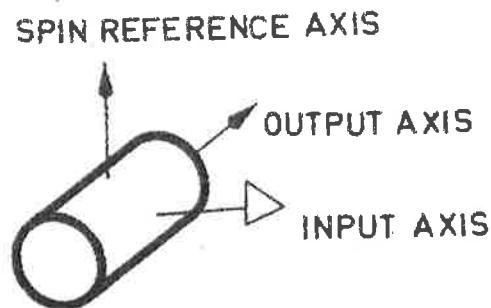
INPUT AXIS

Figure 1.2

The desired performance of the single axis gyro is to give an output signal for rotations around the input axis. For further details of the single axis floated gyro, see reference 1.

In order to maintain the reference, three single axis gyros are mounted to a stable element with mutually orthogonal input axes. The stable element is supported for three degrees of freedom to the carrier frame, for example by a system of gimbals. The main feature of the stable element and the gimbal system is shown in figure 1.3.
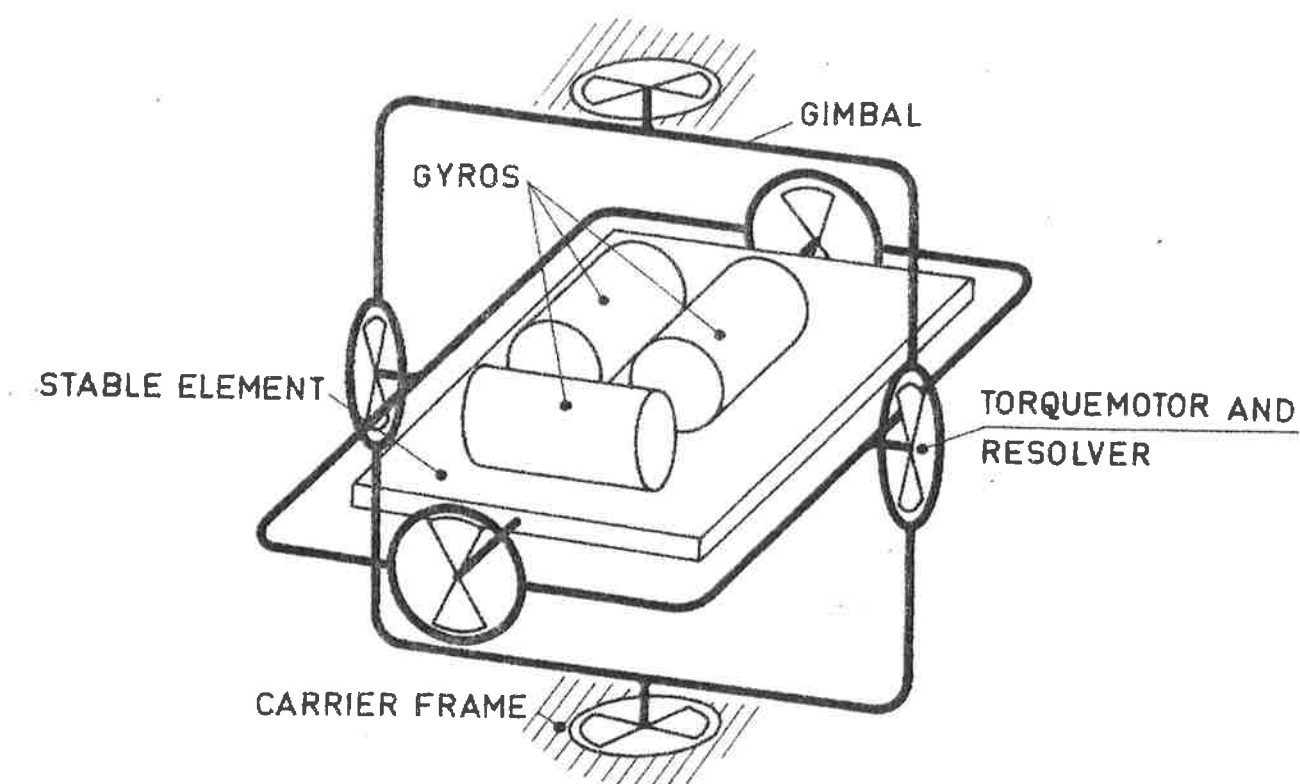
FIG. 1. 3

The rotations of the stable element is sensed by the gyros. The output
signals of the gyros are amplified, filtered and distributed to torque
motors on the gimbals and in the gyros. By the proper choice of the
transfer functions from the gyros to the torque motors it is possible
for the stable element to maintain the desired reference.

2. **Description of the arrangement of the gyros and definition of the coordinate sets.**

The center of mass of the stable element is $O$. Introduce a right-handed orthogonal coordinate system $Oy_1 y_2 y_3$ fixed to the stable element. This coordinate set will be referred to as the y-set. A gyro is named after its input axis, e.g. a gyro with the $y_m$-axis as input axis is called a m-gyro. The input axes of the three gyros are parallel to the axes of the y-set.

To each of the three gyros we associate a coordinate set $O^{(m)} x_1^{(m)} x_2^{(m)} x_3^{(m)}$, referred to as the $x^{(m)}$-set. The point $O^{(m)}$ is the center of the m-gyro. The $x_1$- and $x_2$-axes coincide with the input and output axes, respectively. The $x_3$-axis coincides with the spin reference axis. Compare figure 1.2.

The position of the m-gyro is given by the point $O^{(m)}$ and the orientation of the m-gyro defined by the transformation of the y-set on the x-set.

$$\overline{x}^{(m)} = P^{(m)} \overline{y} \qquad\qquad 2.1$$

As the input axis of the m-gyro is parallel to the $y_m$-axis the transformation matrix is completely specified by an angle $\theta^{(m)}$.
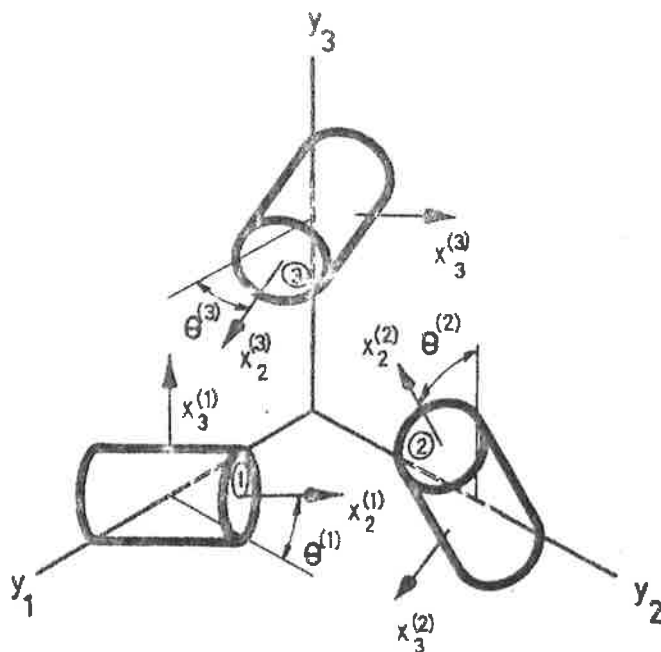
The transformation matrices can be written

$$P^{(1)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta^{(1)} & \sin\theta^{(1)} \\ 0 & -\sin\theta^{(1)} & \cos\theta^{(1)} \end{pmatrix}$$

$$P^{(2)} = \begin{pmatrix} 0 & 1 & 0 \\ \sin\theta^{(2)} & 0 & \cos\theta^{(2)} \\ \cos\theta^{(2)} & 0 & -\sin\theta^{(2)} \end{pmatrix}$$

$$P^{(3)} = \begin{pmatrix} 0 & 0 & 1 \\ \cos\theta^{(3)} & \sin\theta^{(3)} & 0 \\ -\sin\theta^{(3)} & \cos\theta^{(3)} & 0 \end{pmatrix} \qquad 2.2$$

The following conventions are introduced in order to simplify the algebraic manipulations.

(1)    Latin indices used as subscripts will take all values from 1 to 3 unless the contrary is specified.

(2)    If a Latin index is repeated in a term, it is understood that a summation with respect to that index over the range 1, 2, 3 is implied.

Introduce the Kronecker delta $\delta_{ij}$ and the permutation symbol $\epsilon_{ijk}$ defined by

$$\delta_{ij} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases}$$

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if} \quad \text{indices ijk occur in cyclic order} \\ -1 & \text{if} \qquad " \quad " \quad " \quad " \text{ acyclic } \quad " \\ 0 & \text{if} \quad \text{two indices are equal} \end{cases}$$

The transformation 2.1 can be written

$$x_i^{(m)} = P_{ij}^{(m)} y_j \qquad 2.3$$

with the inverse transformation

$$y_i = P_{ji}^{(m)} \, x_j^{(m)} \qquad\qquad 2.4$$

where

$$P_{1i}^{(m)} = \delta_{im} \; ; \; P_{im}^{(m)} = \delta_{1i}$$

$$P_{2, m+1}^{(m)} = \cos \theta^{(m)}$$

$$P_{2, m+2}^{(m)} = \sin \theta^{(m)}$$

$$P_{3, m+1}^{(m)} = -\sin \theta^{(m)}$$

$$P_{3, m+2}^{(m)} = \cos \theta^{(m)} \qquad\qquad 2.5$$

thereby defining

$$P_{i, m+3}^{(m)} = P_{im}^{(m)} \qquad m = 1, 2, 3$$

Introduce the coordinate system $z^{(m)}$ attached to the float of the m-gyro. When the float is in its neutral position, i. e. no output signal, the $z_i^{(m)}$-axis coincides with the $x_i^{(m)}$-axis. The transformation of the $x^{(m)}$-system on the $z^{(m)}$-system is a rotation around the $x_2$-axis. The angle of rotation $\varphi^{(m)}$ is the output signal of the m-gyro. The transformation of the $x^{(m)}$-set on the $z^{(m)}$-set is

$$\overline{z}^{(m)} = R^{(m)} \, \overline{x}^{(m)} \qquad\qquad 2.6$$

$$z_i^{(m)} = r_{ij}^{(m)} \, x_j^{(m)} \qquad\qquad 2.7$$

and the inverse transformation

$$x_i^{(m)} = r_{ji}^{(m)} \, z_j^{(m)} \qquad\qquad 2.8$$

where

$$r_{11}^{(m)} = \cos \varphi^{(m)}$$

$$r_{13}^{(m)} = -\sin \varphi^{(m)}$$

$$r_{31}^{(m)} = \sin \varphi^{(m)}$$

$$r_{33}^{(m)} = \cos \varphi^{(m)}$$

$$r_{2i}^{(m)} = r_{i2}^{(m)} = \delta_{i2} \qquad \text{2.9}$$

Combining equations (2.3) and (2.6) we get

$$z_i^{(m)} = q_{ij}^{(m)} y_j \qquad \text{2.10}$$

where

$$q_{ij}^{(m)} = r_{is}^{(m)} p_{sj}^{(m)} \qquad \text{2.11}$$

Further combining equations (2.4) and (2.7) we get

$$y_i = q_{ji}^{(m)} z_j^{(m)} \qquad \text{2.12}$$

The transformation matrices $Q^{(m)} = R^{(m)} P^{(m)}$ are

$$Q^{(1)} = \begin{pmatrix} \cos \varphi^{(1)} & \sin \varphi^{(1)} \sin \theta^{(1)} & -\sin \varphi^{(1)} \cos \theta^{(1)} \\ 0 & \cos \theta^{(1)} & \sin \theta^{(1)} \\ \sin \varphi^{(1)} & -\cos \varphi^{(1)} \sin \theta^{(1)} & \cos \varphi^{(1)} \cos \theta^{(1)} \end{pmatrix}$$

$$Q^{(2)} = \begin{pmatrix} -\sin \varphi^{(2)} \cos \theta^{(2)} & \cos \varphi^{(2)} & \sin \varphi^{(2)} \sin \theta^{(2)} \\ \sin \theta^{(2)} & 0 & \cos \theta^{(2)} \\ \cos \varphi^{(2)} \cos \theta^{(2)} & \sin \varphi^{(2)} & -\cos \varphi^{(2)} \sin \theta^{(2)} \end{pmatrix}$$

$$Q^{(3)} = \begin{pmatrix} \sin \varphi^{(3)} \sin \theta^{(3)} & -\sin \varphi^{(3)} \cos \theta^{(3)} & \cos \varphi^{(3)} \\ \cos \theta^{(3)} & \sin \theta^{(3)} & 0 \\ -\cos \varphi^{(3)} \sin \theta^{(3)} & \cos \varphi^{(3)} \cos \theta^{(3)} & \sin \varphi^{(3)} \end{pmatrix}$$

$$\text{2.13}$$

3.  Classification and analysis of the arrangement of the gyros.

It was shown in paragraph 2 that the arrangement of the gyros is completely specified by three angles $\theta^{(1)}$, $\theta^{(2)}$, and $\theta^{(3)}$. The arrangements can there fore be classified according to the properties of the triplet $\theta^{(1)}$, $\theta^{(2)}$, $\theta^{(3)}$.

The total angular momentum of the gyros is

$$H \cdot \sum_{m=1}^{3} \hat{x}_3^{(m)} \qquad 3.1$$

where H is the angular momentum of one gyro rotor. The quantity

$$s = \left| \sum_{m=1}^{3} \hat{x}_3^{(m)} \right|^2 \qquad 3.2$$

is called the **spin** of the platform. We get from equation (2.2)

$$s = (\sin \theta^{(1)} - \cos \theta^{(3)})^2 + (\sin \theta^{(2)} - \cos \theta^{(1)})^2 + (\sin \theta^{(3)} - \cos \theta^{(2)})^2$$

$$3.3$$

The spin greately influences the performance of the platform system. Another quantity of significance is the **output axis orientation number 1**, which is defined as the triple scalar product

$$1 = \left[ \hat{x}_2^{(1)}, \hat{x}_2^{(2)}, \hat{x}_2^{(3)} \right] \qquad 3.4$$

The output axis orientation number can be interpreted geometrically as the volume of the parallelepiped with the output axes unit vectors $\hat{x}_2^{(1)}$, $\hat{x}_2^{(2)}$ and $\hat{x}_2^{(3)}$ as concurrent sides. Also notice that the output axis orientation number can be expressed as

$$1 = \det \mathbb{L}$$

where $\mathbb{L}$ is the matrix defined by equation (6.4).

Equation (2. 2) gives

$$1 = \sin \theta^{(1)} \sin \theta^{(2)} \sin \theta^{(3)} + \cos \theta^{(1)} \cos \theta^{(2)} \cos \theta^{(3)} \qquad 3.5$$

The significance of s and l will be shown in paragraph 8. We get from equations (3. 3) and (3. 4)

$$0 \leq s \leq 6 \qquad 3.6$$

$$-1 \leq l \leq 1 \qquad 3.7$$

The cases of equality in equation (3. 6) are of special interest. Zero spin is obtained for

$$\sin \theta^{(1)} - \cos \theta^{(3)} = 0$$

$$\sin \theta^{(2)} - \cos \theta^{(1)} = 0$$

$$\sin \theta^{(3)} - \cos \theta^{(2)} = 0 \qquad 3.8$$

These equations have eight solutions

| $\theta^{(1)}$ | $\theta^{(2)}$ | $\theta^{(3)}$ | | |
|---|---|---|---|---|
| $-\dfrac{3\pi}{4}$ | $-\dfrac{3\pi}{4}$ | $-\dfrac{3\pi}{4}$ | $l = -\dfrac{\sqrt{2}}{2}$ | |
| $-\dfrac{\pi}{4}$ | $\dfrac{\pi}{4}$ | $\dfrac{3\pi}{4}$ | $l = -\dfrac{\sqrt{2}}{2}$ | cycl. |
| $\dfrac{\pi}{4}$ | $\dfrac{\pi}{4}$ | $\dfrac{\pi}{4}$ | $l = \dfrac{\sqrt{2}}{2}$ | |
| $\dfrac{3\pi}{4}$ | $-\dfrac{3\pi}{4}$ | $-\dfrac{\pi}{4}$ | $l = \dfrac{\sqrt{2}}{2}$ | cycl. $\qquad 3.9$ |

The corresponding configurations of the gyros are shown in plate 3. 1. It is obvious that the spin axes are in the same plane. The solution $\left[ \dfrac{\pi}{4}, \dfrac{\pi}{4}, \dfrac{\pi}{4} \right]$ e. g. has the following spin axes

$$\hat{x}_3^{(1)} = \frac{1}{\sqrt{2}} (0, -1, 1)$$

$$\hat{x}_3^{(2)} = \frac{1}{\sqrt{2}} (1, 0, -1)$$

$$\hat{x}_3^{(3)} = \frac{1}{\sqrt{2}} (-1, 1, 0) \qquad\qquad 3.10$$

The angle between the plane through the spin axes and each coordinate axis is $\text{arctg} \sqrt{2} = 54.73^{\circ}$.

The conditions for maximum spin, $s = 6$, are

$$\sin \theta^{(1)} + \cos \theta^{(3)} = 0$$

$$\sin \theta^{(2)} + \cos \theta^{(1)} = 0$$

$$\sin \theta^{(3)} + \cos \theta^{(2)} = 0 \qquad\qquad 3.11$$

The solutions of this system can be obtained from the solution (3.9) by changing the signs of all angles. Hence

| $\theta^{(1)}$ | $\theta^{(2)}$ | $\theta^{(3)}$ | | | |
|---|---|---|---|---|---|
| $\dfrac{3\pi}{4}$ | $\dfrac{3\pi}{4}$ | $\dfrac{3\pi}{4}$ | $1 = 0$ | | |
| $\dfrac{\pi}{4}$ | $-\dfrac{\pi}{4}$ | $-\dfrac{3\pi}{4}$ | $1 = 0$ | cycl. | |
| $-\dfrac{\pi}{4}$ | $-\dfrac{\pi}{4}$ | $-\dfrac{\pi}{4}$ | $1 = 0$ | | |
| $-\dfrac{3\pi}{4}$ | $\dfrac{3\pi}{4}$ | $\dfrac{\pi}{4}$ | $1 = 0$ | cycl. | 3.12 |

The corresponding configurations of the gyros are shown in plate 3.2. In this case the output axes are in the same plane.

As shown in section 8 systems with

$$\begin{cases} s = 3 \\ 1 = 0 \end{cases} \qquad\qquad 3.13$$

are of special interest.

Equations (3.3) and (3.5) give

$$\sin \theta^{(1)} \cos \theta^{(3)} + \sin \theta^{(2)} \cos \theta^{(1)} + \sin \theta^{(3)} \cos \theta^{(2)} = 0$$

$$\sin \theta^{(1)} \sin \theta^{(2)} \sin \theta^{(3)} + \cos \theta^{(1)} \cos \theta^{(2)} \cos \theta^{(3)} = 0 \qquad 3.14$$

These equations have infinitely many solutions. Given arbitrary $\theta^{(1)}$ we can find four pairs $\theta^{(2)}$, $\theta^{(3)}$, satisfying equation (3.14). The solutions can be obtained from plates 3.11 and 3.12. There are 24 solutions which are multiples of $\frac{1}{4}\pi$. These are cyclic permutations of the following basic solutions.

| $\theta^{(1)}$ | $\theta^{(2)}$ | $\theta^{(3)}$ |
|---|---|---|
| $-\frac{\pi}{2}$ | $0$ | $-\frac{3\pi}{4}$ |
| $-\frac{\pi}{2}$ | $0$ | $\frac{\pi}{4}$ |
| $-\frac{\pi}{2}$ | $\pi$ | $-\frac{\pi}{4}$ |
| $-\frac{\pi}{2}$ | $\pi$ | $\frac{3\pi}{4}$ |
| $\frac{\pi}{2}$ | $0$ | $-\frac{\pi}{4}$ |
| $\frac{\pi}{2}$ | $0$ | $\frac{3\pi}{4}$ |
| $\frac{\pi}{2}$ | $\pi$ | $-\frac{3\pi}{4}$ |
| $\frac{\pi}{2}$ | $\pi$ | $\frac{\pi}{4}$ |

$$3.15$$

The corresponding configurations of the gyros are shown in plates 3.3 and 3.4. Some special arrangements occur frequently. We adopt the following terminology. An arrangement is called <u>cyclic</u> if all angles $\theta^{(m)}$ are equal. An arrangement is called <u>orthogonal</u> if all angles $\theta^{(m)}$ are multiples at $\frac{1}{2}\pi$.

Equation 3.3 gives the following condition on the spin number of the orthogonal arrangements

$$1 \leq s \leq 5 \qquad 3.16$$

There are totally 64 orthogonal arrangements. These arrangements are obtained by cyclic permutation of 24 basic arrangements. The basic arrangements can be divided into three groups of eight arrangements each with spin 1, 3 and 5, respectively. The basic arrangements are

$s = 1$ , $1 = 0$

| $\theta^{(1)}$ | $\theta^{(2)}$ | $\theta^{(3)}$ |
|---|---|---|
| $0$ | $0$ | $\frac{\pi}{2}$ |
| $0$ | $\frac{\pi}{2}$ | $\frac{\pi}{2}$ |
| $0$ | $\frac{\pi}{2}$ | $\pi$ |
| $0$ | $\frac{\pi}{2}$ | $-\frac{\pi}{2}$ |
| $0$ | $\pi$ | $-\frac{\pi}{2}$ |
| $\frac{\pi}{2}$ | $\pi$ | $-\frac{\pi}{2}$ |
| $\pi$ | $\pi$ | $-\frac{\pi}{2}$ |
| $\pi$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{2}$ |

$$3.17$$

$s = 3$ , $1 = -1$

| $\theta^{(1)}$ | $\theta^{(2)}$ | $\theta^{(3)}$ |
|---|---|---|
| $\pi$ | $\pi$ | $\pi$ |
| $-\frac{\pi}{2}$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{2}$ |
| $0$ | $0$ | $\pi$ |
| $\frac{\pi}{2}$ | $\frac{\pi}{2}$ | $-\frac{\pi}{2}$ |

$$3.18$$

$s = 3$ , $1 = 1$

| $\theta^{(1)}$ | $\theta^{(2)}$ | $\theta^{(3)}$ |
|---|---|---|
| $0$ | $0$ | $0$ |
| $\frac{\pi}{2}$ | $\frac{\pi}{2}$ | $\frac{\pi}{2}$ |
| $0$ | $\pi$ | $\pi$ |
| $\frac{\pi}{2}$ | $-\frac{\pi}{2}$ | $-\frac{\pi}{2}$ |

$$3.19$$

$s = 5$ , $1 = 0$

| $\theta^{(1)}$ | $\theta^{(2)}$ | $\theta^{(3)}$ |
|---|---|---|
| $0$ | $0$ | $-\dfrac{\pi}{2}$ |
| $0$ | $\pi$ | $-\dfrac{\pi}{2}$ |
| $0$ | $-\dfrac{\pi}{2}$ | $\dfrac{\pi}{2}$ |
| $0$ | $-\dfrac{\pi}{2}$ | $\pi$ |
| $0$ | $-\dfrac{\pi}{2}$ | $-\dfrac{\pi}{2}$ |
| $\dfrac{\pi}{2}$ | $\dfrac{\pi}{2}$ | $\pi$ |
| $\dfrac{\pi}{2}$ | $\pi$ | $\pi$ |
| $\dfrac{\pi}{2}$ | $-\dfrac{\pi}{2}$ | $\pi$ |

3.20

Notice that the solution with spin 5 can be obtained from the solution with spin 1 by changing the signs of all angles.

The configurations represented by equations (3.17), (3.18), (3.19) and (3.20) are shown in plates 3.5 - 3.10.

## 4. The equation of motion of one gyro.

We will now derive the equation of motion of one of the gyroscopes.

The angular velocity of the stable element is

$$\bar{\Omega} = \Omega_s \, \hat{y}_s \qquad\qquad 4.1$$

The float of the m-gyro has the angular velocity $\bar{\omega}^{(m)}$.

$$\bar{\omega}^{(m)} = \omega_s^{(m)} \, \hat{z}_s^{(m)} \qquad\qquad 4.2$$

where

$$\omega_s^{(m)} = q_{st}^{(m)} \Omega_t + \dot{\phi}^{(m)} \delta_{s2} \qquad\qquad 4.3$$

The angular velocity of the gyroscopic element with respect to the float is $\omega_o$. Let the components of the inertia matrix of the float with respect to the $z^{(m)}$-set be $J A_{kl}$, where $J$ is the moment of inertia of the gyroscopic element with respect to the spin axis, and the quantities $A_{kl}$ defined by

$$J A_{kl} = \begin{cases} \displaystyle\int (x_i^2 + x_j^2) \, dm & k = l \,, \quad i \neq j \neq k \neq i \\[2em] -\displaystyle\int x_k \cdot x_l \, dm & k \neq l \end{cases}$$

The float is supposed to have its center of mass on the output axis.

Let the angular momentum of the float of the m-gyro with respect to its center be $H^{(m)}$.

$$\bar{H}^{(m)} = J A_{rs} \, \omega_s^{(m)} \, \hat{z}_r^{(m)} + J \omega_o \, \hat{z}_3^{(m)} \qquad\qquad 4.4$$

Differentiating with respect to time

$$\dot{\bar{H}}^{(m)} = J \left[ A_{ks} \, \dot{\omega}_s^{(m)} + A_{js} \, \omega_s^{(m)} \, \omega_i^{(m)} \, \epsilon_{ijk} + \omega_o \, \omega_i^{(m)} \, \epsilon_{i3k} \right] \hat{z}_k^{(m)}$$

$$4.5$$

The equations (4.3) and (4.5) give

$$\dot{\overline{H}}^{(m)} = J\left[ A_{k2}\ddot{\varphi}^{(m)} + \epsilon_{23k}\dot{\varphi}^{(m)}\omega_o + A_{ks}q_{st}^{(m)}\dot{\Omega}_t + q_{it}^{(m)}\epsilon_{i3k}\Omega_t\omega_o + \right.$$

$$+ A_{j2}\epsilon_{2jk}\dot{\varphi}^{(m)}\dot{\varphi}^{(m)} + A_{j2}q_{ir}^{(m)}\epsilon_{ijk}\Omega_r\dot{\varphi}^{(m)} + A_{js}q_{st}^{(m)}\epsilon_{2ik}\Omega_t\dot{\varphi}^{(m)} +$$

$$\left. + A_{ks}\dot{q}_{st}^{(m)}\Omega_t + A_{js}q_{st}^{(m)}q_{ir}^{(m)}\epsilon_{ijk}\Omega_t\Omega_r \right]\hat{z}_k^{(m)} \qquad 4.6$$

The component of $\dot{\overline{H}}^{(m)}$ along the output axis is

$$\dot{H}_2^{(m)} = J\left[ A_{22}\ddot{\varphi}^{(m)} + A_{2s}q_{st}^{(m)}\dot{\Omega}_t - q_{1t}^{(m)}\Omega_t\omega_o + A_{j2}q_{ir}^{(m)}\epsilon_{ij2}\Omega_r\dot{\varphi}^{(m)} + \right.$$

$$\left. + A_{2s}\dot{q}_{st}^{(m)}\Omega_t + A_{js}q_{st}^{(m)}q_{ir}^{(m)}\epsilon_{ij2}\Omega_t\Omega_r \right] \qquad 4.7$$

The torque acting on the float of the m-gyro has a component $\mathcal{M}_2^{(m)}$ along the output axis. This component is composed by viscous torque, torque from the torque generator, unbalance torque , etc. Supposing it is possible to control the torque generator of each gyro by signals from all gyros we get

$$\mathcal{M}_2^{(m)} = -JA_{22}\left[ \sigma_{m1}(D)\varphi^{(1)} + \sigma_{m2}(D)\varphi^{(2)} + \sigma_{m3}(D)\varphi^{(3)} \right] - Jm^{(m)}$$

$$4.8$$

The $\sigma_{ij}(D)$:s are differential operators. It is assumed that they are rational functions of $D = \dfrac{d}{dt}$ . Further $Jm^{(m)}$ is the component on the output axis of the disturbing torque acting on the float of the m-gyro. Newton's second Law of Motion gives

$$\dot{H}_2^{(m)} = \mathcal{M}_2^{(m)} \qquad m = 1, 2, 3 \qquad 4.9$$

Introduce the following matrix-notations

$$\overline{\varphi} = \begin{pmatrix} \varphi^{(1)} \\ \varphi^{(2)} \\ \varphi^{(3)} \end{pmatrix} \qquad 4.10$$

$$\widetilde{\Omega} = \begin{pmatrix} \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix} \qquad\qquad 4.11$$

$$\overline{m} = \begin{pmatrix} m^{(1)} \\ m^{(2)} \\ m^{(3)} \end{pmatrix} \qquad\qquad 4.12$$

$$\mathbb{S}(D) = \begin{pmatrix} D^2 + \sigma_{11}(D) & \sigma_{12}(D) & \sigma_{13}(D) \\ \sigma_{21}(D) & D^2 + \sigma_{22}(D) & \sigma_{23}(D) \\ \sigma_{31}(D) & \sigma_{32}(D) & D^2 + \sigma_{33}(D) \end{pmatrix} \qquad 4.13$$

$$\mathbb{Q}_a(\overline{\varphi}) = \begin{pmatrix} q_{a1}^{(1)} & q_{a2}^{(1)} & q_{a3}^{(1)} \\ q_{a1}^{(2)} & q_{a2}^{(2)} & q_{a3}^{(2)} \\ q_{a1}^{(3)} & q_{a2}^{(3)} & q_{a3}^{(3)} \end{pmatrix} \qquad 4.14$$

$$\mathbb{E}_a(\overline{\varphi}) = \frac{1}{A_{22}} A_{ai} \mathbb{Q}_i \qquad\qquad 4.15$$

$$\mathbb{T}_a(\overline{\varphi}, \overline{\Omega}) = \begin{pmatrix} q_{ai}^{(1)}\Omega_i & 0 & 0 \\ 0 & q_{ai}^{(2)}\Omega_i & 0 \\ 0 & 0 & q_{ai}^{(3)}\Omega_i \end{pmatrix} \qquad 4.16$$

$$\mathbb{W}(D, \overline{\varphi}) = \frac{\omega_o}{A_{22}} \mathbb{Q}_1 - \mathbb{E}_2 D - \frac{d\,\mathbb{E}_2}{dt} \qquad\qquad 4.17$$

The equation (4.9) can be written

$$\mathbb{S}(D)\,\overline{\varphi} = \mathbb{W}(D,\overline{\varphi})\overline{\Omega} + \frac{1}{A_{22}}\left(A_{32}\,\mathbb{T}'_1 - A_{12}\,\mathbb{T}'_3\right)D\overline{\varphi} +$$

$$+\left(\mathbb{T}'_1\,\mathbb{E}_3 - \mathbb{T}'_3\,\mathbb{E}_1\right)\overline{\Omega} - \frac{1}{A_{22}}\,\overline{m} \qquad\qquad 4.18$$

This equation is called the <u>signal equation</u> of the gyros. The matrix $\mathbb{S}(D)$ represents the dynamic properties of the gyros and the feedback from the signal generators to the torque generators of the gyros. The matrix $\mathbb{W}(D,\varphi)$ shows how the components of the angular velocity of the stable element is transferred to the output signals of the gyros. The second and third terms depend on crosscoupling between the output signals of the gyros and the angular velocity of the stable element, these terms are of the second order in $\varphi^{(m)}$ and $\Omega_j$, since $\mathbb{T}'_a$ is linear in $\Omega_j$. The last term in the signal equation depends on disturbing torque acting on the float of the gyro.

If the z-axes are principal axes of the float i. e.

$$\begin{cases} A_{11} = a_1 \\ A_{22} = a_2 = a \\ A_{33} = a_3 \\ A_{ij} = 0 \qquad\qquad i \neq j \end{cases} \qquad\qquad 4.19$$

equation (4.18) can be simplified to

$$\mathbb{S}(D)\,\overline{\varphi} = \mathbb{W}'(D,\overline{\varphi})\overline{\Omega} + \frac{a_3 - a_1}{a_2}\cdot\overline{q}(\overline{\varphi},\overline{\Omega}) - \frac{\overline{m}}{A_{22}} \qquad\qquad 4.20$$

where

$$\mathbb{W}'(D,\overline{\varphi}) = \frac{\omega_o}{a}\,\mathbb{Q}_1 - \mathbb{Q}_2\,D - \frac{d\mathbb{Q}_2}{dt} \qquad\qquad 4.21$$

and

$$\overline{q}(\overline{\varphi},\overline{\Omega}) = \mathbb{T}'_1\,\mathbb{Q}_3\,\overline{\Omega} = \mathbb{T}'_3\,\mathbb{Q}_1\,\overline{\Omega} = \begin{pmatrix} q_{1i}^{(1)}\,q_{3j}^{(1)}\Omega_i\,\Omega_j \\ q_{1i}^{(2)}\,q_{3j}^{(2)}\Omega_i\,\Omega_j \\ q_{1i}^{(3)}\,q_{3j}^{(3)}\Omega_i\,\Omega_j \end{pmatrix} \qquad\qquad 4.22$$

## 5.  The equation of motion of the stable element.

Let $J C_{kl}$ be the inertia matrix of the stable element with respect to the y-set. In $J C_{kl}$ are included three mass points, each equal to the mass of one float and situated in the center of the float. The elements $C_{kl}$ are defined by

$$
J C_{kl} = \begin{cases} \int (y_i^2 + y_j^2)\,dm & k = 1 \ , \ i \neq j \neq k \neq i \\[3mm] -\int y_k \cdot y_1 \, dm & k \neq 1 \end{cases}
$$

The integration is carried out over the stable element and the three mass-points. The angular momentum of the platform is

$$
\bar{H} = J C_{1k}\Omega_k \hat{y}_1 + \sum_{m=1}^{3} \bar{H}^{(m)} \tag{5.1}
$$

Differentiating with respect to time

$$
\dot{\bar{H}} = J (C_{1k}\dot{\Omega}_k + C_{jk}\Omega_i\Omega_k \,\epsilon_{ijl}) \hat{y}_1 + \sum_{m=1}^{3} \dot{\bar{H}}^{(m)} \tag{5.2}
$$

From equation (4.6) we get

$$
\begin{aligned}
\dot{\bar{H}} = J \Big[ & A_{k2}\, q_{kl}^{(m)}\, \ddot{\varphi}^{(m)} + \delta_{k1}\, q_{kl}^{(m)}\, \dot{\varphi}^{(m)}\,\omega_o + A_{ks}\, q_{st}^{(m)}\, q_{kl}^{(m)}\,\Omega_t + \\
& + q_{it}^{(m)}\, q_{kl}^{(m)}\, \epsilon_{i3k}\Omega_t\,\omega_o + A_{j2}\, q_{kl}^{(m)}\, \epsilon_{2jk}\,\dot{\varphi}^{(m)}\,\dot{\varphi}^{(m)} + \\
& + A_{j2}\, q_{ir}^{(m)}\, q_{kl}^{(m)}\, \epsilon_{ijk}\Omega_r\,\dot{\varphi}^{(m)} + A_{js}\, q_{st}^{(m)}\, q_{kl}^{(m)}\, \epsilon_{2ik}\Omega_t\,\dot{\varphi}^{(m)} + \\
& + A_{ks}\, \dot{q}_{st}^{(m)}\, q_{kl}^{(m)}\Omega_t + A_{js}\, q_{st}^{(m)}\, q_{ir}^{(m)}\, q_{kl}^{(m)}\, \epsilon_{ijk}\Omega_t\Omega_r \Big] \hat{y}_1
\end{aligned} \tag{5.3}
$$

Introduce the inertia matrix $J B_{kl}(\bar{\varphi})$ of the stable element including the floats with all moving parts fixed in their _actual_ positions.

The elements $B_{kl}$ are defined by

$$J B_{kl} (\overline{\varphi}) = \begin{cases} \displaystyle\int ( y_i^2 + y_j^2 ) \, dm & k = 1 \quad, \quad i \neq j \neq k \neq i \\[2em] -\displaystyle\int y_k \cdot y_l \, dm & k \neq l \end{cases}$$

The integration is carried out over the stable element and the floats, fixed in their actual positions. We get

$$B_{kl} (\overline{\varphi}) = C_{kl} + \sum_{m=1}^{3} A_{st} \, q_{sk}^{(m)} \, q_{tl}^{(m)} \qquad\qquad 5.4$$

Notice that the elements $B_{ij}(\overline{\varphi})$ depend on the output signals of the gyros. However, if the inertia ellipsoids of the gyros are symmetric with respect to the output axes the elements $B_{ij}$ are constants.

Equations (5.3) and (5.4) give

$$\dot{\overline{H}} = J (B_{1k}(\overline{\varphi})\dot{\Omega}_k + B_{jk}(\overline{\varphi})\Omega_i \Omega_k \epsilon_{ijl}) \, \hat{\overline{y}}_1 +$$

$$+ J \sum_{m=1}^{3} \left[ A_{k2} \, q_{kl}^{(m)} \, \ddot{\varphi}^{(m)} + q_{11}^{(m)} \, \dot{\varphi}^{(m)} \, \omega_o + \right.$$

$$+ q_{it}^{(m)} \, q_{kl}^{(m)} \, \epsilon_{i3k}\Omega_t \, \omega_o + A_{j2} \, q_{kl}^{(m)} \, \epsilon_{2jk} \, \dot{\varphi}^{(m)} \, \dot{\varphi}^{(m)} +$$

$$+ A_{j2} \, q_{ir}^{(m)} \, q_{kl}^{(m)} \, \epsilon_{ijk}\Omega_r \, \dot{\varphi}^{(m)} + A_{js} \, q_{st}^{(m)} q_{kl}^{(m)} \, \epsilon_{2ik}\Omega_t \, \dot{\varphi}^{(m)} +$$

$$+ A_{ks} \, \dot{q}_{st}^{(m)} \, q_{kl}^{(m)} \, \Omega_t \Big] \hat{\overline{y}}_1 \qquad\qquad 5.5$$

The torque acting on the platform is composed by components from the torque motors $J\overline{T}$ and disturbing torques $J\overline{M}$. Supposing $\overline{T}$ to be controlled by signals from all gyros , we get

$$T_i = - \sum_{j=1}^{3} \tau_{ij} (D) \, \varphi^{(j)} \qquad\qquad 5.6$$

It is assumed that the differential operators $\tau_{ij}(D)$ are rational functions of $D = \dfrac{d}{dt}$ . Further we denote

$$\mathbb{T}(D) = \left\{ \tau_{ij}(D) \right\} \qquad\qquad 5.7$$

Newtons second law of motion gives

$$B_{lm}(\overline{\phi})\dot{\Omega}_m + B_{jk}(\overline{\phi})\,\epsilon_{ijl}\Omega_i\Omega_k + \sum_{m=1}^{3}\left[ A_{k2}\,q_{kl}^{(m)}\,\ddot{\phi}^{(m)} + q_{11}^{(m)}\,\dot{\phi}^{(m)}\,\omega_o + \right.$$

$$+\, \tau_{lm}(D)\,\phi^{(m)} + q_{it}^{(m)}\,q_{kl}^{(m)}\,\epsilon_{i3k}\Omega_t\,\omega_o + A_{j2}\,q_{kl}^{(m)}\,\epsilon_{2jk}\,\dot{\phi}^{(m)}\,\dot{\phi}^{(m)} +$$

$$+\, A_{j2}\,q_{ir}^{(m)}\,q_{kl}^{(m)}\,\epsilon_{ijk}\Omega_r\,\dot{\phi}^{(m)} + A_{js}\,\dot{q}_{st}^{(m)}\,q_{kl}^{(m)}\,\epsilon_{2ik}\Omega_t\,\dot{\phi}^{(m)} +$$

$$\left. +\, A_{ks}\,\dot{q}_{st}^{(m)}\,q_{kl}^{(m)}\,\Omega_t \right] = M_l \qquad\qquad 5.8$$

## 6.    The linear approximation of the equations of motion.

Taking only the terms of equation (4.18) which are linear in $\overline{\varphi}$ and we get

$$\mathbb{S}(D)\,\overline{\varphi} = \mathbb{W}(D)\,\overline{\Omega} - \frac{1}{A_{22}}\,\overline{m} \qquad\qquad 6.1$$

where

$$\mathbb{W}(D) = \mathbb{U}(D,0) \qquad\qquad 6.2$$

Introduce the unit matrix $\mathbb{I}$ and the matrices $\mathbb{L}$ and $\mathbb{N}$ defined by

$$\mathbb{L} = \begin{pmatrix} 0 & \cos\theta^{(1)} & \sin\theta^{(1)} \\ \sin\theta^{(2)} & 0 & \cos\theta^{(2)} \\ \cos\theta^{(3)} & \sin\theta^{(3)} & 0 \end{pmatrix} \qquad\qquad 6.3$$

$$\mathbb{N} = \begin{pmatrix} 0 & -\sin\theta^{(1)} & \cos\theta^{(1)} \\ \cos\theta^{(2)} & 0 & -\sin\theta^{(2)} \\ -\sin\theta^{(3)} & \cos\theta^{(3)} & 0 \end{pmatrix} \qquad\qquad 6.4$$

and their transponates $\widetilde{\mathbb{L}}$ and $\widetilde{\mathbb{N}}$ we get from equation (4.18)

$$\mathbb{W}(D) = \left(\frac{\omega_o}{A_{22}} - A_{21}\,D\right)\mathbb{I} - D\,\mathbb{L} - \frac{A_{23}}{A_{22}}\,D\,\mathbb{N} \qquad\qquad 6.5$$

Equation (6.1) gives

$$\overline{\varphi} = \mathbb{S}^{-1}(D)\cdot\mathbb{W}(D)\,\overline{\Omega} - \mathbb{S}^{-1}(D)\cdot\frac{1}{A_{22}}\,\overline{m} \qquad\qquad 6.6$$

Taking only the linear terms of equation (5.8) we get

$$B_{1m}\dot{\Omega}_m + \omega_o \sum_{m=1}^{3}\left[P_{2t}^{(m)}\cdot P_{11}^{(m)} - P_{1t}^{(m)}\cdot P_{21}^{(m)}\right]\Omega_t +$$

$$+ \sum_{m=1}^{3}\left[A_{k2}\,P_{kl}^{(m)}\,\ddot{\varphi}^{(m)} + P_{11}^{(m)}\,\dot{\varphi}^{(m)}\,\omega_o + \tau_{1m}\,\varphi^{(m)}\right] = M_1$$

$$6.7$$

where

$$B_{ij} = B_{ij}(0) \qquad \qquad 6.8$$

i.e. $JB_{ij}$ is the inertia tensor of the stable element including all moving parts fixed in their <u>null positions</u>. Further denote

$$\mathbb{B} = \left\{ B_{ij} \right\} \qquad \qquad 6.9$$

Equation (6.7) then gives

$$\mathbb{F}(D) \, \overline{\Omega} = \overline{M} - \mathbb{G}(D) \, \overline{\varphi} \qquad \qquad 6.10$$

where

$$\mathbb{F}(D) = D \, \mathbb{B} + \omega_o \, (\mathbb{L} - \widetilde{\mathbb{L}}) \qquad \qquad 6.11$$

$$\mathbb{G}(D) = \mathbb{T}(D) + (A_{12} \, D^2 + \omega_o D) \, \mathbb{I} + A_{22} \, D^2 \, \widetilde{\mathbb{L}} - A_{32} \, D^2 \, \mathbb{N} \qquad \qquad 6.12$$

Equation (6.10) is referred to as the equation of motion of the stable element. The term $J\mathbb{F}(D)\,\overline{\Omega}$ is the time derivative of the angular momentum of the stable element, $J\overline{M}$ is the disturbing torque and $J\mathbb{G}(D)\,\overline{\varphi}$ is the control-torque, i.e. the torque given by the torque-motors which are controlled by the gyros. Elimination of $\overline{\varphi}(t)$ between the signal equation and the equation of motion of the stable element gives

$$\mathbb{K}(D) \, \overline{\Omega} = \overline{M} + \frac{1}{A_{22}} \, \mathbb{G}(D) \, \mathbb{S}^{-1}(D) \, \overline{m} \qquad ℝ \qquad 6.13$$

where

$$\mathbb{K}(D) = \mathbb{F}(D) + \mathbb{G}(D) \, \mathbb{S}^{-1}(D) \, \mathbb{W}(D) \qquad \qquad 6.14$$

The signal equation, the equation of motion of the stable element and equation (6.13) are linear with constant coefficients. Assuming all initial conditions to be zero and Laplace-transforming these equations, we get

$$\mathbb{S}(p) \, \overline{\varphi}(p) = \mathbb{W}(p) \, \overline{\Omega}(p) - \frac{1}{A_{22}} \, \overline{m}(p) \qquad \qquad 6.15$$

$$\mathbb{F}(p)\,\widetilde{\Omega}(p) \;=\; \overline{M}(p) \,\cdot\, \mathbb{G}(p)\,\overline{\phi}(p) \qquad\qquad 6.16$$

$$\mathbb{K}(p)\,\widetilde{\Omega}(p) \;=\; \overline{M}(p) \,+\, \frac{1}{A_{22}}\;\mathbb{G}(p)\,\mathbb{S}^{-1}(p)\,\overline{\phi}(p) \qquad 6.17$$

The transformed variables are denoted by writing p for the argument. In the following we always write the arguments p or t in order to avoid ambiguity.

6.2   For a stable element with only one gyro the equations (6.15), (6.16) and (6.17) are still valid if the matrices are interpreted in the following way.

$$\mathbb{G}'(p) \;=\; \begin{pmatrix} A_{12}\,p^2 + \omega_o + \tau_{11} & 0 & 0 \\[4pt] (A_{22}\cos\theta^{(1)} - A_{32}\sin\theta^{(1)})\,p^2 + \tau_{21} & 0 & 0 \\[4pt] (A_{22}\sin\theta^{(1)} + A_{32}\cos\theta^{(1)})\,p^2 + \tau_{31} & 0 & 0 \end{pmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 6.21$$

$$\mathbb{S}'(p) \;=\; \begin{pmatrix} p^2 + \sigma(p) & 0 & 0 \\[4pt] 0 & 0 & 0 \\[4pt] 0 & 0 & 0 \end{pmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 6.22$$

$$\mathbb{W}'(p) \;=\; \frac{1}{A_{22}} \begin{pmatrix} \omega_o - A_{21}\,p & -(A_{22}\cos\theta^{(1)} - A_{23}\sin\theta^{(1)})p & -(A_{22}\sin\theta^{(1)} + A_{23}\cos\theta^{(1)})p \\[4pt] 0 & 0 & 0 \\[4pt] 0 & 0 & 0 \end{pmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 6.23$$

$$\mathbb{F}'(p) \;=\; \begin{pmatrix} B_{11}\,p & B_{12}\,p + \omega_o\cos\theta^{(1)} & B_{13}\,p + \omega_o\sin\theta^{(1)} \\[4pt] B_{21}\,p - \omega_o\cos\theta^{(1)} & B_{22}\,p & B_{23}\,p \\[4pt] B_{31}\,p - \omega_o\sin\theta^{(1)} & B_{32}\,p & B_{33}\,p \end{pmatrix}$$

$$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 6.24$$

It is here assumed that the input axis of the gyro coincides with the $y_1$-axis of the stable element.

If the motion of the stable element is restricted to rotations around the $y_1$-axis we obtain

$$\Omega_1(p) = \frac{1}{B_{11}\,p(1+Y_o)}\ M_1(p) + \frac{A_{22}}{\omega_o - A_{21}\,p} \cdot \frac{Y_o}{1+Y_o}\ m_1(p) \qquad 6.25$$

where

$$Y_o(p) = \frac{(A_{12}\,p^2 + \omega_o\,p + \tau_{11})(\omega_o - A_{21}\,p)}{A_{22}\,B_{11}\,p\,(p^2 + \sigma_{11})} \qquad 6.26$$

If the output axis coincides with one of the principal axes of the float we get

$$\Omega_1(p) = \frac{1}{B_{11}\,p(1+Y_o)}\ M_1(p) + \frac{a}{\omega_o}\ \frac{Y_o}{1+Y_o}\ m(p) \qquad 6.27$$

where

$$Y_o(p) = \frac{\omega_o}{a\,B_{11}}\ \frac{\tau_{11} + \omega_o\,p}{p\,(p^2 + \sigma_{11})} \qquad 6.28$$

7.   The concept of platform.  Classification of platform systems.

In the previous sections we obtained the following equations

$$\mathbb{S}(D)\,\overline{\varphi}(t) = \mathbb{W}(D)\,\overline{\Omega}(t) - \frac{1}{A_{22}}\,\overline{m}(t) \tag{6.1}$$

$$\mathbb{F}(D)\,\overline{\Omega}(t) = \overline{M}(t) - \mathbb{G}(D)\,\overline{\varphi}(t) \tag{6.10}$$

These equations were referred to as the signal equation and the equation of motion of the stable element, respectively.  Eliminating $\overline{\varphi}(t)$ between these equations we obtain

$$\mathbb{K}(D)\,\overline{\Omega}(t) = \overline{M}(t) + \frac{1}{A_{22}}\,\mathbb{G}(D)\,\mathbb{S}^{-1}(D)\,\overline{m}\,t \tag{6.13}$$

This equation gives the motion of the "stable element when the servoloop is closed".  We shall now give a physical interpretation of this equation. For the sake of convenience the concept platform is introduced.

The platform is an object to which is attributed: structure, attitude, mass, center of mass, center of gravity, angular velocity and angular momentum, defined in the following way.

The structure, attitude, mass, center of mass, center of gravity and angular velocity of the platform are equal to the corresponding properties of the stable element.

If the angular momentum of the platform with respect to a point P is $J\,\overline{H}_p$, where J is the moment of inertia of the gyrorotor with respect to the spin axis, then

$$\frac{d}{dt}\,\overline{H}_p = \mathbb{K}_p(D)\,\overline{\Omega}(t)$$

where $\overline{\Omega}(t)$ is the angular momentum of the platform and

$$\mathbb{K}_p(D) = \mathbb{F}_p(D) + \mathbb{G}(D)\,\mathbb{S}^{-1}(D)\,\mathbb{W}(D)$$

$$\mathbb{F}_p(D) = D\,\mathbb{B}_p + \omega_o\,(\mathbb{L} - \widetilde{\mathbb{L}})$$

Here $J\mathbb{B}_p$ is the matrix of inertia of the stable element (including all moving parts fixed in their null positions) with respect to the point $P$. The matrices $\mathbb{G}(D)$, $\mathbb{L}$, $\mathbb{S}(D)$ and $\mathbb{W}(D)$ are defined in section 6.

From mechanical point of view the concept platform is thus equivalent to "the stable element with the servoloop closed".

In order to guide those of the readers who favour thinking in physical concepts we will now give a physical interpretation of the terms of the $\mathbb{K}(D)$-matrix.

Non-diagonal elements of $\mathbb{K}(D)$ means crosscoupling.

A term $D\mathbb{C}$ of $\mathbb{K}(D)$, where $\mathbb{C}$ is a constant diagonal matrix, means moment of inertia of the platform.

A term $\mathbb{C}$ of $\mathbb{K}(D)$, means a velocity-proportional damping of the oscillations of the platform with respect to inertial space.

A term $\frac{1}{D}$ of $\mathbb{K}(D)$ means that the platform with respect to angular displacements is spring-restrained to inertial space.

The platform systems can be classified according to the properties of the reference they establish.

A platform system arranged in such a way that the platform will maintain its attitude with respect to inertial space is called an <u>inertial platform system</u>.

In addition to the inertial platform systems the <u>vertical indicating</u> or <u>Schuler-tuned</u> platform systems are of great importance in navigational and fire-control equipments.

As the dynamic properties of the platform is _are_ closely related to the matrix $\mathbb{K}(D)$ the platform systems can also be classified with respect to the properties of the matrix $\mathbb{K}(D)$.

1.  A platform system is said to be <u>cyclic</u> if the matrix $\mathbb{K}(D)$ is cyclic.

2.  A platform system is said to be <u>diagonal</u> if $\mathbb{K}(D)$ is diagonal.

3.  A platform system is said to be <u>isotropic</u> if $\mathbb{K}(D)$ is diagonal with equal elements.

8.  Analysis of the stability of inertial stabilized platform systems.

In this section we will analyse the stability of some inertial stabilized platform systems. In part 8. 1 the concept of stability is defined. Some general theorems concerning the stability of platform systems are also given. In parts 8. 2, 8. 3 and 8. 4 are discussed the stability of some special platform systems. The section ends with a short discussion of some questions of interest for the practical applications.

The equation of motion of the platform (equation 6. 17)

$$\overline{\Omega}(p) = \mathbb{K}^{-1}(p)\,\overline{M}(p) + \frac{1}{A_{22}}\,\mathbb{G}(p)\,\mathbb{S}^{-1}(p)\,\mathbb{W}(p)\,\overline{m}(p)$$

The platform is thus disturbed by $\overline{M}(t)$ and $\overline{m}(t)$, who are referred to as distrubing torque acting on the stable element and on the floats of the gyros, respectively. We adopt the following definitions

Definition 8. 11

A platform system is said to be stable if a proper torque puls acting on the stable element or on the float of a gyro gives a finite angular displacement of the stable element.

Definition 8. 12

A platform system is said to be strictly stable if a proper torque puls acting on the stable element gives a displacement error which tends to zero and a proper torque puls acting on the float of a gyro gives a finite angular displacement of the stable element.

By a proper torque puls we mean a disturbing torque, with so small a magnitude that the servos are not saturated, acting for a short time.

We will now analyse the stability of some inertial stabilized platform systems. We have

Definition 8. 13

A stable platform system is inertial stabilized or stabilized with respect to inertial space if a constant torque acting on the stable element gives a finite angular displacement of the stable element.

The definition 8.13 gives the following condition on the $\mathbb{K}(p)$-matrix of an inertial stabilized platform system.

Lemma 8.11

For an inertial stabilized platform system the matrix $\mathbb{K}(p)$ has the property

$$\mathbb{K}(p) = \frac{1}{p^n} \mathbb{C} + \underset{p \to 0}{O} \left( \frac{1}{p^{n-1}} \right)$$

where $n \geq 1$ is an integer and $\mathbb{C}$ is a diagonal matrix with constant non-vanishing diagonal elements.

In physical terms the lemma 8.11 means that an inertial stabilized platform is at least spring-restrained to inertial space with respect to angular displacements. Compare section 7. Before continuing we introduce some notations.

Definition 8.14

An equation is said to be stable if it has no roots in the open right half plane. The equation is said to be strictly stable if it has no roots in the closed right half plane. The function $f(z)$ is said to be (strictly) stable if the equation $f(z) = 0$ is (strictly) stable.

We will now give two conditions for stability.

Theorem 8.11

A necessary and sufficient condition that an inertial platform system should be stable is that the equations

$$\det \left\{ p \, \mathbb{K}(p) \right\} = 0 \tag{8.101}$$

and

$$\det \left\{ p \, \mathbb{S}(p) \, \mathbb{G}^{-1}(p) \, \mathbb{K}(p) \right\} = 0 \tag{8.102}$$

are stable.

## Theorem 8. 12

A necessary and sufficient condition that an inertial platform should be strictly stable is that the equation

$$\det \left\{ p \; \mathbb{K}(p) \right\} = 0$$

is strictly stable and that the equation

$$\det \left\{ p \; \mathbb{S}(p) \; \mathbb{G}^{-1}(p) \; \mathbb{K}(p) \right\} = 0$$

is stable.

The proof is left for the reader.

The equation (8. 101) is referred to as the characteristic equation of the system. The roots of the characteristic equation determines the way the displacement error fades out after a torque puls disturbance on the stable element. If all the roots of the characteristic equation are in the left half plane the displacement error is exponentially damped. If the characteristic equation has pure imaginary roots the displacement error will oscillate with constant amplitude. A single root at the origin but no other roots in the closed right half plane means that the displacement error tends to a constant etc. Because of lemma 8. 11 the characteristic equation has no root at the origin. Similarly the roots of the equation (8. 102) determines the way the displacement error after a torque puls on one of the gyro-floats fades out. Instability of the equation (8. 102) means that a torque puls acting on one of the gyro-floats will give an exponentially increasing angular displacement of the stable element.

Although a system is strictly stable according to the above definitions the displacement error obtained after a torque puls disturbance on the stable element may not tend to zero fast enough. Therefore in an actual application there may be further restrictions on the characteristic equation of the system. Compare appendix section A 1.

Although it is possible to claim that the characteristic equation is strictly stable we cannot claim strict stability of the equation (8. 102). This is obvious from the following lemma.

Lemma 8.12

For an inertial stabilized platform system equation (8.102) has always one single root $p = 0$.

The proof is obvious from the equation (6.14) and the definition of an inertial stabilized platform system. The lemma implies that the displacement error obtained after a proper torque puls disturbance on the float of a gyro tends to a constant. Some other consequences of the lemma are discussed in section 9.

We will now discuss some consequences of the theorem 8.11. We have the following sufficient condition for stability.

Corollary 8.11

An inertial stabilized platform system is (strictly) stable if

(i)     The arrangement of the gyros is chosen in such a way that $s \geq 3$ and $1 = 0$.

(ii)    The characteristic equation of the system, $\det \left\{ p\, \mathbb{K}(p) \right\} = 0$ is (strictly) stable.

(iii)   The function $\det \left\{ \mathbb{K}(p) - \mathbb{F}(p) \right\}$ has no poles in the right half-plane.

Proof

Equation (6.14) gives

$$\mathbb{S}(p)\, \mathbb{G}^{-1}(p) = \mathbb{W}(p)\, (\mathbb{K}(p) - \mathbb{F}(p))^{-1}$$

According to (iii) the function $\det (\mathbb{K}(p) - \mathbb{F}(p))^{-1}$ has no zeros in the right half plane. Further is

$$\det \mathbb{W}(p) = \left( \frac{\omega_o}{a} \right)^3 + \frac{\omega_o (s - 3)}{2\,a}\, p^2 - 1 p^3$$

i.e. the function $\det \mathbb{W}(p)$ has no zeros in the right half plane.

Hence the function

$$\det \left\{ p\ \mathbb{S}(p)\ \mathbb{G}^{-1}(p)\ \mathbb{K}(p) \right\} = \det \left\{ p\ \mathbb{K}(p) \right\} \cdot \det\ \mathbb{W}(p) \cdot \det \left[ \mathbb{K}(p) - \mathbb{F}(p) \right]^{-1}$$

is stable. The system is then stable according to theorem 8.11. If $\det \left\{ p\ \mathbb{K}(p) \right\}$ is strictly stable the system is strictly stable according to theorem 8.12.

An inertial stabilized platform system where the arrangement of the gyros is of the definite stable type is thus certainly stable if the characteristic equation of the system is stable and the elements of the $\mathbb{K}(p)$ matrix have no poles in the right half plane.

If the condition (i) is dropped the function $\det\ \mathbb{W}(p)$ is unstable. The system then must be heavily restricted in order to assure stability. This is illustrated by the following lemma.

Lemma 8.13

For a stable platform system the functions $\det\ \mathbb{W}(p)$ and $\det \left\{ \mathbb{K}(p) - \mathbb{F}(p) \right\}$ have the same zeros in the right half plane.

The proof is left for the reader.

This lemma means that if the function

$$\det\ \mathbb{W}(p) = \left( \frac{\omega_o}{a} \right)^3 + \frac{\omega_o(s-3)}{2a}\ p^2 - 1 p^3$$

is not stable i.e. $1 \neq 0$ or $1 = 0$ and $s < 3$ the matrices $\mathbb{K}(p)$ and $\mathbb{F}(p)$ must be chosen in a very special way if a torque puls acting on one of the gyro floats should not give an exponentially increasing angular displacement of the stable element.

We will now further analyse the stability conditions for some special inertial platform systems. Suppose that

$$B_{ij} = b\ \delta_{ij} \qquad\qquad\qquad 8.103$$

$$A_{21} = A_{23} = 0\ ;\quad A_{22} = a \qquad\qquad 8.104$$

$$\tau_{ij}(p) = \tau(p)\,\delta_{ij} \qquad\qquad 8.105$$

$$\sigma'_{ij}(p) = \sigma'(p)\,\delta_{ij} \qquad\qquad 8.106$$

The assumptions (8.103) - (8.106) are equivalent to the following physical conditions.

1. The inertia ellipsoid of the stable element is a sphere.

2. The inertia ellipsoids of the gyro floats are symmetric with respect to the output axes.

3. The torque generator of each gyro is only controlled by signals from the gyro itself. The feedback characteristics are the same for all gyros.

4. The m-component of the torque acting on the stable element is only controlled by the m-gyro. The same feedback characteristic is used in all channels.

Using the assumptions (8.103) - (8.106) we get from the equations (4.13), (6.11), (6.12) and (6.6)

$$\mathbb{S}^{-1}(p) = \left[p^2 + \sigma'(p)\right]^{-1}\mathbb{I} \qquad\qquad 8.107$$

$$\mathbb{F}(p) = bp\,\mathbb{I} + \omega_o\,(\mathbb{L} - \widetilde{\mathbb{L}}) \qquad\qquad 8.108$$

$$\mathbb{G}(p) = \tau(p) + \omega_o p\,\mathbb{I} + ap^2\,\widetilde{\mathbb{L}} \qquad\qquad 8.109$$

$$\mathbb{W}(p) = \frac{\omega_o}{a}\,\mathbb{I} - p\,\mathbb{L} \qquad\qquad 8.110$$

Equation (6.14) gives

$$\mathbb{K}(p) = \left[bp + \frac{\omega_o}{a}\,\frac{\tau(p) + \omega_o p}{p^2 + \sigma'(p)}\right]\mathbb{I} + \left[\omega_o - \frac{p(\tau(p) + \omega_o p)}{p^2 + \sigma'(p)}\right]\mathbb{L} -$$

$$- \frac{\omega_o\,\sigma'(p)}{p^2 + \sigma'(p)}\,\widetilde{\mathbb{L}} - \frac{ap^3}{p^2 + \sigma'(p)}\,\widetilde{\mathbb{L}}\,\mathbb{L} \qquad\qquad 8.111$$

Introducing

$$Y_1(p) = p + \frac{\omega_o}{ab} \frac{\tau(p) + \omega_o p}{p^2 + \sigma'(p)} = p\left[1 + Y_o\right] \qquad 8.112$$

we get

$$\mathbb{K}(p) = b \left\{ Y_1(p) \, \mathbb{I} - \frac{ap}{\omega_o} \left[ Y_1(p) - p \right] \mathbb{L} + \frac{\omega_o}{b} \mathbb{L} - \right.$$

$$\left. - \frac{\omega_o \sigma'(p)}{b(p^2 + \sigma'(p))} \, \widetilde{\mathbb{L}} - \frac{a p^3}{b(p^2 + \sigma'(p))} \, \widetilde{\mathbb{L}} \, \mathbb{L} \right\} \qquad 8.113$$

The fact that $\mathbb{K}$ is not a diagonal matrix means that we have interaction between the three channels. The interaction is referred to a crosscoupling. Putting (8.107) and (8.110) into equation (6.8) we get

$$\overline{\varphi}(p) = \frac{1}{p^2 + \sigma'(p)} \left( \frac{\omega_o}{a} \mathbb{I} - p \, \mathbb{L} \right) \overline{\Omega}(p) + \frac{1}{a(p^2 + \sigma'(p))} \, \overline{m}(p)$$

A gyro thus senses the component of the angular velocity on the output axis as well as the component on the input axis. This phenomena is referred to as the output axis sensitivity of a gyro.

In equation (8.113) the second term is due to the output axis sensitivity of the gyros. The third and fourth terms are due to secondary reaction torques of the gyros, i.e. the components of the reaction torque on the output and spin reference axes. The last term is due to a combination of output axis sensitivity and gyro reaction torques.

These cross coupling phenomenas strongly affects the dynamic properties of the system. If a platform system is designed on a single axis basis and the cross coupling effects are neglected, the dynamic properties of the complete system can differ widely from those predicted by neglecting the crosscoupling effects. In some cases the complete system can even be unstable.

8.2 Make the assumption that the moment of inertia of the stable element is much greater than that of the gyro rotor, i.e.

$$b \gg 1 \qquad\qquad 8.201$$

Equation (8.113) then gives

$$\mathbb{K}(p) = b \left\{ Y_1(p) \, \mathbb{I} - \frac{ap}{\omega_o} \left[ Y_1(p) - p \right] \mathbb{L} \right\} \qquad\qquad 8.202$$

where the function $Y_1(p)$ is given by the equation (8.112).

Equation (8.202) means that the secondary reaction torques of the gyros are neglected. The cross-coupling is thus entirely caused by the output axis sensitivity of the gyros. Compare the discussion at the end of section 8.1.

Equations (8.107), (8.201) and (8.202) gives

$$\mathbb{K}(p) - \mathbb{F}(p) = \frac{ab}{\omega_o} \left[ Y_1(p) - p \right] \mathbb{W}(p) \qquad\qquad 8.203$$

hence

$$p \, \mathbb{S}(p) \, \mathbb{G}^{-1}(p) \, \mathbb{K}(p) = \frac{\omega_o \, p}{ab \left[ Y_1(p) - p \right]} \, \mathbb{K}(p)$$

Suppose that the function $Y_1(p)$ has no poles in the right half plane. The stability of the function $\det \left\{ p \, \mathbb{K}(p) \right\}$ then implies stability of the function $\det \left\{ p \, \mathbb{S}(p) \, \mathbb{G}^{-1}(p) \, \mathbb{K}(p) \right\}$. <u>For the system discussed it is thus sufficient to analyse the stability of the characteristic equation.</u>

$$\det \left\{ p \, \mathbb{K}(p) \right\} = 0$$

Compare theorem 8.11.

The characteristic equation of the system can be reduced to

$$p \, Y_1(p) - t_i \, \frac{ap^2}{\omega_o} \left[ Y_1(p) - p \right] = 0 \qquad i = 1, 2, 3 \qquad 8.204$$

where the $t_i$:s are the roots of the equation

$$t^3 + \frac{s - 3}{2} t - 1 = 0 \qquad\qquad 8.205$$

s is the spin number and l is the output axis orientation number introduced in paragraph 3.

The roots of the equation (8.205) for integral values of the parameters s and l are given in table 8.21.

Table 8.21

| s \ l | - 1 | 0 | + 1 |
|---|---|---|---|
| 0 | - 1.477<br>+ 0.738 ± i 0.361 | 0<br>± 1.225 | 1.477<br>- 0.738 ± i 0.361 |
| 1 | - 1.326<br>0.662 ± i 0.563 | 0<br>± 1.000 | 1.326<br>- 0.662 ± i 0.563 |
| 2 | - 1.165<br>0.583 ± i 0.720 | 0<br>± 0.707 | 1.165<br>- 0.583 ± i 0.720 |
| 3 | - 1<br>0.500 ± i 0.865 | 0<br>± 0 | 1<br>- 0.500 ± i 0.865 |
| 4 | - 0.836<br>0.418 ± i 0.739 | 0<br>± i 0.707 | 0.836<br>- 0.418 ± i 0.739 |
| 5 | - 0.682<br>0.233 ± i 0.961 | 0<br>± i | 0.682<br>- 0.233 ± i 0.961 |
| 6 | - 0.554<br>0.277 ± i 1.315 | 0<br>± i 1.225 | 0.554<br>- 0.277 ± i 1.315 |

We have the following condition on the stability of the characteristic equation of the system discussed.

### Lemma 8.21

A sufficient condition that the characteristic equation (8.204) should be strictly stable for any $Y_1(p)$ with no zeros in the closed right half plane is

$$\begin{cases} s = 3 \\ 1 = 0 \end{cases} \qquad\qquad 8.206$$

### Proof

If the condition (8.206) is satisfied we obtain

$$t_i = 0 \qquad\qquad i = 1, 2, 3$$

The characteristic equation is then

$$p\,Y_1(p) = 0$$

The function $Y_1(p)$ has no zeros in the right half plane. As the system is inertial stabilized we have

$$\lim_{p \to 0} p\,Y_1(p) \ne 0$$

which implies that the characteristic equation is strictly stable.

The system discussed is thus certainly strictly stable for any $Y_1(p)$ with no poles or zeros in the closed right half plane, if the arrangement of the gyros is of the definite stable configuration. Compare section 3 and plate

Some questions now arises. Is it possible to obtain a stable system if the arrangement of the gyros is not of the definite stable configuration? Although a system with $s = 3$ and $1 = 0$ is strictly stable, is it sufficiently damped to be of practical use?

Before answering these questions we will further discuss the properties of the actual $Y_1(p)$-functions.

From physical point of view the function $J b Y_1(p)$ is the transfer function
from the component of stable element angular velocity on one input axis
to the component of the disturbing torque on the same axis. According to
lemma 8.11 the $Y_1(p)$-functions must also have a pole at the origin of the
order $n \geq 1$. In order to assure stability of the equation (8.102) we must
also require that the $Y_1(p)$-functions have no poles or zeros in the open
right half plane. If the error obtained after a torque puls disturbance on
the stable element should be sufficiently damped, we must require that
the zeros of $Y_1(p)$ have a sufficient distance from the imaginary axis.
This question is discussed in the appendix.

Equations (A 23), (A 309) and (A 407) give the following possible transfer
functions

$$Y_1(p) = \frac{p^2 + 2\zeta \beta p + \beta^2}{p}$$

$$Y_1(p) = \frac{(p^2 + 2\zeta \beta p + \beta^2)(p + p_1)}{p(p + p_2)}$$

$$Y_1(p) = \frac{(p^2 + 2\zeta \beta p + \beta^2)(p + p_1)(p + p_2)}{p^2(p + p_3)}$$

We will now discuss the properties of the system obtained if the first of
these functions is chosen. The characteristic equation is then

$$p^2 + 2\zeta \beta p + \beta^2 - t_i \cdot \gamma \left[ 2\zeta p^2 + \beta p \right] = 0 \quad i = 1, 2, 3 \qquad 8.208$$

The $t_i$:s satisfy equation (8.205) and $\gamma$ is the cross coupling coefficient
defined by

$$\gamma = \frac{a\beta}{\omega_o} \qquad\qquad 8.209$$

Numerical values of the cross coupling coefficient can be obtained from
table 8.22 and figure 8.21. Notice that the cross coupling coefficient
increases with the bandwidth of the servo system.

Table 8.22

| Gyro | $\dfrac{a}{\omega_o}$ |
|---|---|
| MIT $10^4$ Integrating gyro unit | 0.0034 |
| HIG 6 GG 12 C-2 | 0.0012 |
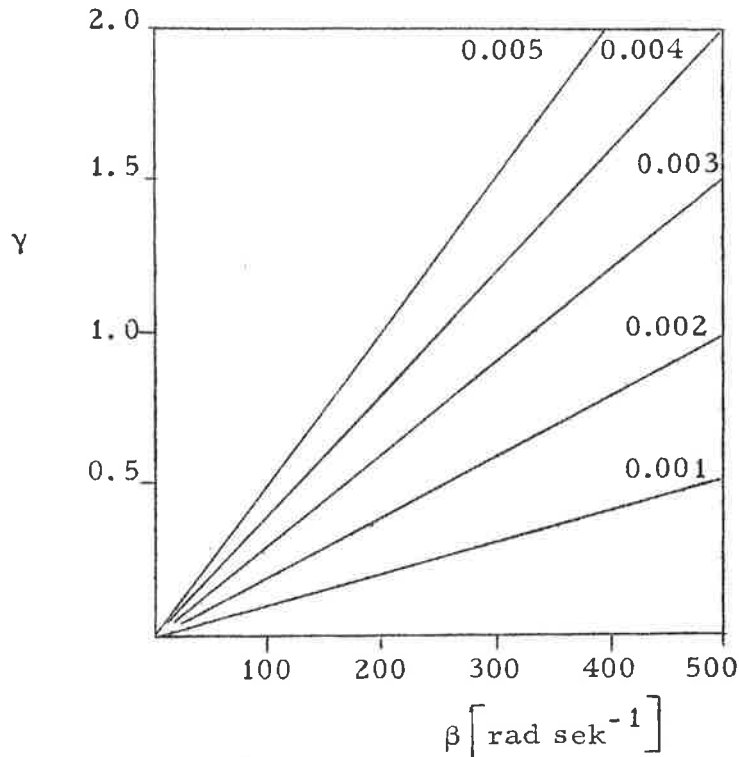


Figure 8.21

Coupling coefficient $\gamma$ as function of $\beta$ and $\dfrac{a}{\omega_o}$

Combining two of the equations (8.208) obtained with complex conjugated $t_i$-values we obtain an equation of the fourth degree with real coefficients. Applying the theorem of Hurwitz on this we obtain the following condition for stability

$$
\begin{cases}
4\zeta^2 - 4\zeta\,\gamma(1+2\zeta^2)\mathrm{Re}\{t_i\} + \gamma^2(1+8\zeta^2)(\mathrm{Re}\{t_i\})^2 - \gamma^3\,2\zeta\,|t_i|^2\,\mathrm{Re}\{t_i\} \geq 0 \\[2mm]
2\zeta - \gamma\,\mathrm{Re}\{t_i\} \geq 0 \qquad\qquad\qquad i = 1, 2, 3
\end{cases}
\qquad 8.210
$$

In case of equality in the first of the equations (8.210) the characteristic equation has two pure imaginary roots

$$p = \pm i \, \frac{2\zeta - \gamma \operatorname{Re}\{t_i\}}{2\zeta \, \gamma \operatorname{Im}\{t_i\}} \, \beta \qquad\qquad 8.211$$

The stability condition (8.210) is obviously satisfied for all $\gamma$-values if

$$\operatorname{Re}\{t_i\} \leq 0 \qquad i = 1, 2, 3 \qquad\qquad 8.212$$

Equation (8.205) then gives the following condition on the numbers $s$ and $l$

$$\begin{cases} l = 0 \\ s \geq 3 \end{cases} \qquad\qquad 8.213$$

If this condition is not satisfied the system is at least stable for sufficiently small values of the cross coupling coefficient $\gamma$.

If either $l \neq 0$ and arbitrary $s$, or $l = 0$ and $s < 3$ equation (8.208) has at least one root in the right half plane. Let $t_o$ be the root in the first quadrant or on the real axis. The condition of stability (8.210) gives

$$\begin{cases} (2\zeta - \gamma \operatorname{Re}\{t_o\})^2 (1 - 2\zeta \, \gamma \operatorname{Re}\{t_o\}) - 2\zeta \, \gamma^3 (\operatorname{Im}\{t_o\})^2 \operatorname{Re}\{t_o\} \geq 0 \\ 2\zeta - \gamma \operatorname{Re}\{t_o\} \geq 0 \end{cases} \qquad 8.214$$

Equations (8.214) are satisfied if

$$\gamma \operatorname{Re}\{t_o\} \leq f(\zeta, a_o) \qquad\qquad 8.215$$

where

$$f(\zeta, a_o) = \min(2\zeta, z_o) \qquad\qquad 8.216$$

and $z_o$ is the smallest positive root of the equation.

$$\begin{cases} 2\zeta (1 + a_o) z^3 - (1 + 8\zeta^2) z^2 + 4\zeta (1 + 2\zeta^2) z - 4\zeta^2 = 0 \\ a_o = \left(\dfrac{\operatorname{Im}\{t_o\}}{\operatorname{Re}\{t_o\}}\right)^2 \end{cases}$$

Systems with $1 > 0$ and arbitrary $s$, or $1 = 0$ and $s < 3$ have

$$\text{Im}\left\{t_o\right\} = 0$$

hence

$$a_o = 0$$

Equation (8.216) then reduces to

$$f(\zeta, 0) = \min\left(2\zeta, \frac{1}{2\zeta}\right) \qquad\qquad 8.217$$

A graph of the function $f(\zeta, 0)$ is given in figure 8.22.
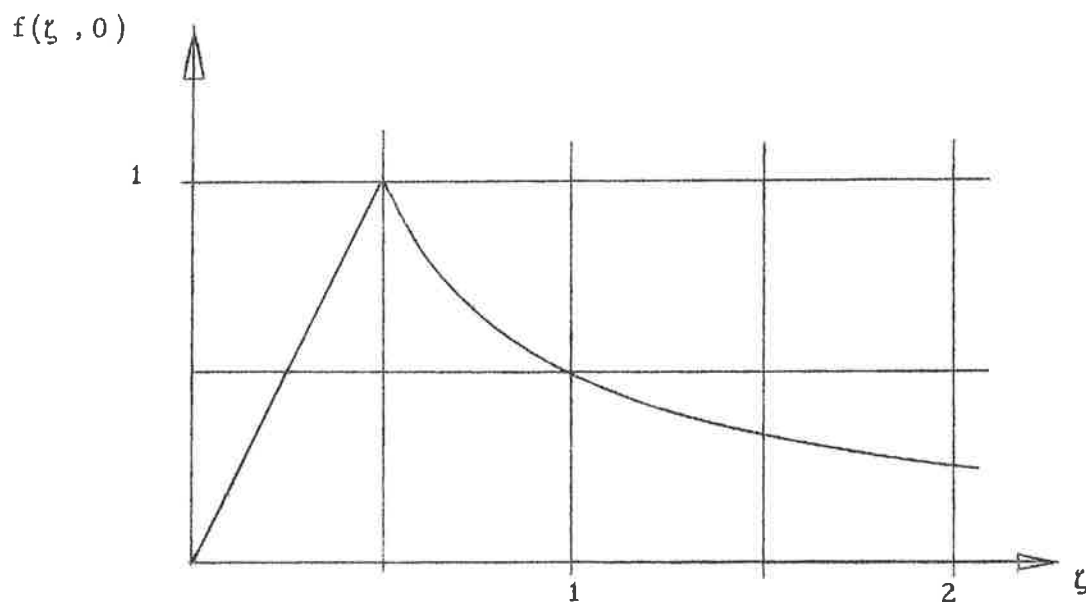


Figure 8.22

Summarizing the stability conditions for a system with

$$Y_1(p) = \frac{p^2 + 2\zeta\,\beta p + \beta^2}{p}$$

we get

    I.    Systems with $1 > 0$ are stable if

$$\gamma\,\text{Re}\left\{t_o\right\} \leq f(\zeta, a_o)$$

II. Systems with $l = 0$ and $s < 3$ are stable if

$$\gamma \, \mathrm{Re}\left\{t_o\right\} \leq f(\zeta, 0)$$

III. Systems with $l = 0$ and $s \geq 3$ are stable for all values of $\gamma$

IV. Systems with $l > 0$ are stable if

$$\gamma \, \mathrm{Re}\left\{t_o\right\} \leq f(\zeta, 0)$$

If the cross-coupling coefficient $\gamma$ is sufficiently small the system discussed is thus stable, independent of the orientation of the gyros. The upper limit of $\gamma$ for a stable system is given by the equation (8.215).

When the cross-coupling coefficient $\gamma$ is increased over the critical value given by equation (8.215) the characteristic equation of the systems I. has two complex roots in the right half plane while the characteristic equation of the systems II. and IV. has one real root in the right half plane. These cases are referred to as oscillating and pure exponential instabilities, respectively.

Example.

Give the stability conditions for systems with the following arrangements of the gyros

A. $\left[ 0, \quad 0, \quad \dfrac{\pi}{2} \right]$

B. $\left[ \pi, \quad \pi, \quad \pi \right]$

C. $\left[ 0, \quad 0, \quad 0 \right]$

D. $\left[ \dfrac{\pi}{2}, \quad \dfrac{\pi}{2}, \quad \pi \right]$

The $s$, $l$ and $t_o$ numbers are obtained from equations (3.3), (3.5) and table (8.31). We get

A.   $s = 1,\ 1 = 0,\ t_o = 1$

B.   $s = 3,\ 1 = -1,\ t_o = \frac{1}{2} + i\ \frac{1}{2}\sqrt{3}$

C.   $s = 3,\ 1 = 1,\ t_o = 1$

D.   $s = 5,\ 1 = 0,\ t_o = 0$

According to (8.217) the systems A and C are stable if

$$\gamma < f(\zeta, 0)$$

The system B is stable if

$$\gamma < 2 \cdot f(\zeta, 3)$$

where

$$f(\zeta, 3) = \min\left[2\zeta, z_o\right]$$

and $z_o$ the smallest positive root of the equation

$$8\zeta\ z^3 - (1 + 8\zeta^2)\ z^2 + 4\zeta\ (1 + 2\zeta^2)\ z - 4\zeta^2 = 0$$

Figure 8.23 shows a graph of the function $2f(\zeta, 3)$.
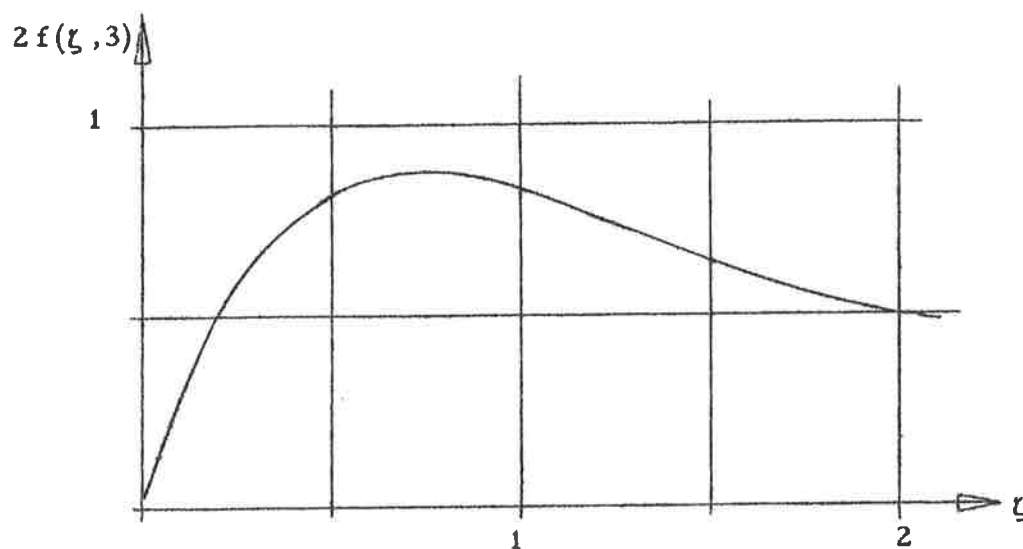


Figure 8.23

Although a system is stable if equation (8.210) is satisfied it may be too oscillative for practical use. To judge this we have to solve the characteristic equation. This is most conveniently carried out with the graphical method of Evans (ref. 2). This method gives directly the root locus of the characteristic equation with respect to the coupling coefficient.

Plates 8.21 - 8.24 shows the root loci for the characteristic equations of the systems treated in the example.

In an actual application we have to consider $Y_1(p)$-functions considerably more complicated than the one just dealt with. The analysis can though be carried out in a straight forward way following the scheme of the simple example. The algebraic conditions have a rather formidable appearance in case of a complicated $Y_1(p)$-function why it seems wise to use the graphical methods to solve the characteristic equation.

Consider e.g. a system with

$$s = 3 \qquad 1 = -1$$

and

$$Y_1(p) = \frac{(p^2 + 1.41 \beta p + \beta^2)(p + 0.05 \beta)}{p(p + 1.46 \beta)}$$

The characteristic equation of the system is of the 9th degree. According to equation (8.204) it can be reduced to

$$(p^2 + 1.41 \beta p + \beta^2)(p + 0.05 \beta) + t_i \; 1.1 \; \frac{a \beta^2}{\omega_o} p(p + 0.0467) = 0 \qquad i = 1, 2, 3$$

where the $t_i$:s are the roots of the equation (8.205).

Introducing $s = 3$ and $1 = -1$ in equation (8.204) we get

$$t_1 = -1$$
$$t_2 = 0.5 + i \, 0.865$$
$$t_3 = 0.5 - i \, 0.865$$

Introducing the cross coupling coefficient $\gamma$

$$\gamma = \frac{a\beta}{\omega_o}$$

we get

$$(p^2 + 1.41\,\beta p + \beta^2)(p + 0.05\,\beta) + 1.1\,t_i\,\gamma\,\beta p\,(p + 0.0467) = 0$$

The root locus of this equation with respect to the cross coupling coefficient $\gamma$ is shown in plate 8.25.

8.3 The assumption (8.201), $b \gg 1$ means that the moments of inertia of the stable element are much greater than those of the gyro rotor. Systems where the moments of inertia of the platform are much greater than those of the gyro rotor behave in a similar way. In order to show this we will analyse a special case. Suppose that the system is arranged according to the equations (8.103) - (8.106). Assume further that

$$\sigma(p) = a p + \mathcal{H} \qquad\qquad 8.301$$

this implies that the floats of the gyros are "spring-restrained" to their neutral positions, or "rate-coupled".

Equation (8.301) gives

$$bp + \frac{\omega_o}{a}\,\frac{\tau(p) + \omega_o p}{p^2 + \sigma(p)} = b'\left[\frac{p + \dfrac{\omega_o}{ab}\cdot\tau(p)}{p^2 + ap + \mathcal{H}} + \frac{bp^2}{b'}\,\frac{p + a}{p^2 + ap + \mathcal{H}}\right] \qquad 8.302$$

where

$$b' = b + \frac{\omega_o^2}{a\mathcal{H}} \qquad\qquad 8.303$$

Introduce

$$Y_1'(p) = \frac{\mathcal{H}p + \dfrac{\omega_o}{ab}\cdot\tau(p)}{p^2 + ap + \mathcal{H}} + \frac{bp^2}{b'}\,\frac{p + a}{p^2 + ap + \mathcal{H}} \qquad 8.304$$

Equation (8.111) gives

$$\mathbb{K}(p) = b'\left\{ Y_1'(p)\,\mathbb{I} - \frac{ap}{\omega_0}\left[ Y_1'(p) - \frac{b}{b'}\,p \right]\mathbb{L} + \frac{\omega_0}{b'}\,\mathbb{L} - \right.$$
$$\left. - \frac{\omega_0(ap+\mathcal{H})}{b'(p^2+ap+\mathcal{H})}\,\widetilde{\mathbb{L}} - \frac{ap^3}{b'(p^2+ap+\mathcal{H})}\,\widetilde{\mathbb{L}}\,\mathbb{L} \right\} \qquad 8.305$$

Further is

$$\mathbb{K}(p) - \mathbb{F}(p) = b'\left\{ \left[ Y_1'(p) - \frac{b}{b'}\,p \right]\mathbb{I} - \frac{ap}{\omega_0}\left[ Y_1'(p) - \frac{b}{b'}\,p \right]\mathbb{L} + \right.$$
$$\left. + \frac{\omega_0\,p^2}{b'(p^2+ap+\mathcal{H})}\,\mathbb{L} - \frac{a\,p^3}{b'(p^2+ap+\mathcal{H})}\,\widetilde{\mathbb{L}}\,\mathbb{L} \right\}$$
$$8.306$$

Supposing

$$b' \gg 1 \qquad 8.307$$

we get

$$\mathbb{K}(p) = \mathbb{K}(p) - \mathbb{F}(p) = b'\,Y_1''\left\{ \mathbb{I} - \frac{ap}{\omega_0}\,\mathbb{L} \right\} \qquad 8.308$$

where

$$Y_1''(p) = \frac{\mathcal{H}p + \dfrac{\omega_0}{ab}\,\tau(p)}{p^2 + ap + \mathcal{H}} \qquad 8.309$$

Equation (8.307) implies that the secondary reaction torques of the gyros are neglected. The cross coupling is thus caused by the output axis sensitivity of the gyros. Compare section 8.2.

According to the equations (8.308) and (8.309) the moment of inertia of the platform is

$$Jb' = Jb + J\,\frac{\omega_0^2}{\mathcal{H}} \qquad 8.310$$

The moments of inertia of the platform are thus greater than those of the stable element.

We have the following condition for the stability of the system described by equation (8. 308).

### Theorem 8. 31

The necessary and sufficient condition for the system to be stable respectively strict stable for any $Y_1''(p)$ with no poles or zeros in the open right half plane and no zeros on the imaginary axis are

$$\begin{cases} s \geq 3 \\ 1 = 0 \end{cases} \qquad \qquad 8.311$$

and

$$\begin{cases} s = 3 \\ 1 = 0 \end{cases} \qquad \qquad 8.312$$

### Proof

We prove the first part of the theorem.

Equation (8. 308) gives

$$\det p\, \mathbb{K}(p) = b\, p\, Y_1''(p) \left(\frac{a}{\omega_o}\right)^3 \det \mathbb{W}(p)$$

This equation is stable if the condition (8. 311) is satisfied. The system is thus stable according to lemma 8. 15.

Further is

$$\det \left\{ p\, \mathbb{S}(p)\, \mathbb{G}^{-1}(p)\, \mathbb{K}(p) \right\} = p \det \mathbb{W}(p)$$

according to the equations (6. 14) and (8. 308). If $1 \neq 0$ or $1 = 0$ and $s < 3$ the function $\det \mathbb{W}(p)$ has at least one zero in the right half plane and the system is thus unstable according to lemma 8. 12 which proves the first part of the theorem. The second part is proved similarly.

For the special system discussed it is thus necessary to have a definite stable arrangement of the gyros if the system should be strictly stable. Compare the system analysed in section 8. 2. As was already pointed out in section 8. 1 it is very difficult to obtain a stable system if the condition (8. 311) is not satisfied. If the gyros are not arranged according to

equation (8.311) the function det $\mathbb{W}(p)$ has at least one zero in the right half plane. According to lemma 8.15 we must then require that the function det $\left\{ \mathbb{K}(p) - \mathbb{F}(p) \right\}$ has the same zeros in the right half plane.

8.4 In section 8.2 we showed that the cross-coupling due to the output axis sensitivity of the gyros can cause undesired effects, such as instability. This cross-coupling can of cause be eliminated by the proper choice of the matrix $\mathbb{S}$.

If

$$\mathbb{S} = f(p) \, \mathbb{W}(p) \qquad\qquad 8.401$$

equation (6.8) gives

$$\bar{\varphi} = \frac{1}{f(p)} \, \mathbb{I} \, \bar{\Omega} - \frac{1}{a f(p)} \, \mathbb{W}^{-1}(p) \, \bar{m}$$

This means that the m-component of the angular velocity only affects the m-gyro. The output axis sensitivity is thus eliminated.

Suppose

$$f(p) = \frac{a}{\omega_o} \, (p^2 + \sigma(p))$$

Equations (8.401) and (8.110) gives

$$\mathbb{S}(p) = (p^2 + \sigma(p)) \, \mathbb{I} - \frac{a}{\omega_o} \, p \, (p^2 + \sigma(p)) \, \mathbb{L} \qquad\qquad 8.402$$

thus

$$\sigma_{11}(p) = \sigma_{22}(p) = \sigma_{33}(p) = \sigma(p)$$

$$\sigma_{12}(p) = - \frac{a}{\omega_o} \, p \, (p^2 + \sigma(p)) \cos \theta^{(1)}$$

$$\sigma_{13}(p) = - \frac{a}{\omega_o} \, p \, (p^2 + \sigma(p)) \sin \theta^{(1)}$$

$$\sigma_{21}(p) = - \frac{a}{\omega_o} \, p \, (p^2 + \sigma(p)) \sin \theta^{(2)}$$

$$\sigma_{23}(p) = - \frac{a}{\omega_o} \, p \, (p^2 + \sigma(p)) \cos \theta^{(2)}$$

$$\sigma_{31}(p) = - \frac{a}{\omega_o} \, p \, (p^2 + \sigma(p)) \cos \theta^{(3)}$$

$$\sigma_{32}(p) = - \frac{a}{\omega_o} \, p \, (p^2 + \sigma(p)) \sin \theta^{(3)}$$

Make the assumptions (8.103) - (8.105) and suppose $\mathbb{S}$ to be chosen according to equation (8.401) then

$$\mathbb{K}(p) = b \left\{ Y_1(p) \, \mathbb{I} + Y_2(p) \, \mathbb{L} - Y_3(p) \, \widetilde{\mathbb{L}} \right\} \qquad 8.403$$

where

$$Y_1(p) = p + \frac{\omega_o}{ab} \cdot \frac{\tau(p) + \omega_o \, p}{p^2 + \sigma(p)} \qquad 8.404$$

$$Y_2(p) = \frac{\omega_o}{b} \qquad 8.405$$

$$Y_3(p) = \frac{\omega_o \, \sigma(p)}{b(p^2 + \sigma(p))} \qquad 8.406$$

*are not to*

If disturbing torques on the floats of the gyros ~~should not~~ give errors increasing exponentially with time, it is necessary that the equation (8.102) is stable. Equation (8.402) gives

$$\det \left\{ \mathbb{G}^{-1}(p) \, \mathbb{S}(p) \right\} = \frac{a}{\omega_o} (p^2 + \sigma(p)) \, \det \left\{ \mathbb{G}^{-1}(p) \, \mathbb{W}(p) \right\}$$

Hence a necessary condition for stability of the systems is that the equation

$$\det \mathbb{W}(p) = \left( \frac{\omega_o}{a} \right)^3 + \frac{s-3}{2} \cdot \frac{\omega_o}{a} p^2 - 1 \, p^3 = 0$$

is stable. There are a few exceptions from this case, namely when the positive zeros of $\mathbb{W}(p)$ are cancelled by poles of $\mathbb{G}^{-1}(p)$. This occurs only when

$$\begin{cases} \tau(p) = 0 \\ 1 = 0 \end{cases}$$

Compare lemma 8.15.

Excluding this case, the systems with $s \geq 3$ and $1 = 0$ are the only platform systems where the output axis sensitivity of the gyros can be successfully eliminated by the proper choice of $\mathbb{S}(p)$. Arguing in a similar way we come to the same conclusion if the output axis sensitivity is eliminated by the proper choice of $\mathbb{T}(p)$.

Equation (8.403) gives

$$\det \mathbb{K}(p) = Y_1{}^3 + 1\,(Y_2{}^3 - Y_3{}^3) + \frac{s-3}{2}\,Y_1\,(Y_2{}^2 + Y_3{}^2) +$$

$$+ \frac{1}{2}\,f\,Y_2\,Y_3 + 3\,Y_1\,Y_2\,Y_3 \qquad\qquad 8.407$$

where

$$f = \sin 2\,\theta^{(1)}\,(Y_3 \sin \theta^{(3)} - Y_2 \cos \theta^{(2)}) +$$
$$+ \sin 2\,\theta^{(2)}\,(Y_3 \sin \theta^{(1)} - Y_2 \cos \theta^{(3)}) +$$
$$+ \sin 2\,\theta^{(3)}\,(Y_3 \sin \theta^{(2)} - Y_2 \cos \theta^{(1)})$$

$$8.408$$

Suppose that the arrangement of the gyros is definite stable, i.e. $s = 3$ and $1 = 0$, then

$$\det \mathbb{K}(p) = Y_1{}^3 + \frac{1}{2}\,f\,Y_2\,Y_3 + 3\,Y_1\,Y_2\,Y_3 \qquad\qquad 8.409$$

The definite stable arrangement of the gyros is no longer sufficient for stability of the system. It is therefore necessary to analyse the characteristic equation of the system.

## Example

Suppose that the arrangement of the gyros is one of the definite stable arrangements of table 3.15 then

$$f = 0$$

The characteristic equation of the system is thus

$$Y_1(p)\left[Y_1(p)^2 + 3\,Y_2(p)\,Y_3(p)\right] = 0 \qquad\qquad 8.410$$

Suppose that $Y_1(p)$ is chosen according to equation (A 23) i.e.

$$Y_1(p) = \frac{p^2 + 2\zeta\,\beta p + \beta^2}{p}$$

and further that

$$\sigma(p) = a\,p$$

Equations (8.405) and (8.406) gives

$$Y_2(p) = \frac{\omega_o}{b}$$

$$Y_3(p) = \frac{\omega_o \, a}{b(p + a)}$$

Introducing the cross coupling coefficient $\rho$

$$\rho = \frac{\omega_o}{b \beta}$$

Equation (8.410) gives

$$\left( \frac{p^2 + 2\zeta p + \beta^2}{\rho p} \right)^2 + 3\beta^2 \, \frac{a}{p + a} = 0 \qquad\qquad 8.411$$

The stability of this equation is most conveniently analysed by the graphical method of Evans.

Introduce the following numerical values

$$\beta = 1$$
$$\zeta = 0.707$$
$$a = 0.707$$

Equation (8.411) gives

$$\left( \frac{p^2 + 1.41 p + 1}{p} \right)^2 + \rho^2 \, \frac{2.12}{p + 0.707} = 0$$

Plate 8.41 shows the root-locus of this equation with respect to the cross coupling coefficient $\rho$ .

8.5   The condition

$$\begin{cases} s = 3 \\ 1 = 0 \end{cases}$$

plays an important rôle in the previous discussions of stability. The above

condition was a consequence of the necessity for the equation det $\mathbb{W}(p)$ to be stable. As the condition depends on the orientation of mechanical axes it is impossible to satisfy the condition exactly. The reader might therefore expect that the systems whose stability depend on the above condition in practice are unstable. This is not necessarily the fact. In an actual application we have to consider the fact that the condition cannot be exactly satisfied, but we must also notice that the input axes of the gyros are not mutually orthogonal. We have in general

$$\left\{ \mathbb{W}(D) \right\}_{mt} = \frac{\omega_o}{a} P_{1t}^{(m)} - D P_{2t}^{(m)} \qquad\qquad 8.51$$

Compare equation (4.7). The matrix $P^{(m)}$ is defined by the transformation (2.1) but in the general case the elements of the $P^{(m)}$-matrix are not given by the equation (2.2).

We have

$$\det \mathbb{W}(D) = \left\{ \mathbb{W}(D) \right\}_{1i} \left\{ \mathbb{W}(D) \right\}_{2j} \left\{ \mathbb{W}(D) \right\}_{3k} \epsilon_{ijk}$$

hence

$$\det \mathbb{W}(D) = V_1 \left( \frac{\omega_o}{a} \right)^3 + V_2 \left( \frac{\omega_o}{a} \right)^2 D + V_3 \left( \frac{\omega_o}{a} \right) D^2 + V_4 D^3 \qquad 8.52$$

where

$$V_1 = \left[ \hat{x}_1^{(1)}, \ \hat{x}_1^{(2)}, \ x_1^{(3)} \right]$$

$$V_2 = - \left[ \hat{x}_2^{(1)}, \ \hat{x}_1^{(2)}, \ \hat{x}_1^{(3)} \right] - \left[ \hat{x}_1^{(1)}, \ \hat{x}_2^{(2)}, \ \hat{x}_1^{(3)} \right] - \left[ \hat{x}_1^{(1)}, \ \hat{x}_1^{(2)}, \ \hat{x}_2^{(3)} \right]$$

$$V_3 = + \left[ \hat{x}_1^{(1)}, \ \hat{x}_2^{(2)}, \ \hat{x}_2^{(3)} \right] + \left[ \hat{x}_2^{(1)}, \ \hat{x}_1^{(2)}, \ \hat{x}_2^{(3)} \right] + \left[ \hat{x}_2^{(1)}, \ \hat{x}_2^{(2)}, \ \hat{x}_1^{(3)} \right]$$

$$V_4 = - \left[ x_2^{(1)}, \ x_2^{(2)}, \ x_2^{(3)} \right]$$

The scalar triple products above can be interpreted geometrically as the volume of the parallelepiped which has the vectors for concurrent sides. The quantity $V_1$ is thus the volume of the parallelepiped formed by the

input axes etc. The stability of the function det $W(p)$ gives

$$
\begin{cases}
V_1 > 0 \\
V_2 > 0 \\
V_3 > 0 \\
V_2 \, V_3 > V_4 > 0
\end{cases}
$$

which replaces the condition $s = 3$, $l = 0$ in case of non-orthogonal input axes.

In an actual application we have to consider variations in the angular velocities of the gyros as well.

9. The angular deviation of the stable element caused by disturbances.

9.1    There are many reasons why the stable element should deviate from its
desired orientation. In order to obtain a complete picture of the deviation
we have to consider the details of the motion of the carrying vehicle, the
temperature distributions, the elastic deformations, the gimbal errors,
the friction torques, etc. Because of the enormous number of quantities
which are necessary for a complete description of the state of the system
we group all quantities together and treat the problem of the deviation of
the platform with statistical methods.

For the sake of convenience, the disturbances are divided into two groups
$\overline{m}(t)$ and $\overline{M}(t)$ referred to as disturbing torques on the gyrofloats and on
the stable element, respectively.

Introduce a coordinate set $O\xi_1 \xi_2 \xi_3$ fixed to inertial space and initially
coincident with the y-set. The transformation of the $\xi$-set on the y-set is

$$\overline{y} = \mathbb{C}(t)\, \overline{\xi} \qquad\qquad 9.11$$

where

$$\mathbb{C}(0) = \mathbb{I} \qquad\qquad 9.12$$

The orientation of the stable element is thus completely determined by the
transformation matrix $\mathbb{C}(t)$. According to Eulers theorem of a rigid body,
an orthogonal transformation can be interpreted as a rotation around the
eigenvector of the transformation matrix. The angle of rotation $\emptyset(t)$ is
used to specify the angular deviation of the stable element. The angle $\emptyset(t)$
is related to the matrix $\mathbb{C}(t)$ by the relation

$$\emptyset(t) = \text{arc cos } \frac{1}{2}\left[\text{Tr } \mathbb{C}(t) - 1\right] \qquad\qquad 9.13$$

leaving ambiguity to the sign of $\emptyset(t)$. Tr $\mathbb{C}$ is the trace of the matrix $\mathbb{C}$.
We obtain the following equation for $\mathbb{C}(t)$

$$\mathbb{C}(t) = \mathbb{I} + \int_0^t \Omega(t')\, \mathbb{C}(t')\, dt' \qquad\qquad 9.14$$

where

$$(\tilde{\Omega})_{jk} = \Omega_i \, \epsilon_{ijk}$$

and $\overline{\Omega}(t)$ the angular velocity of the stable element

$$\overline{\Omega} = \Omega_i \, \hat{y}_i$$

Introduce the matrix sequence

$$\mathbb{C}_o = \mathbb{I}$$

$$\mathbb{C}_n(t) = \mathbb{I} + \int_o^t \tilde{\Omega}(t') \, \mathbb{C}_{n-1}(t') \, dt'$$

This sequence converges to a limit $\mathbb{C}$ which is the solution of the equation (9.14) when all the elements of the $\tilde{\Omega}$-matrix are bounded in a compact t-set including $(0, t)$. As

$$\tilde{\Omega} + \widetilde{\tilde{\Omega}} = 0$$

the solution $\mathbb{C}$ is an orthogonal matrix.

Further is

$$\mathrm{Tr}(\mathbb{C} - \mathbb{C}_2) < \frac{(at)^4}{4!} \, (e^{at} - 1)$$

where

$$a = \sup_{0 < t < T} \sum_i \sum_j |\Omega_{ij}(t)|$$

In the first approximation we neglect the right-hand side of the above equation\*, then

$$\emptyset^2(t) = e_i(t) \cdot e_i(t)$$

---

\* In actual applications $\emptyset$ is of the order of magnitude of milliradians. The approximation is made throughout this chapter.

where

$$e_i(t) = \int_0^t \Omega_i(t') \, dt'$$

9.16

Equations (6.17) and (9.15) give

$$\overline{e}(p) = \frac{1}{p} \mathbb{K}^{-1}(p) \overline{M}(p) + \frac{1}{ap} \mathbb{K}^{-1}(p) \mathbb{G}(p) \mathbb{S}^{-1}(p) \overline{m}(p)$$

9.17

By this equation it is possible to calculate the error if the disturbances $\overline{M}(t)$ and $\overline{m}(t)$ are known.

For an inertial stabilized platform system the function

$$\det \left\{ p \, \mathbb{S}(p) \, \mathbb{G}^{-1}(p) \, \mathbb{K}(p) \right\}$$

has a simple zero at the origin. Compare Lemma 8.13. This implies that a constant disturbing torque $m_o$ acting on a gyrofloat will give the stable element a constant angular velocity

$$\Omega_o = \frac{m_o}{a \omega_o}$$

This phenomena is referred to as drift of the platform system. A disturbing torque can e.g. be obtained if the center of gravity and center of buoyance of the gyrofloat do not coincide with the output axis of the gyro. The constant drift, which is of the order of magnitude of 0.01 $^o$/h for a good gyro, can to some extent be eliminated by proper design and careful compensation of the gyro. In the following we assume that the constant drift is eliminated, i.e. the disturbing moments acting on the gyrofloats have zero ensemble averages, hence

$$E \left\{ \overline{m}(t) \right\} = 0$$

It is also assumed that

$$E \left\{ \overline{M}(t) \right\} = 0$$

Equation (9. 17) then gives

$$E \left\{ \bar{e} (t) \right\} = 0$$

Because of the complicated nature of the disturbances $\bar{m}(t)$ and $\overline{M}(t)$ we cannot expect to have a detailed knowledge of them. We thus have to find refuge in a statistical description of the disturbances.

The problem is to determine some measure of the angular deviation $\emptyset (t)$ when the statistical character of the disturbing torques is known.

To characterize the angular deviation of the stable element we choose the variance of the angular deviation, i. e.

$$E \left\{ \emptyset (t)^2 \right\}$$

The problem stated is complicated by the initial condition

$$\bar{e} (t) = 0 \quad \text{for} \quad t = 0$$

and the fact that the function

$$\det \left\{ p \ \mathbb{S} (p) \ \mathbb{G}^{-1} (p) \ \mathbb{K} (p) \right\}$$

is not strictly stable. This implies that the function $\bar{e} (t)$ is not a stationary process even if $\bar{m}(t)$ and $\overline{M}(t)$ are stationary processes, which means that the Wiener theory of stationary processes cannot be used. Before continuing we state two theorems.

9. 2   Consider a linear system with n input signals $x_i(t)$ and m output signals $y_i(t)$ related by

$$\bar{y} (p) = \mathbb{Y} (p) \ \bar{x} (p) \qquad\qquad 9.21$$

where $\bar{x} (t)$ and $\bar{y} (t)$ are the column vectors formed by $x_i(t)$ and $y_i(t)$, respectively. Suppose that $E \left\{ \bar{x} (t) \right\} = 0$

The correlation matrix $\mathbb{R}(\bar{f}, t, \tau)$ of a vector $\bar{f}(t)$ is defined by

$$(\mathbb{R}(\bar{f}, t, \tau))_{ij} = E\left\{f_i\left(t - \frac{\tau}{2}\right) f_j\left(t + \frac{\tau}{2}\right)\right\} \qquad 9.22$$

The Fourier-Laplace transform of $\mathbb{R}(\bar{f}, t, \tau)$ is denoted by $\mathbb{F}(\bar{f}, p, \omega)$. If the function $\bar{f}(t)$ is sufficiently wellbehaved the inversion formulas

$$\mathbb{F}(\bar{f}, p, \omega) = \frac{1}{2\pi} \int_0^\infty e^{-pt} \, dt \int_{-\infty}^\infty \mathbb{R}(\bar{f}, t, \tau) \, e^{-i\omega\tau} \, d\tau \qquad 9.23$$

$$\mathbb{R}(\bar{f}, t, \tau) = \frac{1}{2\pi i} \int_T e^{pt} \, dt \int_{-\infty}^\infty \mathbb{F}(\bar{f}, p, \omega) \, e^{i\omega\tau} \, d\omega \qquad 9.24$$

hold.

For a stationary random process $\bar{g}(t)$ we have

$$\mathbb{R}(\bar{g}, t, \tau) = \mathbb{R}(\bar{g}, \omega) \qquad \text{independent of } t$$

hence

$$\mathbb{F}(\bar{g}, p, \omega) = \frac{1}{p} \, \mathbb{F}(\bar{g}, \omega)$$

Suppose that equations (9.23) and (9.24) are valid for the functions $\bar{x}(t)$ and $\bar{y}(t)$.

### Theorem 9.21

If $E\left\{\bar{x}(t)\right\} = 0$ the correlation functions of the input and output signals are related by

$$\mathbb{F}(\bar{y}, p, \omega) = \mathbb{W}\left(\frac{p}{2} - i\omega\right) \mathbb{F}(\bar{x}, p, \omega) \, \widetilde{\mathbb{W}}\left(\frac{p}{2} + i\omega\right) \qquad 9.25$$

For a system with one input signal and one output signal equation (9.25) reduces to

$$F(y, p, \omega) = Y\left(\frac{p}{2} - i\omega\right) Y\left(\frac{p}{2} + i\omega\right) F(x, p, \omega)$$

If the input signal is a stationary random process we have

$$F(x, p, \omega) = \frac{1}{p} \emptyset_{xx}(\omega)$$

where $\emptyset_{xx}(\omega)$ is the power spectrum of the signal. For systems with one input signal and one output signal we have

Theorem 9.22

If the transfer function $Y(p)$ has a pole of the order $n$ at the origin with

$$\lim_{p \to 0} p^n Y(p) = 1, \qquad n \geq 1$$

and no other poles with non-negative real part then

$$\lim_{t \to \infty} R(y, t, \tau) = \frac{1}{(2n-1)((n-1)!)^2} \int_{-\infty}^{\infty} R_{xx}(\tau) \, d\tau$$

where $R_{xx}(\tau)$ is the autocorrelation function of the output signal, i.e. the Fourier-transform of the power spectrum $\emptyset_{xx}(\omega)$.

Corollarium

If the transfer function $Y(p)$ is strictly stable and $Y(0) = 1$, we get

$$\lim_{t \to \infty} R(y, t, \tau) = \int_{-\infty t}^{\infty t} Y(i\omega) \, \emptyset_{xx}(\omega) \, Y(-i\omega) \, e^{i\omega\tau} \, d\omega$$

The time required to reach the asymptotic values is of the order of magnitude of the step-function-response-time of the transfer function $Y(p)$.

For the proof of these theorems we refer to reference 3.

9.3    The theorems of section 9.2 form suitable tools for solving the problem stated in section 9.1. Introduce the column vector $\overline{1}$ define by

$$\overline{1}(t) = \begin{pmatrix} \overline{M}(t) \\ \overline{m}(t) \end{pmatrix}$$                    9.31

and the $3 \times 6$ matrix $\mathbb{Z}(p)$ defined by

$$\mathbb{Z}(p) = (\frac{1}{p} \mathbb{K}^{-1}(p), \quad \frac{1}{ap} \mathbb{K}^{-1}(p) \mathbb{G}(p) \mathbb{S}^{-1}(p)) \qquad 9.32$$

Equation (9.17) gives

$$\overline{e}(p) = \mathbb{Z}(p) \overline{I}(p) \qquad 9.33$$

Equation (9.16) and (9.22) gives

$$E\left\{\phi(t)^2\right\} = \operatorname{Tr} \mathbb{R}(\overline{e}, t, 0)$$

Using theorem 9.21 we get

$$\operatorname{Tr} \mathbb{R}(\overline{e}, t, 0) = \frac{1}{2\pi i} \int_{\Gamma} e^{pt} dp \int_{-\infty}^{\infty} \operatorname{Tr} \mathbb{F}(\overline{e}, p, \omega) d\omega$$

where

$$\mathbb{F}(\overline{e}, p, \omega) = \mathbb{Z}(\frac{p}{2} - i\omega) \mathbb{F}(\overline{I}, p, \omega) \widetilde{\mathbb{Z}}(\frac{p}{2} + i\omega)$$

hence

$$E\left\{(\phi(t)^2\right\} = \frac{1}{2\pi i} \int_{\Gamma} e^{pt} dp \int_{-\infty}^{\infty} \operatorname{Tr}\left[\mathbb{Z}(\frac{p}{2} - i\omega) \mathbb{F}(\overline{I}, p, \omega) \widetilde{\mathbb{Z}}(\frac{p}{2} + i\omega)\right] d\omega$$

$$9.34$$

The problem stated in section 9.1 can thus be solved if the correlation function of the input signal is known.

The asymptotic properties of $E\left\{\phi(t)^2\right\}$ will now be analysed. We start with an example.

Example

Suppose that the platform servos are perfect meaning that

$$\overline{\phi}(t) = 0 \qquad \text{all } t$$

Equation (6.1) gives

$$\bar{\Omega}(p) = \frac{1}{a}\ \mathbb{W}^{-1}(p)\ \bar{m}(p)$$

hence

$$\bar{e}(p) = \frac{1}{pa}\ \mathbb{W}^{-1}(p)\ \bar{m}(p) \qquad\qquad 9.35$$

The only way to obtain a system whose errors do not increase exponentially with time is by choosing an arrangement of the gyros which gives

$$\det \mathbb{W}(p) \neq 0 \quad \text{for} \quad \text{Re}\left\{p\right\} \geq 0$$

i.e. it is necessary to use a definite stable arrangement of the gyros. Compare section 3.

For small p we get the following asymptotic expansion of $\mathbb{W}(p)$

$$\mathbb{W}(p) = \frac{\omega_o}{a}\ \mathbb{I} + \underset{p \to 0}{O(p)}$$

Suppose further that the disturbing moment $\bar{m}(t)$ is a stationary random process with zero average whose autocorrelation function exists, i.e.

$$\mathbb{R}(\bar{m}, t, \tau) = \mathbb{R}(\bar{m}, \tau) \quad \text{independent of } t$$

Applying theorem 9.22 on the components of the equation (9.35) we get

$$E\left\{\bar{e}(t)^2\right\} = \frac{t}{\omega_o^2} \int_{-\infty}^{\infty} \text{Tr}\left\{\mathbb{R}(\bar{m}, \tau)\right\} d\tau$$

In this special case the variance of the navigation error thus increases linearily with time. This depends on the fact that the diagonal elements of the matrix

$$\frac{1}{p}\ \mathbb{W}^{-1}(p)$$

have a pole at the origin. Physically the property depends on the fact that a gyro responds to a disturbing torque along the output axis in the same

way as to an angular velocity along the input axis. In order to obtain a
system where the variance of the indication error does not increase with
time the gyro must be substituted by a component which does not have
this property.

The result obtained in the example is valid under more general conditions.
If the disturbing torques $\overline{m}(t)$ and $\overline{M}(t)$ are independent functions, the
input function correlation matrix can be partitioned in the following way

$$\mathbb{F}(\overline{I}, p, \omega) = \begin{pmatrix} \mathbb{F}(\overline{M}, p, \omega) & 0 \\ \\ 0 & \mathbb{F}(\overline{m}, p, \omega) \end{pmatrix} \qquad 9.36$$

Introduce

$$\mathbb{Z}_1(p) = \frac{1}{p} \, \mathbb{K}^{-1}(p) \qquad 9.37$$

$$\mathbb{Z}_2(p) = \frac{1}{pa} \, \mathbb{K}^{-1}(p) \, \mathbb{G}(p) \, \mathbb{S}^{-1}(p) \qquad 9.38$$

then

$$\mathbb{Z}(p) = (\mathbb{Z}_1(p), \, \mathbb{Z}_2(p))$$

Equation (9.33) gives

$$\overline{e}(p) = \overline{e}_1(p) + \overline{e}_2(p)$$

where

$$\overline{e}_1(p) = \mathbb{Z}_1(p) \cdot \overline{M}(p)$$

and

$$\overline{e}_2(p) = \mathbb{Z}_2(p) \, \overline{m}(p)$$

We can now state a theorem.

### Theorem 9.31

If the disturbing moments $\overline{M}(t)$ and $\overline{m}(t)$ are independent stationary random processes and

$$E\left\{\overline{M}(t)\right\} = 0 \qquad\qquad E\left\{\overline{M}(t)^2\right\} < C_1$$

$$E\left\{\overline{m}(t)\right\} = 0 \qquad\qquad E\left\{\overline{m}(t)^2\right\} < C_2$$

then

$$E\left\{\emptyset(t)^2\right\} = \frac{t}{\omega_0^2} \int_{-\infty}^{\infty} \text{Tr}\,\mathbb{R}(\overline{m}, \tau)\, d\tau + \underset{t \to \infty}{O(1)}$$

for all platform systems which are stabilized with respect to inertial space.

### Proof.

The disturbing moments $\overline{m}(t)$ and $\overline{M}(t)$ are independent hence $\overline{e}_1(t)$ and $\overline{e}_2(t)$ are also independent, i.e.

$$E\left\{\emptyset(t)^2\right\} = E\left\{\overline{e}_1(t)^2\right\} + E\left\{\overline{e}_2(t)^2\right\}$$

The asymptotic properties of the first term will first be considered. As was shown in section 8, the equation

$$\det p\, \mathbb{K}(p) = 0$$

has no roots with $\text{Re}\left\{p\right\} \geq 0$, hence the elements of the matrix $\mathbb{Z}_1(p)$ are strictly stable. Applying the corollarium of theorem 9.2 on the three components of the equation

$$\overline{e}_1(p) = \mathbb{Z}_1(p) \cdot \overline{M}(p)$$

we get

$$\lim_{t \to \infty} E\left\{e_1(t)^2\right\} = \int_{-\infty}^{\infty} \text{Tr}\left\{\mathbb{Z}_1(i\omega)\, \mathbb{F}(\overline{M}, \omega)\, \mathbb{Z}_1(-i\omega)\right\} d\omega$$

The matrix $\mathbb{Z}_1(p)$ is strictly stable, hence all elements are bounded on the imaginary axis this implies that the right hand side of the above equation is bounded by

$$C \cdot E \left\{ \overline{M}(t)^2 \right\}$$

Equations (6. 14) and (9. 38) gives

$$p \, \mathbb{Z}_2(p) = \frac{1}{a} \left( \mathbb{I} - \mathbb{K}^{-1}(p) \, \mathbb{F}(p) \right) \mathbb{W}^{-1}(p)$$

According to section 11 we have for all platform systems which are stabilized with respect to inertial space.

$$\lim_{p \to 0} \mathbb{K}(p) = p^{-a} \, \mathbb{C} \qquad a \geq 1$$

where $\mathbb{C}$ is a constant diagonal matrix. Hence

$$\lim_{p \to 0} p \, \mathbb{Z}_2(p) = \frac{1}{\omega_o} \, \mathbb{I}$$

The diagonal elements of $\mathbb{Z}_2(p)$ thus have a simple pole at the origin. Applying theorem 9. 22 on the components of the equation

$$\overline{e}_2(p) = \mathbb{Z}_2(p) \, \overline{m}(p)$$

we obtain

$$E \left\{ \overline{e}_2(t)^2 \right\} = \frac{t}{\omega_o^2} \int_{-\infty}^{\infty} \mathrm{Tr} \left\{ \mathbb{R}(\overline{m}, \tau) \right\} d\tau + \underset{t \to \infty}{O(1)}$$

Hence

$$E \left\{ \emptyset(t)^2 \right\} = \frac{t}{\omega_o^2} \int_{-\infty}^{\infty} \mathrm{Tr} \left\{ \mathbb{R}(\overline{m}, \tau) \right\} d\tau + \underset{t \to \infty}{O(1)}$$

which proves the theorem.

If the disturbing torques acting on the gyrofloats are uncorrelated and have the same stocastic properties we obtain

$$\mathbb{R}(\overline{m}, \tau) = R_{mm}(\tau) \cdot \mathbb{I}$$

where $R_{mm}(\tau)$ is the autocorrelation function of the disturbing moments acting on a gyrofloat.

Theorem 9.31 gives

$$E\left\{\emptyset(t)^2\right\} = \frac{3t}{\omega_o^2} \int_{-\infty}^{\infty} R_{mm}(\tau)\, d\tau + \begin{array}{c} O(1) \\ t \to \infty \end{array}$$

Notice that the disturbing moments are normed by the angular momentum of the gyroscopic element.

Theorem 9.31 shows that the standard deviation of the navigation error will increase as the square root of the time coordinate. Notice that we assumed that

$$E\left\{\overline{m}(t)\right\} = 0$$

i.e. the disturbing torques on the gyrofloats have zero averages, which means that the unbalance of the floats is carefully compensated.

Theorem 9.31 thus represents the ultimate navigation accuracy obtained with systems based on gyros, whose unbalance torque is perfectly compensated.

The angular deviation of the platform is then entirely caused by the random variations of the unbalance torques of the gyros.

In order to specify the random drift of the gyros the quality figure $Q$ is suggested. *

$$Q = \frac{1}{\omega_o} \sqrt{\int_{-\infty}^{\infty} R_{mm}(\tau)\, d\tau} \qquad \text{rad sec}^{-1/2}$$

---

* Notice that $Q$ depends on the environmental conditions.

Notice that the m is the normed disturbing torque. Introducing the torques themselves m′ we get instead

$$Q = \frac{1}{H} \sqrt{\int_{-\infty}^{\infty} R_{m'm'}(\tau) \, d\tau}$$

Introducing this quality figure, the standard deviation of the angular deviation of the platform is,

$$\sqrt{E\left\{\phi^2(t)\right\}} = D\left\{\phi(t)\right\} = Q\sqrt{3t}$$

For a single axis system we obtain in a similar way

$$D\left\{\phi(t)\right\} = Q\sqrt{t}$$

## 10. Techniques for the synthesis of inertial platform systems.

10.1 The design procedure will always start from some kind of specifications. Depending on the specifications the design will take different lines. The synthesis can e.g. start from scratch or start with a given stable element with gimbals and gyros. In the first case the designer can choose both the arrangement of the gyros, the weight of the different parts, etc. while in the second case the designer can only choose the transfer functions between the different parts of the system. The following scheme is suggested for the synthesis procedure.

    1.    Choose a matrix $\mathbb{K}(p)$ which satisfies the specifications.

    2.    Design a system which has the $\mathbb{K}(p)$-matrix obtained above.

    3.    Check if it is possible to change the $\mathbb{K}(p)$-matrix in order to simplify the instrumentation without overriding the specifications.

10.2 The first part of the synthesis is the classical problem on servomecanisms. Although no complete solution is yet obtained the problem is solved for certain classes of specifications in ordinary textbooks on automatic control. Let it suffice by mentioning a few things about this special problem. With the specifications ordinarily given, the problem usually has no unique solution. The choice between the different solutions is governed by instrumentational considerations. Compare part 10.4. It is often favourable, however, to use a diagonal system or a system with small cross-couplings. These systems can be synthesized essentially on a single axis basis which means a considerable simplification of the analytical work. Compare appendix.

According to the physical interpretations given in section 7 the $\mathbb{K}(p)$-matrix determines the properties of the platform. The first part of the synthesis of inertial platform systems thus consists of choosing a $\mathbb{K}(p)$-matrix which gives a sufficiently tight coupling between the stable element and inertial space, and a reasonable amount of damping. Compare section 7 and the definition (8.13).

Example

Consider an isotropic platform system, i.e.

$$\mathbb{K}(p) = k(p)\,\mathbb{I} \qquad\qquad 10.21$$

As there is no cross-couplings in the system the synthesis can be carried out following the scheme given in appendix. Suppose e.g. that we obtain

$$k(p) = b\,\frac{p^2 + 2\zeta\,\beta p + \beta^2}{p} = b(p + 2\zeta\,\beta + \frac{\beta^2}{p}) \qquad\qquad 10.22$$

Interpreting the different terms of the $\mathbb{K}(p)$-matrix according to section 7, we get

$J\,b\,\mathbb{I}$      the term corresponding to moment of inertia of the platform. The inertia matrix of the platform is $J\,b\,\mathbb{I}$

$2\,J\,\zeta\,\beta\,b\,\mathbb{I}$      the term corresponding to camping of the platform with respect to inertial space. The damping coefficient is $2\,J\,\zeta\,\beta\,b\,\left[\mathrm{Nms\ rad}^{-1}\right]$.

$J\,b\,\dfrac{\beta^2}{p}\,\mathbb{I}$      the stabilizing term which implies that the stable element is "spring-restrained" to inertial space. The spring-coefficient is $J\,b\,\beta^2\left[\mathrm{Nm\ rad}^{-1}\right]$.

10.3 For the second part of the synthesis we start with equation (6.14) i.e.

$$\mathbb{K}(p) = \mathbb{F}(p) + \mathbb{G}(p)\,\mathbb{S}^{-1}(p)\,\mathbb{W}(p) \qquad\qquad 10.31$$

The matrix $\mathbb{K}(p)$ is given by the first step of the synthesis procedure. The equation above then gives 9 equations for determining the 18 feedback operators $\sigma_{ij}$ and $\tau_{ij}$, the orientation angles of the gyros $\theta^{(1)}$, $\theta^{(2)}$ and $\theta^{(3)}$, and the 18 components of the inertia tensors $A_{ij}$ and $B_{ij}$. Besides we have to consider the stability conditions of section 8. Anyway the problem is highly indetermined and we can impose several other conditions. Some examples are given below.

## Example 1

Suppose that the gyros, their orientation, the stable element and all $\sigma_{ij}$:s are specified.

Equations (6.12) and (6.14) give

$$\mathbb{T}(p) = \mathbb{G}(p) - (A_{12} p^2 + \omega_0 p) \, \mathbb{I} - A_{22} p^2 \, \widetilde{\mathbb{L}} + A_{32} p^2 \, \widetilde{\mathbb{N}}$$

and

$$\mathbb{G}(p) = \mathbb{K}(p) - \mathbb{F}(p) \; \mathbb{W}^{-1}(p) \, \mathbb{S}(p)$$

Assume further that the output axis sensitivity of the gyros is eliminated. (This usually requires a definite stable arrangement of the gyros. Compare section 8.4), i.e.

$$\mathbb{W}^{-1}(p) \, \mathbb{S}(p) = \frac{\omega_0}{a(p^2 + \sigma(p))} \; \mathbb{I}$$

hence

$$\mathbb{G}(p) = \frac{\omega_0}{a(p^2 + \sigma(p))} \left[ \mathbb{K}(p) - \mathbb{F}(p) \right]$$

A system with the desired properties is thus obtained if the transfer functions from the gyros to the torquers $\tau_{ij}$ are chosen in the following way

$$\mathbb{T}(p) = \frac{\omega_0}{a(p^2 + \sigma(p))} \left[ \mathbb{K}(p) - \mathbb{F}(p) \right] - (A_{12} p^2 + \omega_0 p) \, \mathbb{I} - A_{22} p^2 \, \widetilde{\mathbb{L}} +$$
$$+ A_{32} p^2 \, \widetilde{\mathbb{N}}$$

## Example 2

Suppose that the gyros, their orientation, the inertia of the stable element and all $\tau_{ij}$:s are given. Equations (6.12) and (6.14) give

$$\mathbb{S}(p) = \mathbb{W}(p) \left[ \mathbb{K}(p) - \mathbb{F}(p) \right]^{-1} \mathbb{G}(p)$$

where

$$\mathbb{G}(p) = \mathbb{T}(p) + (A_{12} p^2 + \omega_0 p) \, \mathbb{I} + A_{22} p^2 \, \widetilde{\mathbb{L}} - A_{32} p^2 \, \widetilde{\mathbb{N}}$$

Suppose e.g. that

$$\mathbb{T}(p) = 0$$

This corresponds to a system without gimbal torquers which means that the gyros are used in the double purpose of sensing device and torque actuators. The desired properties of the system are obtained by feeding the output signals of the gyros to the torquers of the gyros through suitable networks whose characteristics are given by the matrix $\mathbb{S}(p)$.

10.4 The analytical problems of the last part of the synthesis are essentially to estimate the errors obtained when the $\mathbb{K}(p)$-matrix deviates from the ideal character. Because of man's limited analytical ability it seems wise to use analogue computer methods.

It is impossible to give any rules how to change the system in order to simplify the instrumentation. The designer thus have to rely upon his intuition. However, there is one thing we would like to point out. It is necessary to analyse the order of magnitude of the output signals of the gyros . This is necessary as one of the assumptions made in the beginning of section 6 was that the output signals were small quantities. If this is not true the analysis is not valid.

We will end this chapter with a discussion of the synthesis of diagonal platform systems.

10.5 Suppose we want to synthesize a diagonal platform system. Before starting the analysis we will give a physical interpretation of the non-diagonal elements of $\mathbb{K}(p)$.

1. $B_{ij} \neq 0 \qquad i \neq j$

   This means that the inertia ellipsoid of the stable element is not symmetric with respect to the $y_k$-axes.

2. $\mathbb{L} \neq \widetilde{\mathbb{L}}$

   This means that the spin number s of the platform is not zero,

3. $A_{22} p^2 \widetilde{\mathbb{L}} - A_{32} p^2 \widetilde{\mathbb{N}}$

   Secondary reaction torques.

   (When the gyros give signals they give rise to reaction torques on the stable element.)

4. Non-diagonal elements of $\mathbb{S}^{-1}(p) \, \mathbb{W}(p)$ means that the gyros are sensitive to angular velocities along axes orthogonal to the input axis, e.g. output axis sensitivity.

5. Non-diagonal elements of $\mathbb{T}$ implies that the component of the torque produced by the torque motors on the $y_m$-axis is not only controlled by the m-gyro.

In order to obtain a diagonal platform system we can let non-diagonal elements cancel each other, or try to make all non-diagonal elements zero.

The effect of the non-diagonal elements can be eliminated in the following way.

1.´ Making a stable element whose principal axes coincide with the $y_k$-axes.

2.´ Choosing a zero-spin arrangement of the gyros.

3.´ Making the gyrofloats symmetric with respect to the output axis gives $A_{32} = 0$. Increasing the moments of inertia of the platform decreases the influence of the secondary reaction torques. The high moments of inertia of the platform can be obtained by making a large heavy stable element.

4.´ The output axis sensitivity of the gyros can be eliminated by the proper choice of the matrix $\mathbb{S}(p)$. Compare section 8.4.

5.´ Choose $\mathbb{T}$ diagonal !

The conditions $1'$ - $5'$ are unfortunately inconsistent. Condition $2'$ implies a zero-spin system, according to section 8.4. the output axis sensitivity cannot be eliminated in a zero-spin system without overriding the stability condition. Further condition $4'$ requires a system with $s = 3$ and $1 = 0$, which contradicts condition $2'$. If the secondary reaction torques are eliminated by the proper choice of the internal feedback we must also have a system with $s = 3$ and $1 = 0$.

## Example

Consider a system according to equations (8.103), (8.104), (8.105) and (8.401) whose gyros have a definite stable arrangement. Assume further that

$$\sigma(p) = a\,p + \mathcal{H}$$

and that

$$b' = b + \frac{\omega_o^2}{a\,\mathcal{H}} \;>\!>\; 1$$

then

$$\mathbb{K}(p) = b \left\{ Y_1'(p)\,\mathbb{I} + Y_2'(p)\,\mathbb{L} - Y_3'(p)\,\widetilde{\mathbb{L}} \right\}$$

where

$$Y_1'(p) = \frac{\mathcal{H}\,p + \dfrac{\omega_o}{ab'}\,\tau(p)}{p^2 + a\,p + \mathcal{H}} + \frac{bp^2}{b'}\,\frac{p + a}{p^2 + a\,p + \mathcal{H}}$$

$$Y_2'(p) = \frac{\omega_o}{b'}$$

$$Y_3'(p) = \frac{\omega_o\,(a\,p + \mathcal{H})}{b'\,(p^2 + a\,p + \mathcal{H})}$$

and

$$b' = b + \frac{\omega_o^2}{a\,\mathcal{H}}$$

Hence by making $\mathcal{H}$ sufficiently small the effect of the secondary reaction torques are becoming negligible and the complete system is isotropic. The same effect can be obtained by the proper choice of $\tau(p)$. The details are left for the reader.

## References.

1. Draper, C.S., Wrigley, W., and Grohe, L.R., "The floating integrating gyro and its application to geometrical stabilization problems on moving bases".

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5. "The mechanics of a three-gimbal platform system", TTN-Gruppen, Rapport 591101.

6. "Gimbal-locking", TTN-Gruppen, Report in preparation.

A method for the synthesis of a single axis inertial stabilized platform system.

A. 1  When designing servomechanisms for platform systems we have to consider the following facts.

1.  The indication error caused by disturbing torques acting on the gyrofloats and on the stable element.

2.  The ability of the stable element to follow commanding signals.

The disturbing torques depend on the motion of the vehicle, the vibration level, the elastic properties of the stable element, errors in the gimbal system etc. In order to carry out a successful design it is therefore necessary to have information about the motion of the vehicle, the vibration level, the disturbing torques and the commanding signals. The synthesis of the servomechanisms then consists of choosing the transfer functions from the gyro to the torque generator of the gyro and to that of the stable element.

In order to obtain the main features of the required transfer functions we will make a simplified approach to the design problem. This approach does not require detailed information about the disturbances and the commanding signals. Instead we sacrifice a close-fit between the specifications and the performance of the system. The validity of the simplified approach can be tested by evaluating the error for some characteristic disturbances. We are also supported by the experimental fact that systems designed in this way behave satisfactorily.

Equation (6.27) gives for the single axis platform

$$\Omega(p) = \frac{Y_o}{1 + Y_o} \Omega_o(p) + \frac{1}{b p (1 + Y_o)} M(p) + \frac{a}{\omega_o} \frac{Y_o}{1 + Y_o} m(p)$$

(A. 11)

where

$$Y_o(p) = \frac{\omega_o}{ab} \cdot \frac{\tau(p) + \omega_o p}{p(p^2 + \sigma(p))}$$

(A. 12)

and

$\omega_o$        angular velocity of gyroscopic element

$J$        moment of inertia of the gyroscopic element with respect to spin axis

$aJ$        moment of inertia of the float of the gyro

$bJ$        moment of inertia of the stable element

$\sigma(p)$        transfer function from output signal to torque motor of the gyro

$\tau(p)$        transfer function from output signal to the torque motor of the stable element

$MJ$        disturbing torque acting on the stable element

$mJ$        disturbing torque acting on the float of the gyro

$\varphi$        output signal of the gyro

$\Omega$        angular velocity of the stable element

$\Omega_o$        commanding angular velocity

$\theta$        orientation of the stable element

$\theta_o$        commanding signal

A block-diagram of the system described by equation (A. 11) is shown in figure A. 1.
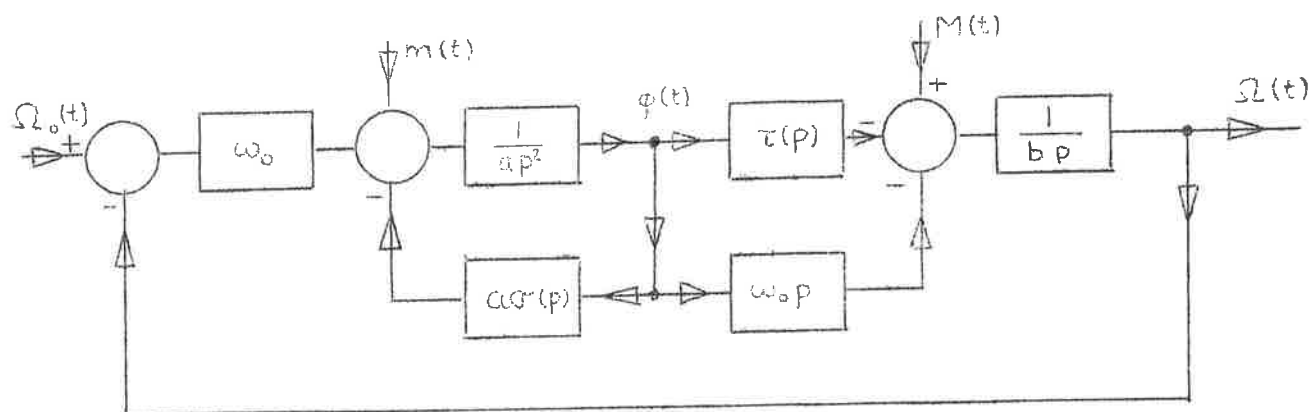


Figure A. 1

Equation (A. 11) gives

$$\theta(p) = \frac{Y_o}{1 + Y_o}\,\theta_o(p) + \frac{1}{bp^2(1 + Y_o)}\,M(p) + \frac{1}{\omega_o p}\frac{Y_o}{1 + Y_o}\,m(p)$$

(A. 13)

The transfer function from torque acting on the stable element to stable element orientation is thus

$$\frac{\theta(p)}{M(p)} = \frac{1}{bp\,Y_1}$$

(A. 14)

where

$$Y_1(p) = p(1 + Y_o(p))$$

(A. 15)

Define the coefficient of stiffness $K_s$ [*]

$$K_s = \lim_{p \to 0} \frac{J\,M(p)}{\theta(p)}$$

(A. 16)

The steady state error when a constant torque is applied is thus

$$\theta(+\infty) = \frac{J\,M_o}{K_s}$$

(A. 17)

where $J\,M_o$ is the magnitude of the disturbing torque.

It is reasonable to assume that a constant torque will result in a finite angular deviation of the stable element i. e. $K_s \neq 0$. This gives the following conditions on the error constants of the servo

$$K_p = \infty$$
$$K_v = \infty$$

If these conditions are satisfied the coefficient of stiffness is proportional to the acceleration constant

$$K_s = J b\,K_a$$

(A. 18)

[*] Compare the "spring coefficient" of section 10. 2.

The permissible steady-state indication error will thus determine the acceleration constant.

The ability of the system to follow commanding signal is essentially determined by choosing a suitable bandwidth and damping ratio of the system.

An analysis of the disturbing torques and the commanding signals will therefore give some preliminary specifications of the following kind.

$$
\begin{array}{lll}
\text{bandwidth} & \omega_B & \\
\text{damping ratio of control poles} & \zeta & \\
\text{positional error constant} & K_p = \infty & \\
\text{velocity error constant} & K_v = \infty & \\
\text{acceleration error constant} & K_a = \dfrac{K_s}{Jb} & \text{(A. 19)}
\end{array}
$$

The order of magnitude of $\beta$, $\zeta$ and $K_s$ is in an airborne application 100 rad/sec, 0.7, 1 Nm/rad., respectively.

A system with these specifications can easily be obtained by the synthesis-procedure given by Truxal. See reference 4, chapter 5.

A. 2  The specifications can be satisfied by a pole-zero configuration of the overall system according to figure A. 2 where the position of the zero is determined by the condition on the velocity constant
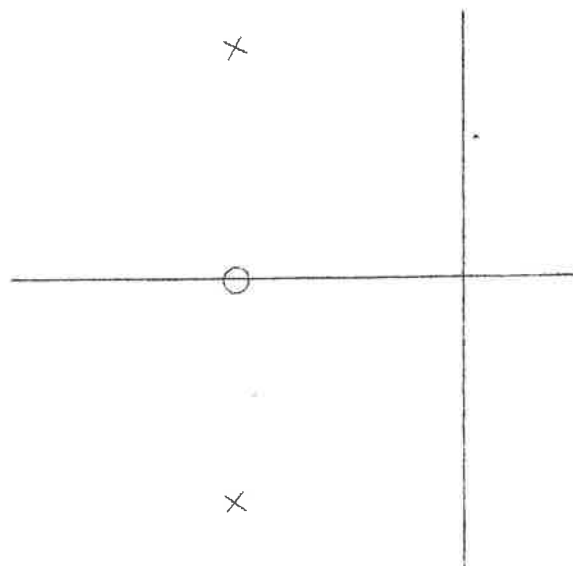


Figure A. 2

Pole-zero location for overall system transfer function $Y(p)$.

The over-all system transfer function is

$$Y(p) = \frac{2\zeta\beta p + \beta^2}{p^2 + 2\zeta\beta p + \beta^2} \qquad (A.21)$$

The constant $\beta$ is chosen so that the bandwidth condition is satisfied. Suppose $\zeta = 0.7$ then $\beta = \frac{1}{2}\omega_B$. Cf. ref. 4, figure 3.5.

The open-loop transfer function is

$$Y_o(p) = \frac{Y(p)}{1 - Y(p)} = \frac{2\zeta\beta p + \beta^2}{p^2} \qquad (A.22)$$

Equation (A.15) gives

$$Y_1(p) = p(1 + Y_o(p)) = p\ \frac{Y_o(p)}{Y(p)} = \frac{p^2 + 2\zeta\beta p + \beta^2}{p} \qquad (A.23)$$

thus

$$p\left[Y_1(p) - p\right] = 2\zeta\beta p + \beta^2 \qquad (A.24)$$

Equation (A.16) gives

$$K_s = Jb\beta^2 \qquad (A.25)$$

The transfer function $\tau(p)$ is obtained from equation (A.12)

$$\tau(p) = \frac{ab}{\omega_o} \cdot \frac{1}{p}\left[p^2 + \sigma(p)\right]\left[2\zeta\beta p + \beta^2\right] - \omega_o p \qquad (A.26)$$

A. 3 It is desirable to have simple expressions for $\tau(p)$ as this transfer function must be realized with some networks.

We obtain an expression simpler than (A. 26) by choosing a pole-zero configuration of the overall system according to figure A. 3.
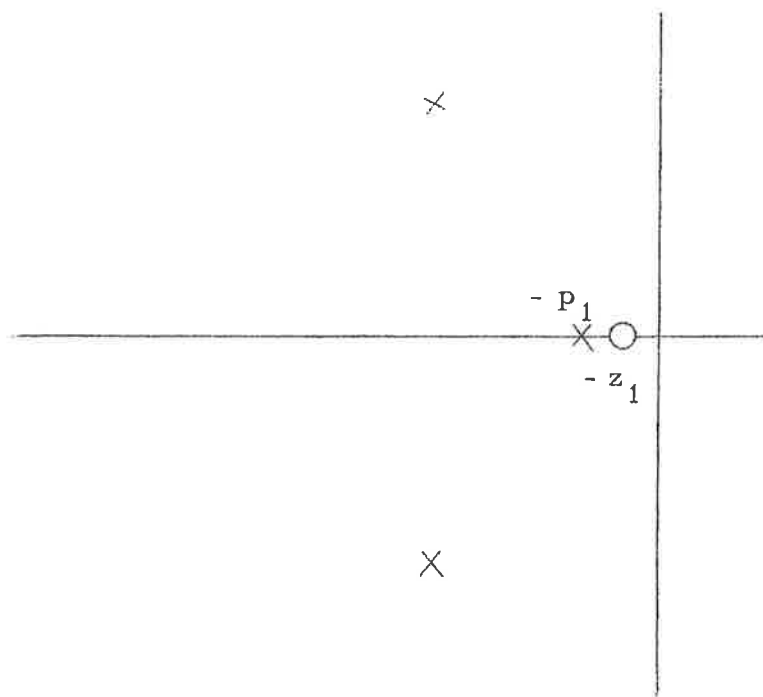


Figure A. 3

Pole-zero location for overall system transfer function $Y(p)$

We obtain

$$Y(p) = \frac{\beta^2 \, p_1}{z_1} \cdot \frac{(p + z_1)}{(p^2 + 2\zeta \, \beta p + \beta^2)(p + p_1)} \tag{A. 301}$$

By proper design of the system the steady-state behaviour is essentially determined by $p_1$ and $z_1$ and the transient behaviour by $\beta$ and $\zeta$. The damping ratio of the control poles $\zeta$ is given directly by the specifications and the constant $\beta$ is chosen in order to satisfy the bandwidth condition. If $p_1$ and $z_1$ should not affect the transient behaviour too much the quotient $p_1/z_1$ should be near 1, say

$$1 < \frac{p_1}{z_1} < 1.1 \tag{A. 302}$$

according to Truxal. After the choice of $P_1/z_1$ the zero $z_1$ is determined by the condition on the velocity coefficient.

$$2 \zeta \beta P_1 + \beta^2 - \frac{\beta^2 P_1}{z_1} = 0 \qquad \text{(A.303)}$$

The open loop transfer function $Y_o(p)$ is

$$Y_o(p) = \frac{Y(p)}{1 - Y(p)} = \frac{\beta^2 P_1}{z_1} \cdot \frac{p + z_1}{p^2 (p + P_2)} \qquad \text{(A.304)}$$

where

$$P_2 = P_1 + 2 \zeta \beta \qquad \text{(A.305)}$$

Equation (A.16) gives

$$K_s = J b \beta^2 \cdot \frac{P_1}{P_2} \qquad \text{(A.306)}$$

From equation (A.14) we obtain

$$Y_1(p) = p \left[ 1 + Y_o(p) \right] = p \, \frac{Y_o(p)}{Y(p)} \qquad \text{(A.307)}$$

thus

$$Y_1(p) = \frac{(p^2 + 2 \zeta \beta p + \beta^2)(p + P_1)}{p (p + P_2)} \qquad \text{(A.309)}$$

and

$$p \left[ Y_1(p) - p \right] = p^2 \, Y_o(p) = \frac{\beta^2 P_1}{z_1} \, \frac{p + z_1}{p + P_2} \qquad \text{(A.310)}$$

The transfer function $\tau(p)$ is obtained from equation (A.13)

$$\tau(p) = \frac{a b \beta^2 P_1}{\omega_o z_1} \, \frac{a_3 p^3 + a_2 p^2 + \sigma(p) (p + z_1)}{p (p + P_2)} \qquad \text{(A.311)}$$

where

$$a_3 = 1 - \frac{\omega_o^2 z_1}{a b \beta^2 p_1}$$

$$a_2 = z_1 - \frac{\omega_o^2 z_1 p_2}{a b \beta^2 p_1}$$

Suppose that we can choose $a_3 = 0$ consistent with equation (A.302) i.e.

$$\frac{p_1}{z_1} = \frac{\omega_o^2}{a b \beta^2} \tag{A.312}$$

and

$$1 < \frac{\omega_o^2}{a b \beta^2} < 1.1 \qquad * \tag{A.313}$$

Equations (A.303), (A.305) and (A.312) give

$$p_1 = \frac{\beta}{2\zeta} \left( \frac{\omega_o^2}{a b \beta^2} - 1 \right) \tag{A.314}$$

$$p_2 = \frac{\beta}{2\zeta} \left[ 4\zeta^2 + \frac{\omega_o^2}{a b \beta^2} - 1 \right] \tag{A.315}$$

$$z_1 = \frac{\beta}{2\zeta} \left( 1 - \frac{a b \beta^2}{\omega_o^2} \right) \tag{A.316}$$

Equation (A.311) then reduces to

$$\tau(p) = \frac{1}{\omega_o} \frac{p^2(z_1 - p_2) + \sigma(p)(p + z_1)}{p(p + p_2)} \tag{A.317}$$

Suppose further that we use an integrating gyro and that its torque ~~motor~~ generator.

---

\* This means that the moment of inertia of the stable element should match the bandwidth of the system.

is not used for feedback purposes, i.e.

$$\sigma(p) = a p$$

then

$$\tau(p) = \frac{1}{\omega_o} \cdot \frac{p(z_1 + a - P_2) + a z_1}{p + P_2} \qquad \text{(A. 318)}$$

A. 4  If the disturbing torques have a high level it may be necessary to claim

$$K_s = \infty \qquad \text{(A. 401)}$$

for proper performance of the platform system. The specifications (A. 17) with this condition added can be met with by choosing a pole-zero configuration of the overall system according to figure A. 41.
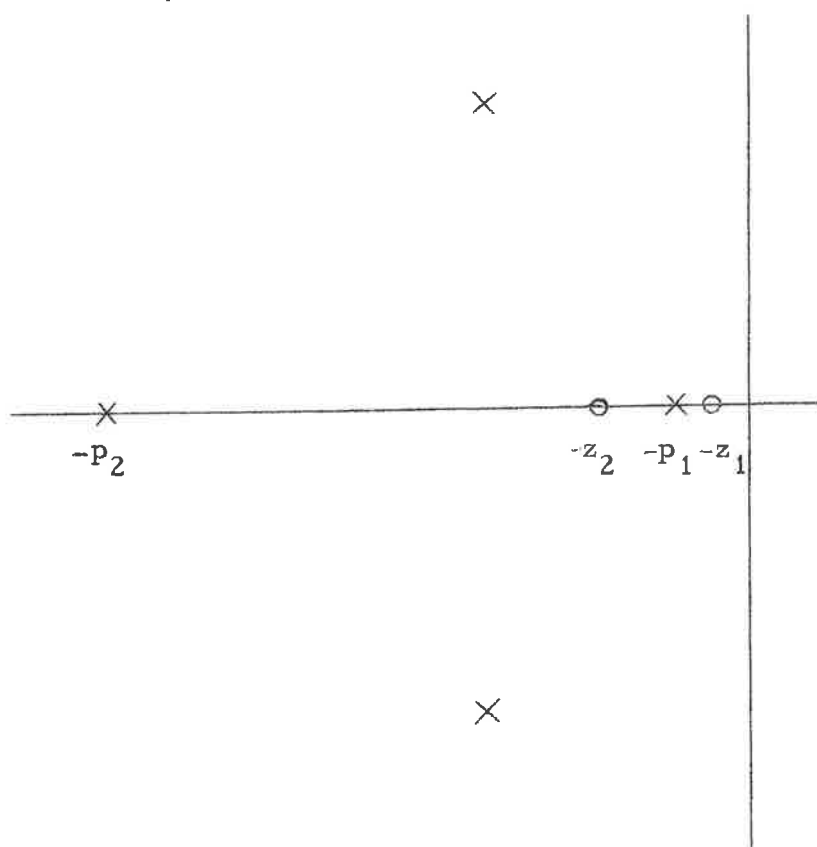


Figure A. 41

Pole-zero configuration of overall system transfer function.

The overall system function is

$$Y(p) = \frac{\beta^2 P_1 P_2}{z_1 z_2} \cdot \frac{(p + z_1)(p + z_2)}{(p^2 + 2\zeta\beta p + \beta^2)(p + P_1)(p + P_2)} \qquad (A.402)$$

The specifications of the velocity and acceleration constants give

$$2\zeta\beta P_1 P_2 + \beta^2 P_1 + \beta^2 P_2 - \frac{\beta^2 P_1 P_2}{z_1} - \frac{\beta^2 P_1 P_2}{z_2} = 0 \qquad (A.403)$$

$$P_1 P_2 + 2\zeta\beta P_1 + 2\zeta\beta P_2 + \beta^2 - \frac{\beta^2 P_1 P_2}{z_1 z_2} = 0 \qquad (A.404)$$

The open-loop transfer function is

$$Y_0(p) = \frac{Y(p)}{1 - Y(p)} = \frac{\beta^2 P_1 P_2}{z_1 z_2} \cdot \frac{(p + z_1)(p + z_2)}{p^3 (p + P_3)} \qquad (A.405)$$

where

$$P_3 = P_1 + P_2 + 2\zeta\beta \qquad (A.406)$$

Equation (A.14) gives

$$Y_1(p) = \frac{(p^2 + 2\zeta\beta p + \beta^2)(p + P_1)(p + P_2)}{p^2 (p + P_3)} \qquad (A.407)$$

thus

$$p\left[Y_1(p) - p\right] = \frac{\beta^2 P_1 P_2}{z_1 z_2} \cdot \frac{(p + z_1)(p + z_2)}{p (p + P_3)} \qquad (A.408)$$

Equation (A.13) gives

$$\tau(p) = \frac{ab\beta^2 P_1 P_2}{\omega_0 z_1 z_2} \cdot \frac{a_4 p^4 + a_3 p^3 + p^2 z_1 z_2 + \sigma(p)(p + z_1)(p + z_2)}{p^2 (p + P_3)}$$

$$(A.409)$$

where

$$a_4 = 1 - \frac{\omega_o^2 z_1 z_2}{a b \beta^2 P_1 P_2} \qquad (A.410)$$

$$a_3 = z_1 + z_2 - \frac{\omega_o^2 z_1 z_2 P_3}{a b \beta^2 P_1 P_2} \qquad (A.411)$$

Choosing

$$\frac{P_1}{z_1} \frac{P_2}{z_2} = \frac{\omega_o^2}{a b \beta^2} \qquad (A.412)$$

we get

$$\tau(p) = \frac{1}{\omega_o} \cdot \frac{p^3(z_1 + z_2 - P_3) + p^2 z_1 z_2 + \sigma(p)(p + z_1)(p + z_2)}{p^2(p + P_3)} \qquad (A.413)$$

We have still one condition left before $P_1 P_2 z_1 z_2$ are determined.

PLATE 3.1

SYSTEMS WITH ZERO SPIN



$$\left[-\frac{3\pi}{4}, -\frac{3\pi}{4}, -\frac{3\pi}{4}\right] \quad S = 0, \ 1 = -\frac{\sqrt{2}}{2}$$

$$\left[-\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}\right] \quad S = 0, \ 1 = -\frac{\sqrt{2}}{2}$$

$$\left[\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}\right] \quad S = 0, \ 1 = \frac{\sqrt{2}}{2}$$
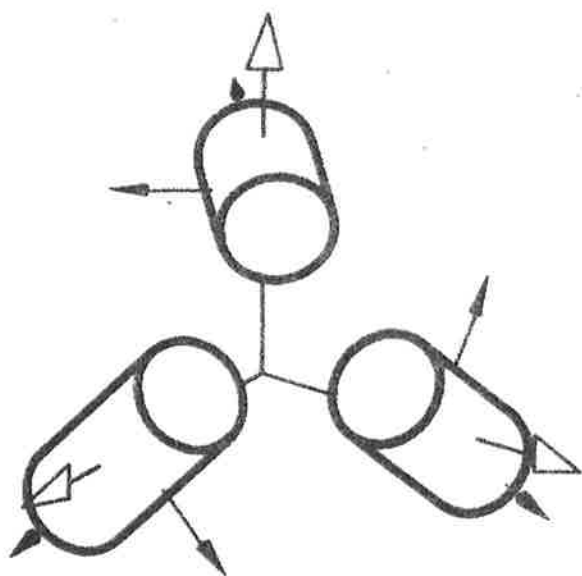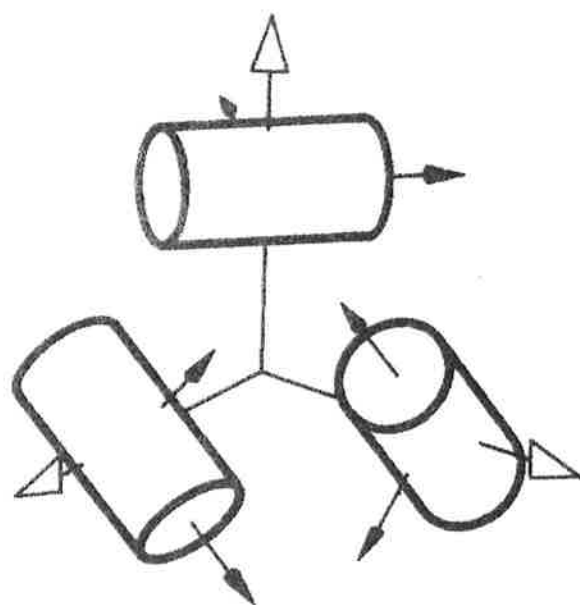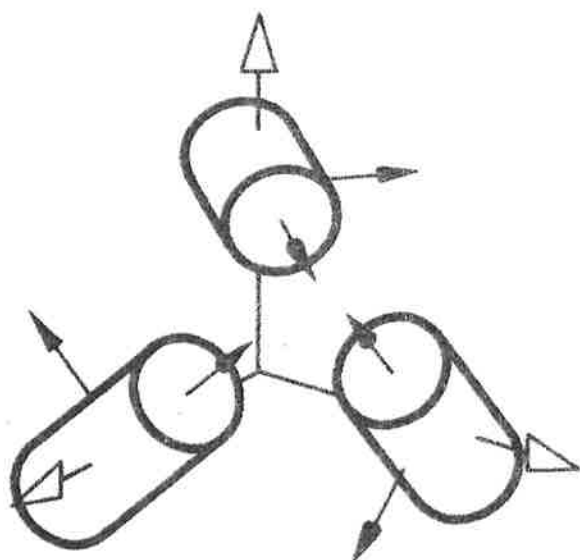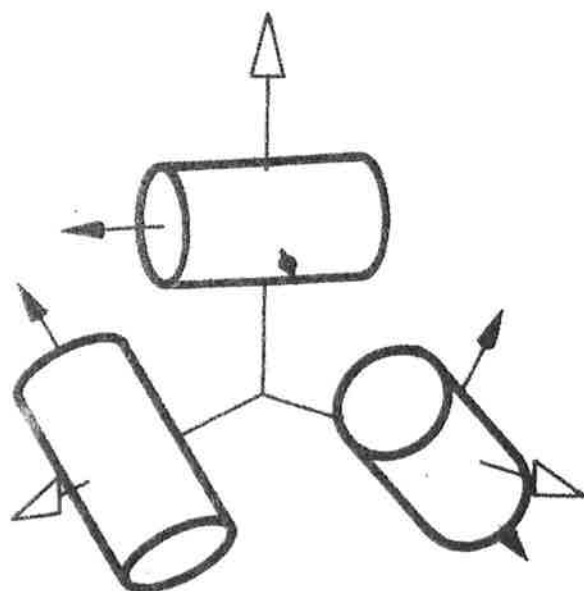
$$\left[\frac{3\pi}{4}, -\frac{3\pi}{4}, -\frac{\pi}{4}\right] \quad S = 0, \ 1 = \frac{\sqrt{2}}{2}$$

PLATE 3.2

SYSTEMS WITH SPIN SIX



$$\left[\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{3\pi}{4}\right] \quad S = 6, I = 0$$

$$\left[\frac{\pi}{4}, -\frac{\pi}{4}, -\frac{3\pi}{4}\right] \quad S = 6, I = 0$$

$$\left[-\frac{\pi}{4}, -\frac{\pi}{4}, -\frac{\pi}{4}\right] \quad S = 6, I = 0$$
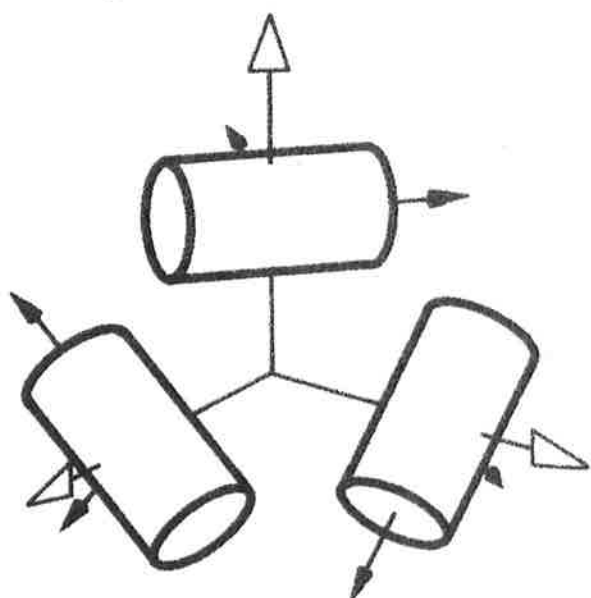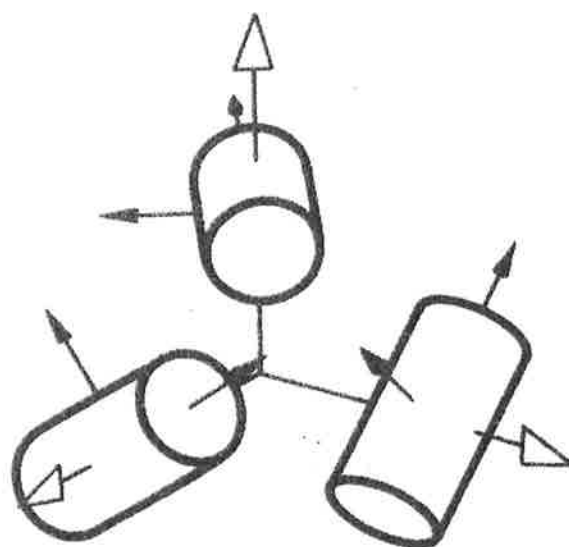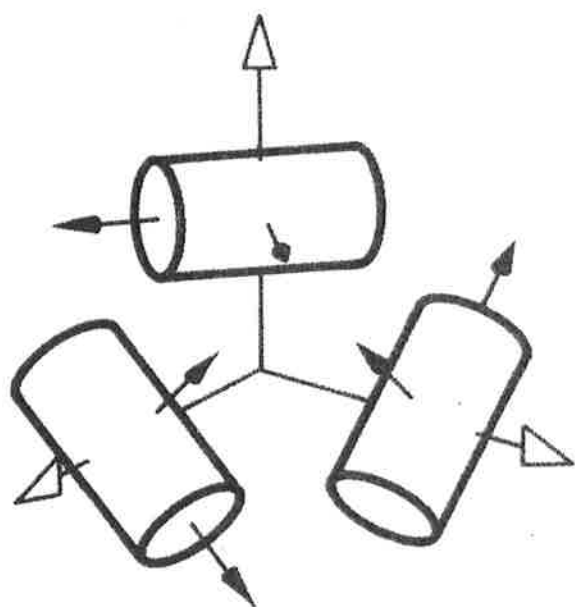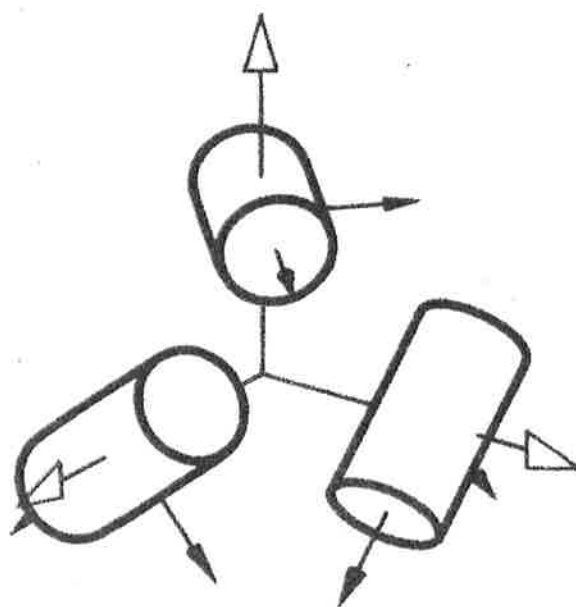
$$\left[-\frac{3\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right] \quad S = 6, I = 0$$

PLATE 3. 3

SYSTEMS WITH SPIN THREE AND OUTPUT AXES IN THE SAME PLANE



$$\left[-\frac{\pi}{2},\ 0\ ,\ -\frac{3\pi}{4}\right]\quad S = 3\ ,\ 1 = 0$$



$$\left[-\frac{\pi}{2},\ 0\ ,\ \frac{\pi}{4}\right]\quad S = 3\ ,\ 1 = 0$$



$$\left[-\frac{\pi}{2},\ \pi\ ,\ -\frac{\pi}{4}\right]\quad S = 3\ ,\ 1 = 0$$
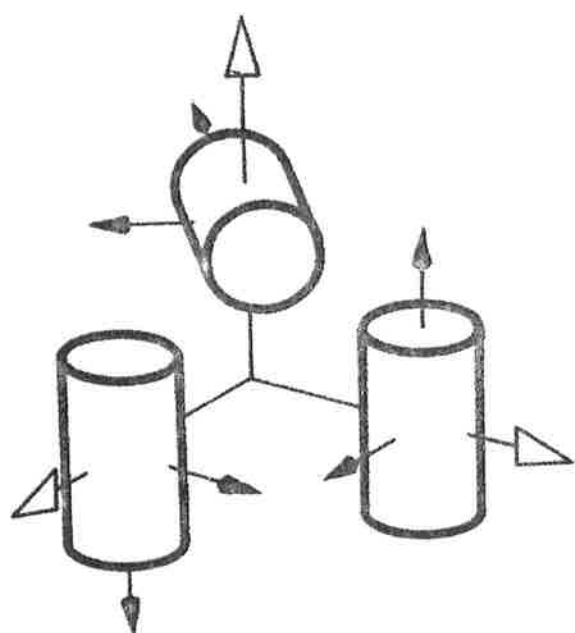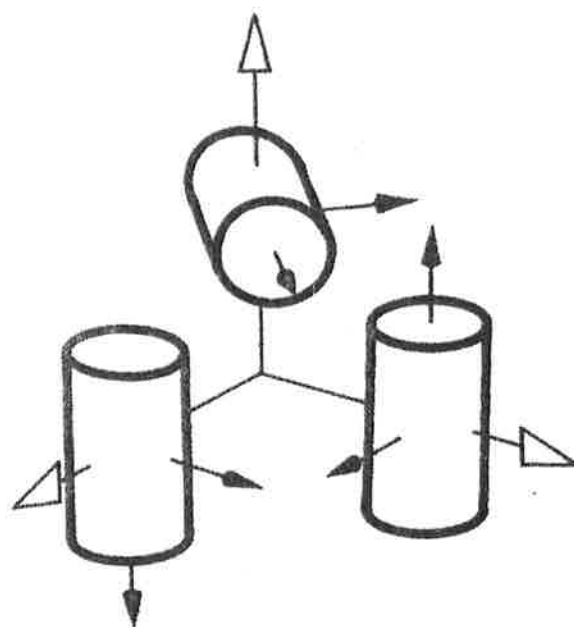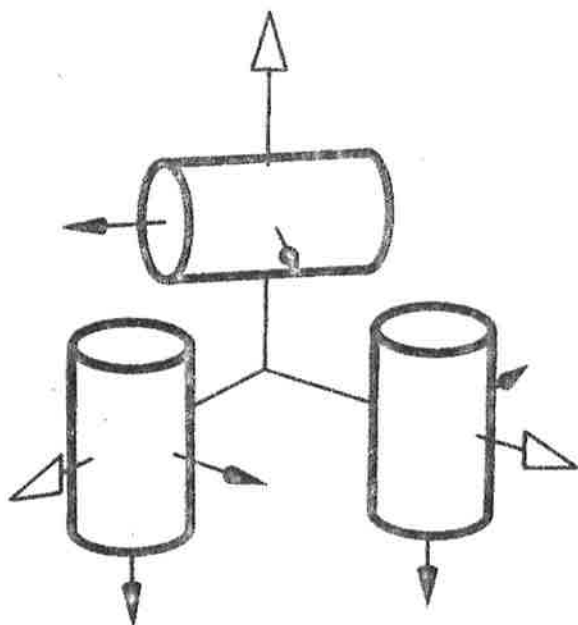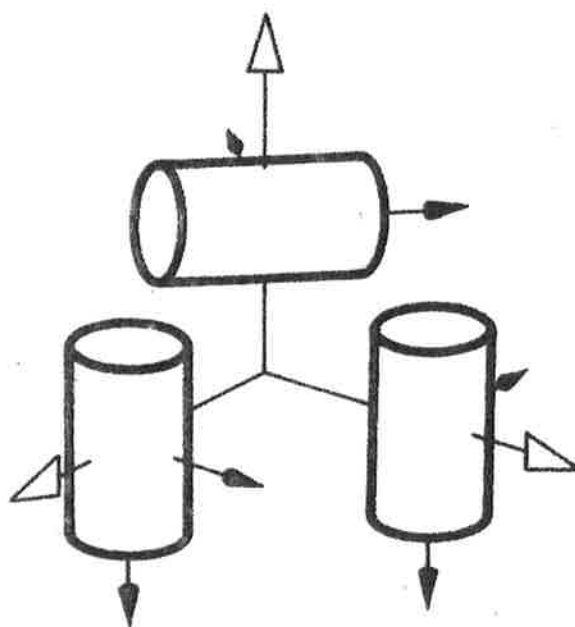


$$\left[-\frac{\pi}{2},\ \pi\ ,\ \frac{3\pi}{4}\right]\quad S = 3\ ,\ 1 = 0$$

PLATE 3. 4

SYSTEMS WITH SPIN THREE AND OUTPUT AXES IN THE SAME PLANE.



$$\left[\frac{\pi}{2}, \ 0 \ , \ -\frac{\pi}{4}\right] \ S = 3 \ , \ I = 0$$

$$\left[\frac{\pi}{2}, \ 0 \ , \ \frac{3\pi}{4}\right] \ \ S = 3 \ , \ I = 0$$



$$\left[\frac{\pi}{2}, \ \pi \ , \ -\frac{3\pi}{4}\right] \ \ S = 3 \ , \ I = 0$$
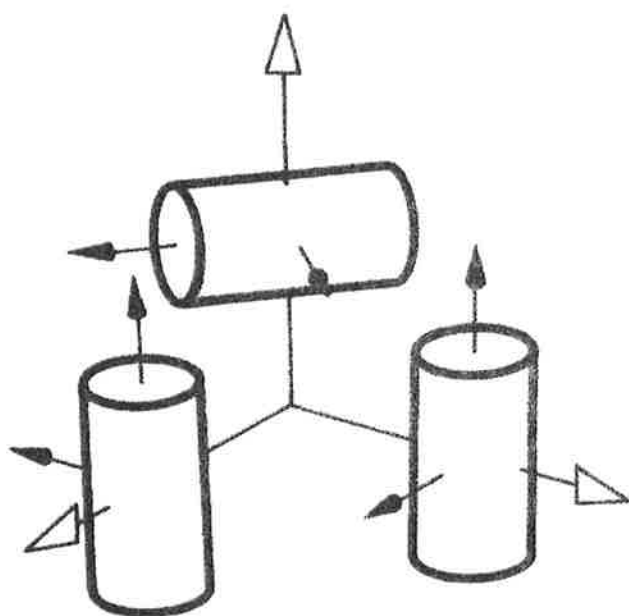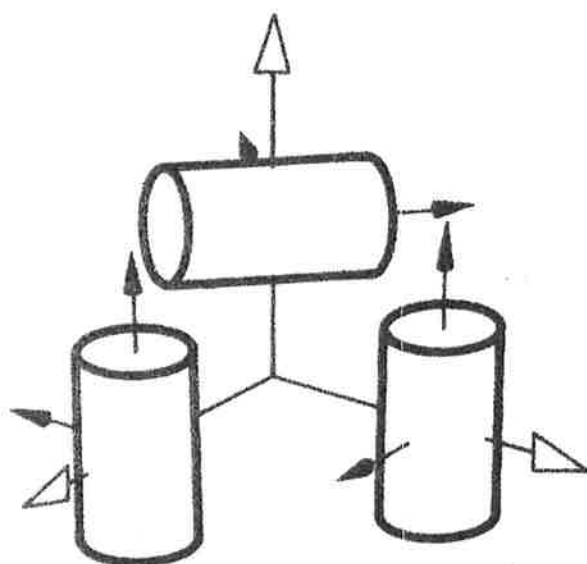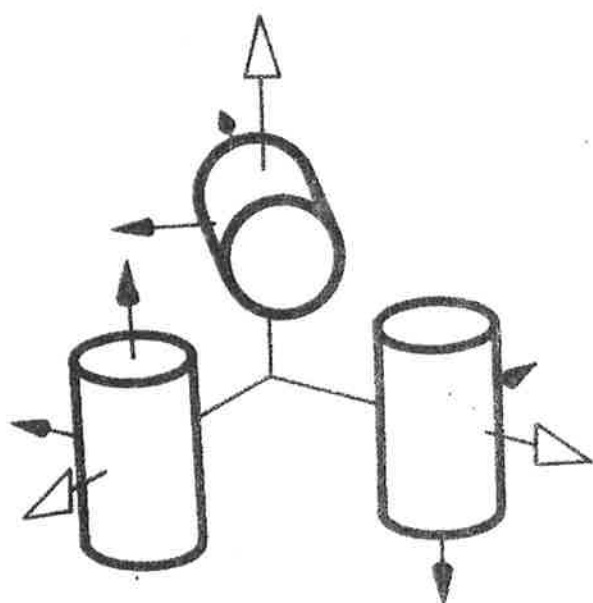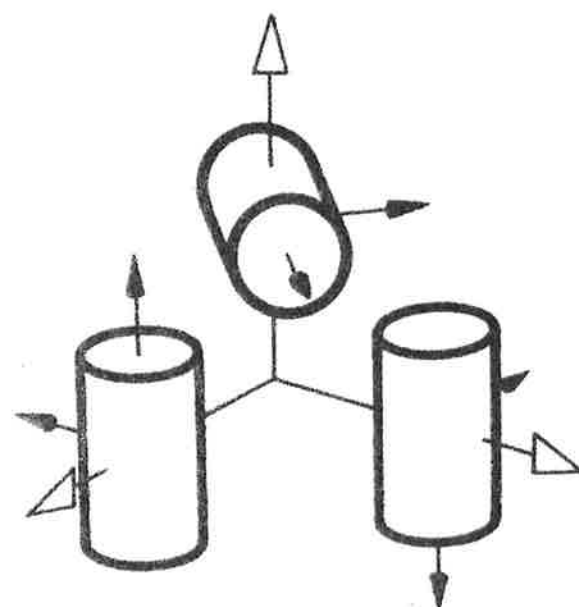
$$\left[\frac{\pi}{2}, \ \pi \ , \ \frac{\pi}{4}\right] \ \ S = 3 \ , \ I = 0$$

## PLATE 3.5

ORTHOGONAL SYSTEMS WITH SPIN ONE



$$\left[0, 0, \frac{\pi}{2}\right] \quad S = 1, \, I = 0$$
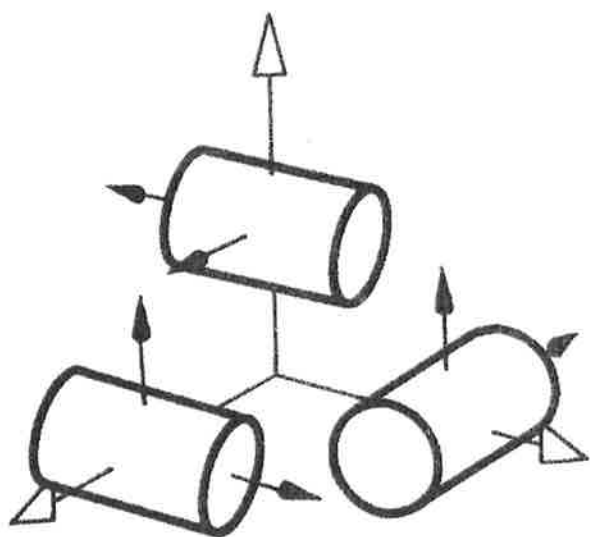
$$\left[0, \frac{\pi}{2}, \frac{\pi}{2}\right] \quad S = 1, \, I = 0$$

$$\left[0, \frac{\pi}{2}, \pi\right] \quad S = 1, \, I = 0$$
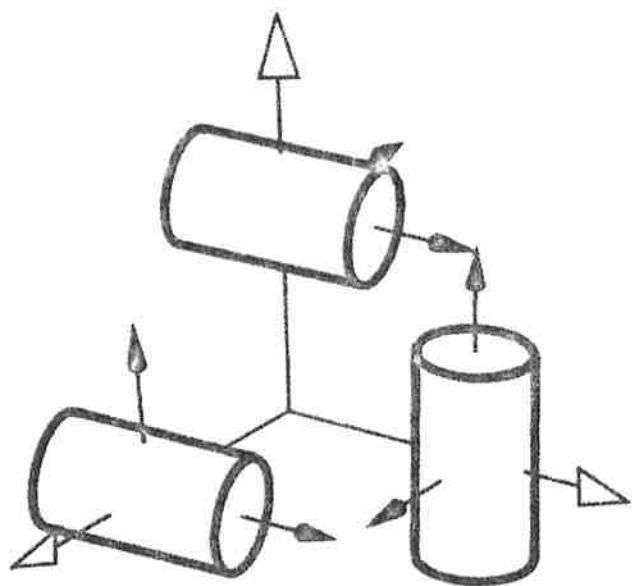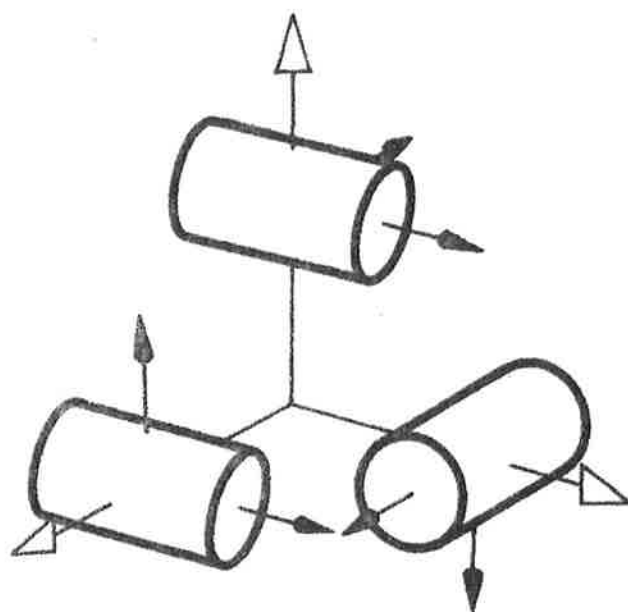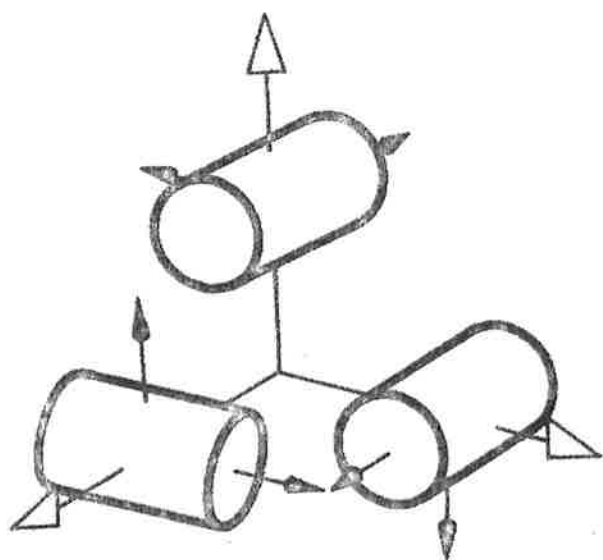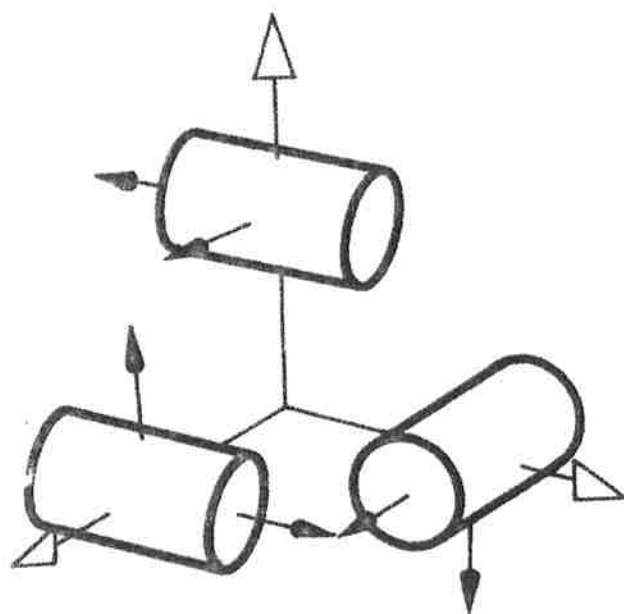
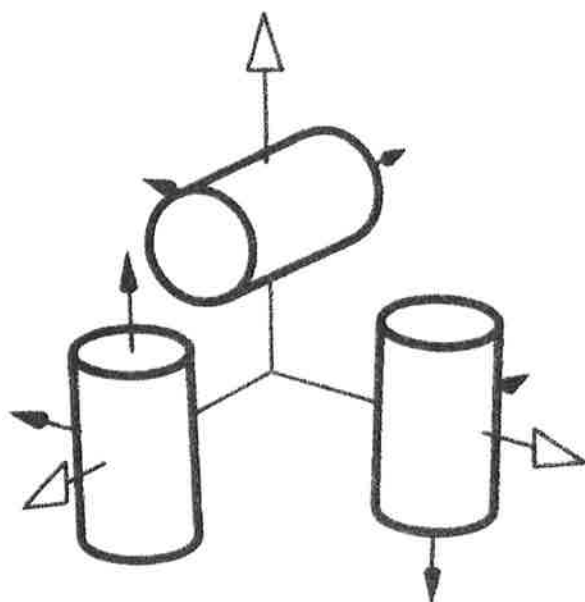$$\left[0, \frac{\pi}{2}, -\frac{\pi}{2}\right] \quad S = 1, \, I = 0$$

PLATE 3. 6

## ORTHOGONAL SYSTEMS WITH SPIN ONE



$$\left[ 0 , \pi , -\frac{\pi}{2} \right] \quad S=1 , I=0$$

$$\left[ \frac{\pi}{2} , \pi , -\frac{\pi}{2} \right] \quad S=1 , I=0$$

$$\left[ \pi , \pi , -\frac{\pi}{2} \right] \quad S=1 , I=0$$
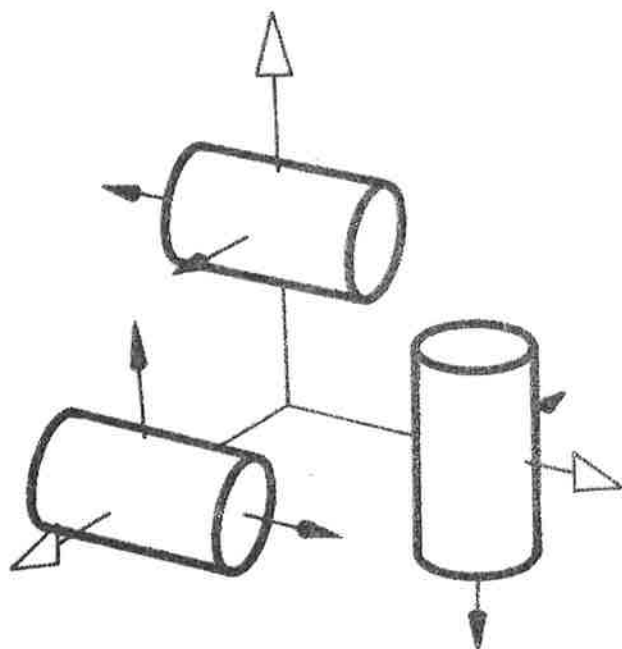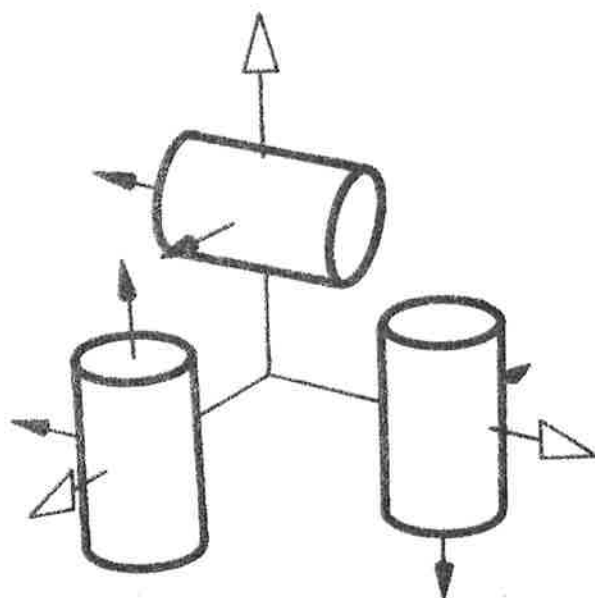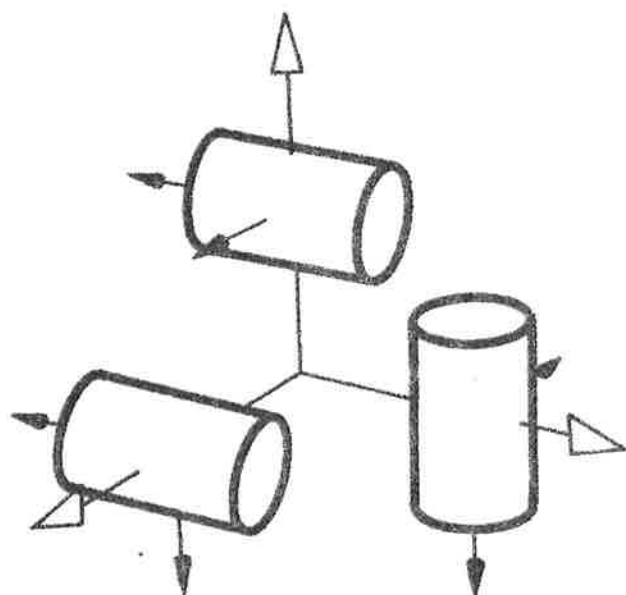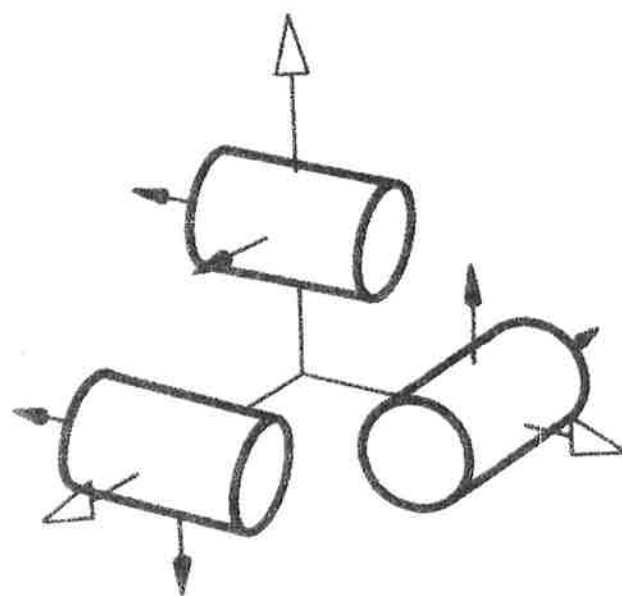
$$\left[ \pi , -\frac{\pi}{2} , -\frac{\pi}{2} \right] \quad S=1 , I=0$$

PLATE 3.7

ORTHOGONAL SYSTEMS WITH SPIN THREE



$$\left[\pi, \pi, \pi\right] \quad S = 3, I = -1$$

$$\left[-\frac{\pi}{2}, -\frac{\pi}{2}, -\frac{\pi}{2}\right] \quad S = 3, I = -1$$

$$\left[0, 0, \pi\right] \quad S = 3, I = -1$$
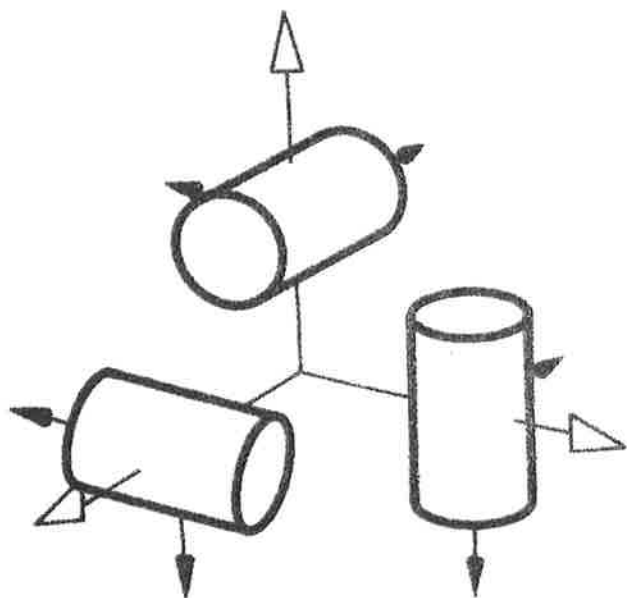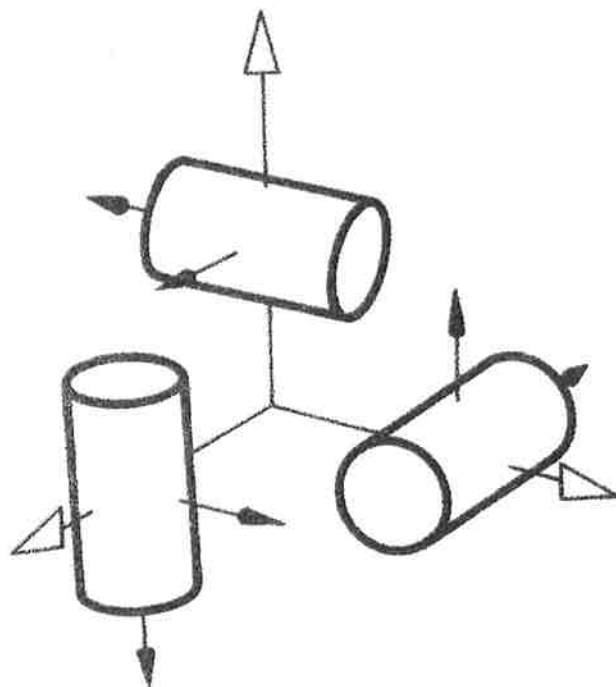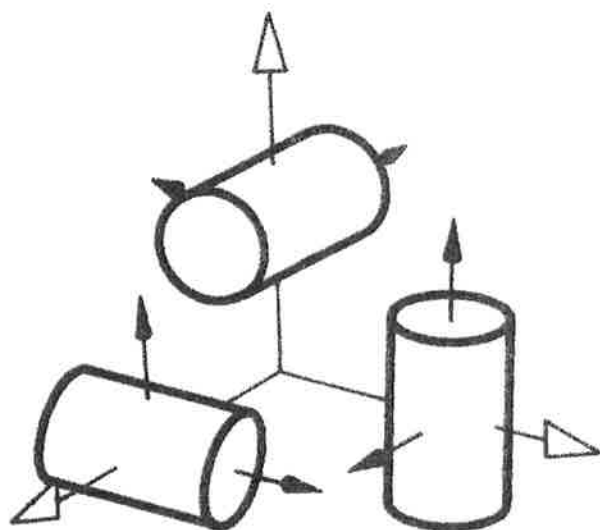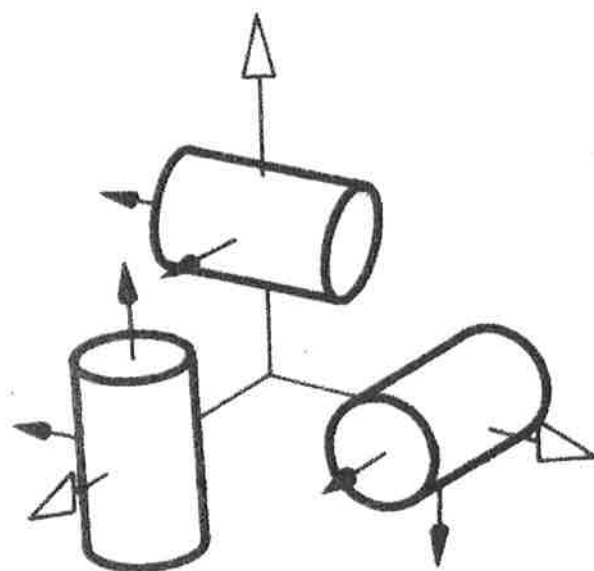
$$\left[\frac{\pi}{2}, \frac{\pi}{2}, -\frac{\pi}{2}\right] \quad S = 3, I = -1$$

PLATE 3.8

ORTHOGONAL SYSTEMS WITH SPIN THREE



$$\begin{bmatrix} 0 , 0 , 0 \end{bmatrix} \quad S = 3 , I = 1$$

$$\begin{bmatrix} \dfrac{\pi}{2}, \dfrac{\pi}{2}, \dfrac{\pi}{2} \end{bmatrix} \quad S = 3 , I = 1$$



ORTHOGONAL SYSTEMS WITH SPIN THREE

$$\begin{bmatrix} 0 , \pi , \pi \end{bmatrix} \quad S = 3 , I = 1$$
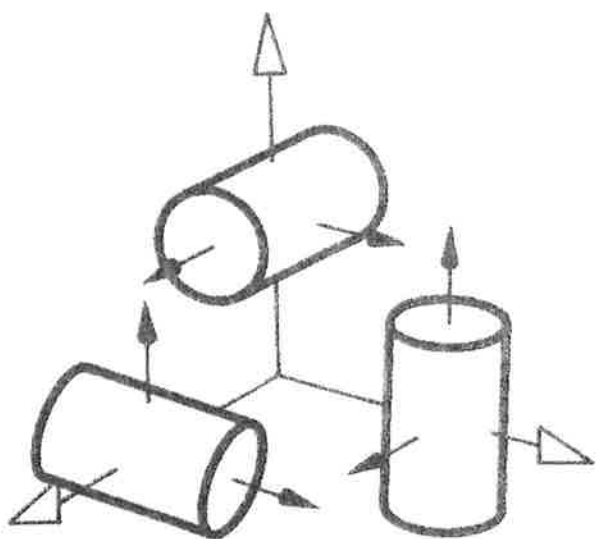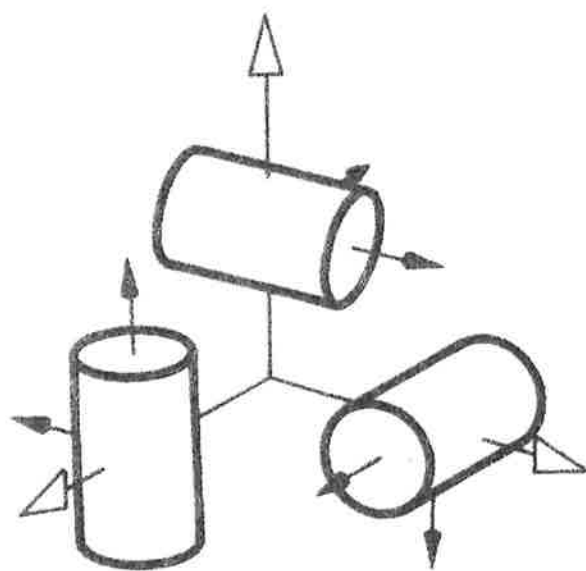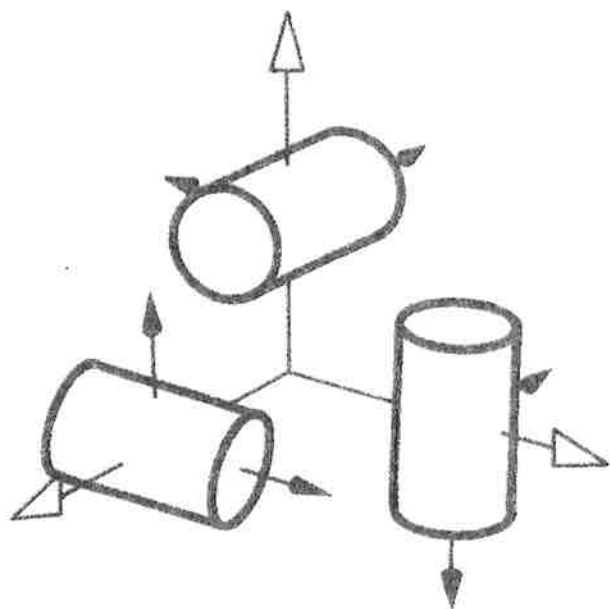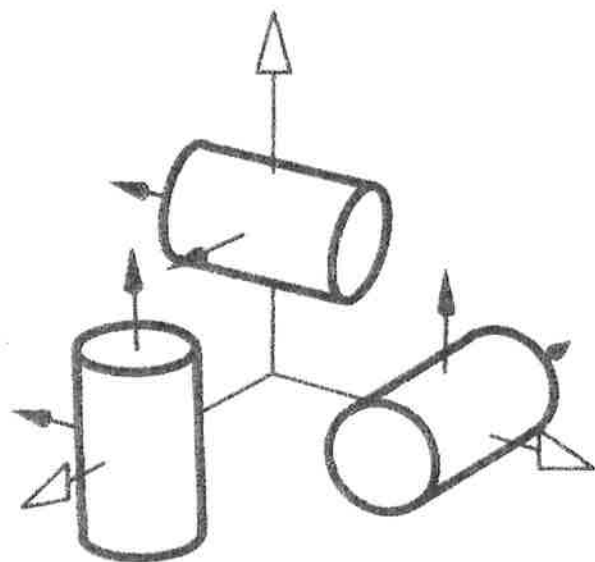
$$\begin{bmatrix} \dfrac{\pi}{2}, -\dfrac{\pi}{2}, -\dfrac{\pi}{2} \end{bmatrix} \quad S = 3 , I = 1$$

# PLATE 3.9

## ORTHOGONAL SYSTEMS WITH SPIN FIVE



$$\left[0, 0, -\frac{\pi}{2}\right] \quad s = 5, \ 1 = 0$$

$$\left[0, \pi, -\frac{\pi}{2}\right] \quad s = 5, \ 1 = 0$$

$$\left[0, -\frac{\pi}{2}, \frac{\pi}{2}\right] \quad s = 5, \ 1 = 0$$
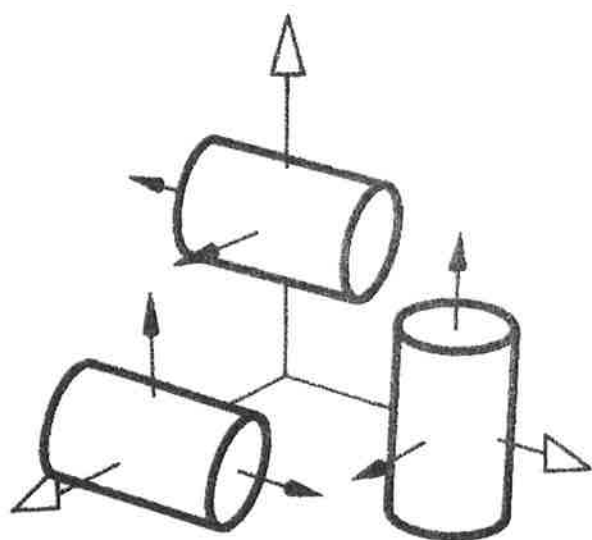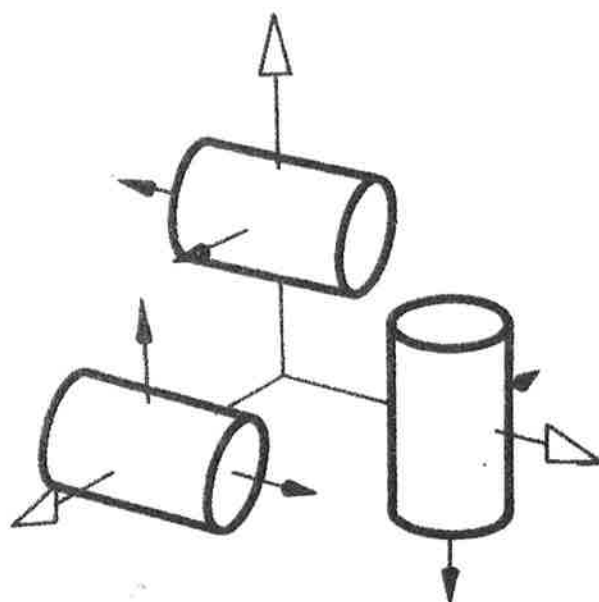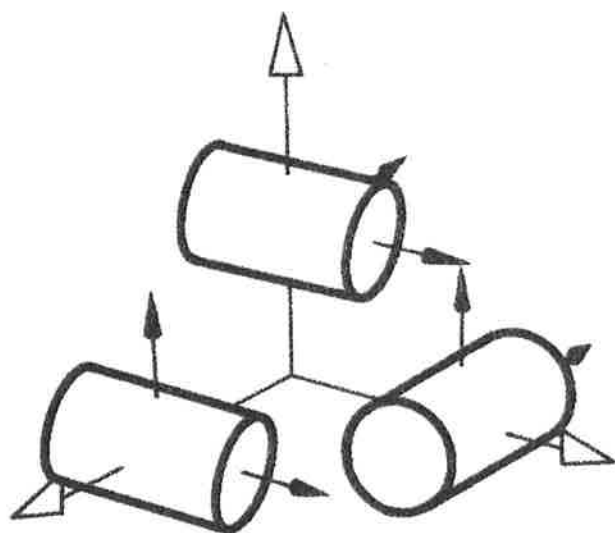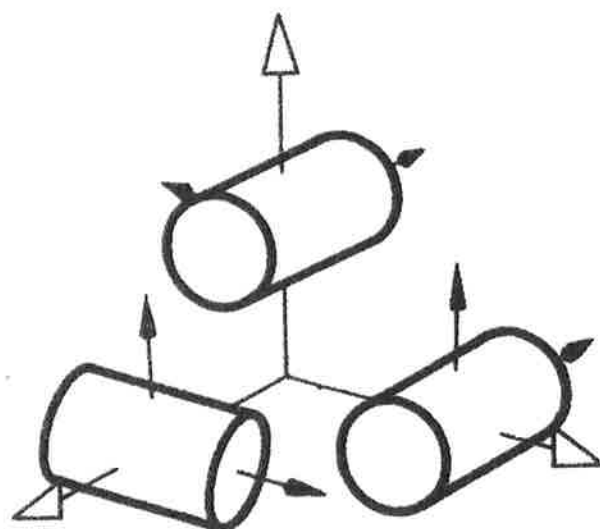
$$\left[0, -\frac{\pi}{2}, \pi\right] \quad s = 5, \ 1 = 0$$

PLATE 3.10

ORTHOGONAL SYSTEMS WITH SPIN FIVE



$$\left[ 0 , -\frac{\pi}{2} , -\frac{\pi}{2} \right] \quad s = 5 , \ 1 = 0$$

$$\left[ \frac{\pi}{2} , \frac{\pi}{2} , \pi \right] \quad s = 5 , \ 1 = 0$$

$$\left[ \frac{\pi}{2} , \pi , \pi \right] \quad s = 5 , \ 1 = 0$$
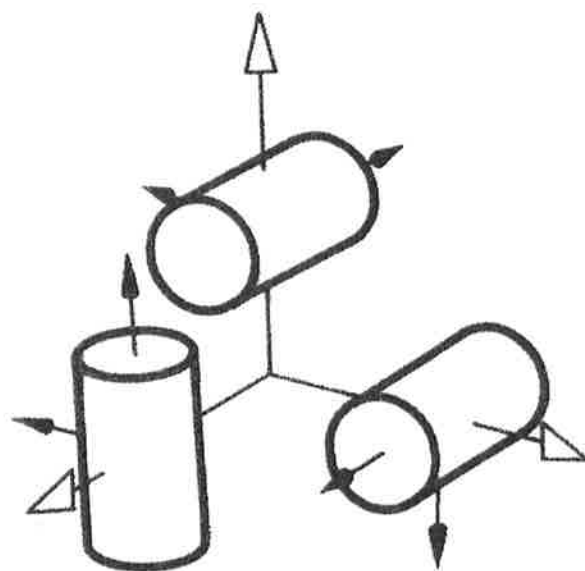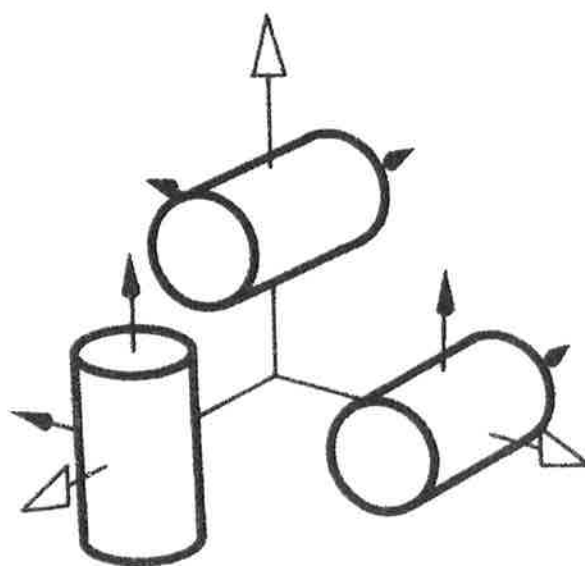
$$\left[ \frac{\pi}{2} , -\frac{\pi}{2} , \pi \right] \quad s = 5 , \ 1 = 0$$
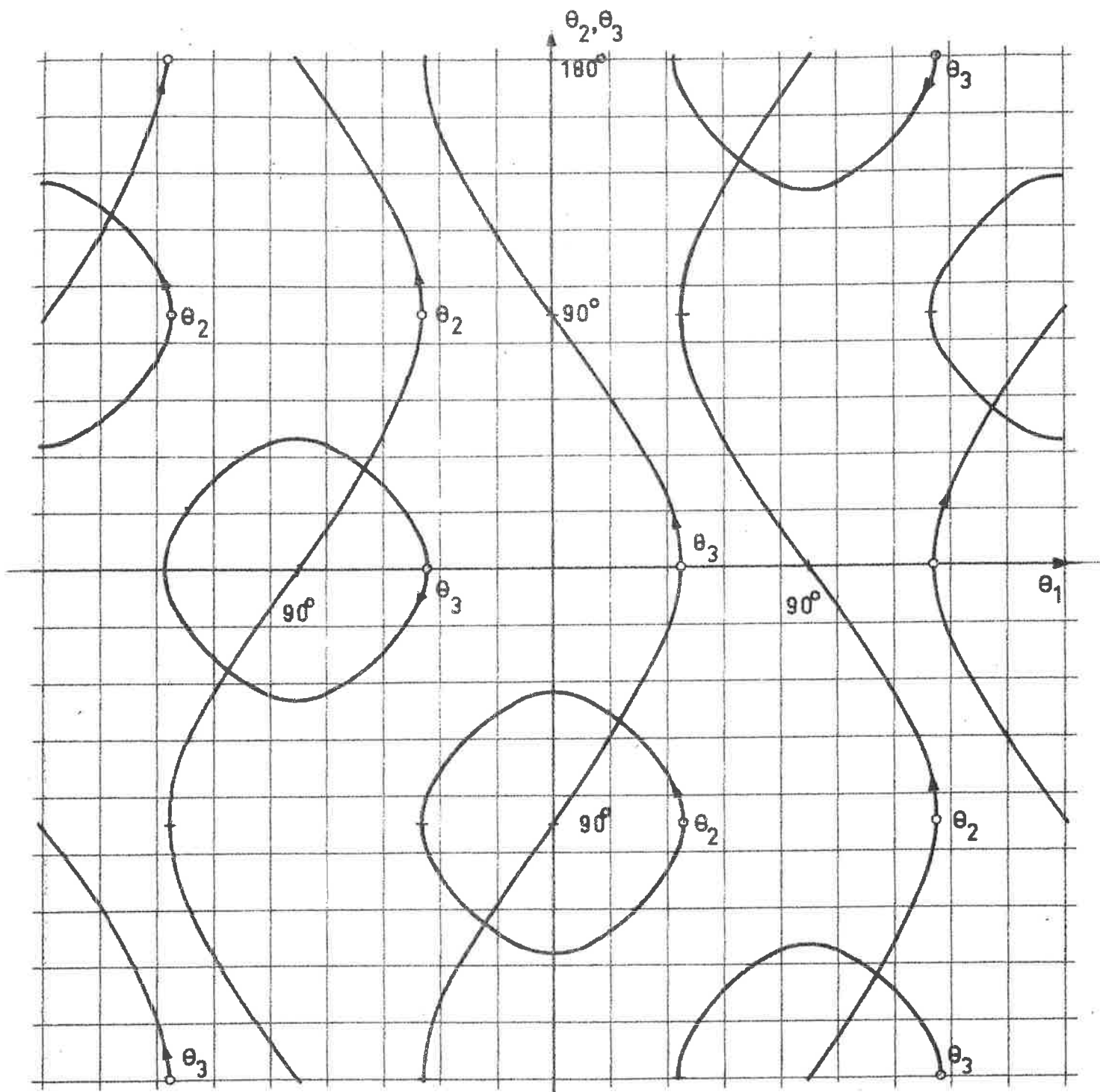
Plate 3.11

Solutions of the equations;

$$\begin{cases} \sin\theta^{(1)}\cos\theta^{(3)} + \sin\theta^{(2)}\cos\theta^{(1)} + \sin\theta^{(3)}\cos\theta^{(2)} = 0 \\ \sin\theta^{(1)}\sin\theta^{(2)}\sin\theta^{(3)} + \cos\theta^{(1)}\cos\theta^{(2)}\cos\theta^{(3)} = 0 \end{cases}$$
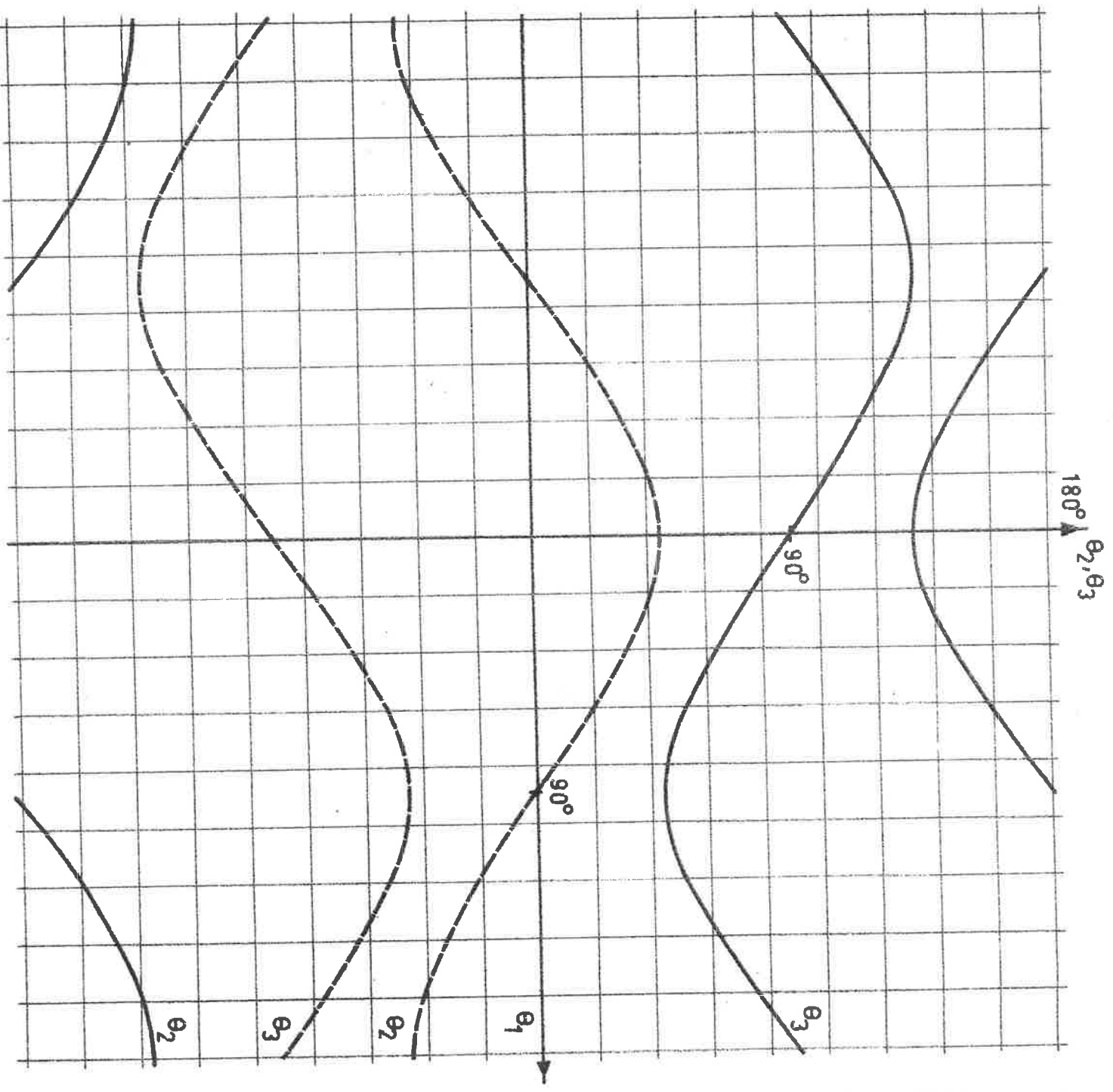
Plate 3.12

Solutions of the equations:

$$\begin{cases} \sin\theta^{(1)}\cos\theta^{(3)} + \sin\theta^{(2)}\cos\theta^{(1)} + \sin\theta^{(3)}\cos\theta^{(2)} = 0 \\ \sin\theta^{(1)}\sin\theta^{(2)}\sin\theta^{(3)} + \cos\theta^{(1)}\cos\theta^{(2)}\cos\theta^{(3)} = 0 \end{cases}$$

PLATE 8.2

ROOTLOCUS WITH RESPECT TO THE CROSSCOUPLING
COEFFICENT $\gamma$ FOR THE CHARACTERISTIC EQUATION
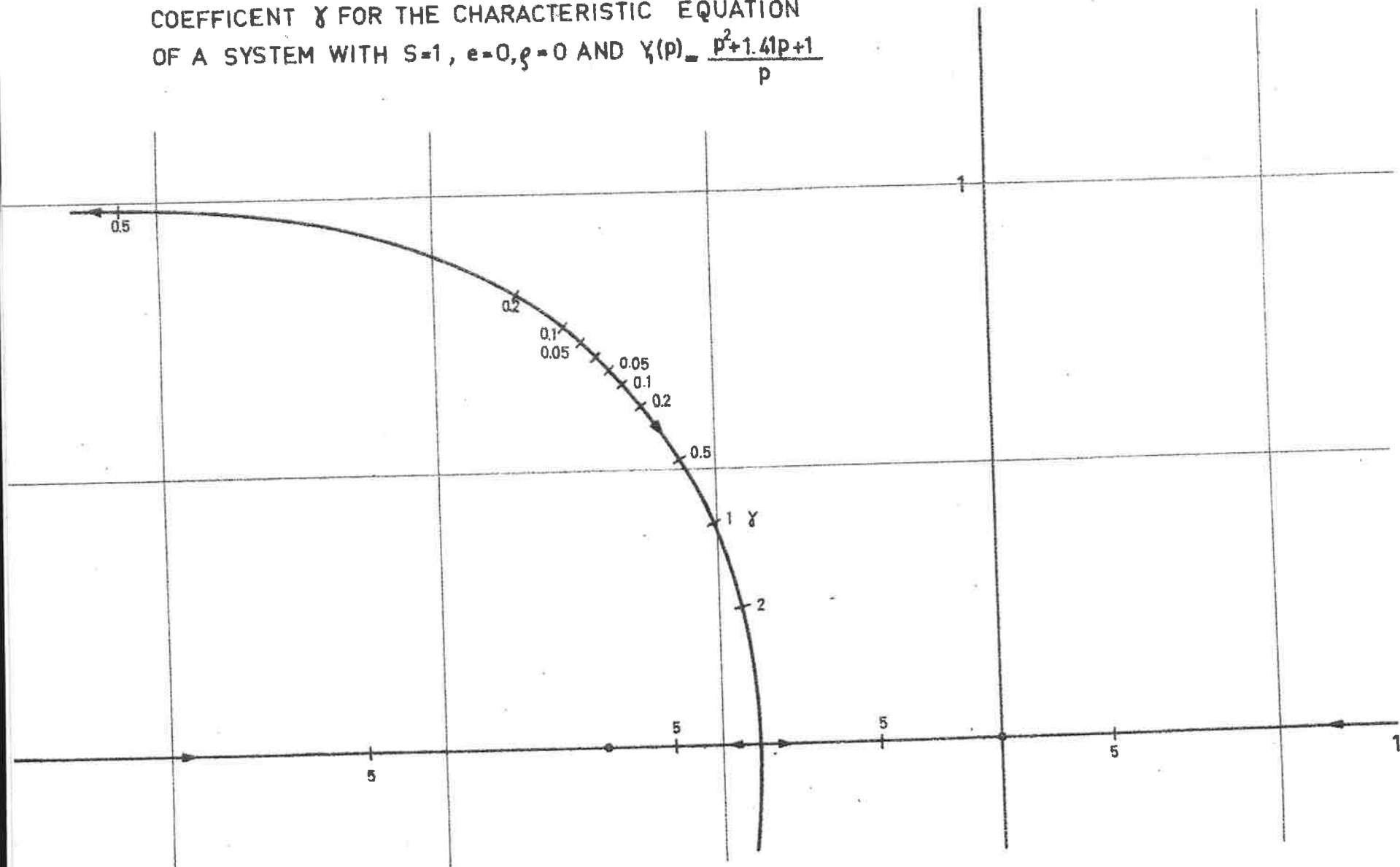OF A SYSTEM WITH $S=1$, $e=0, \varrho=0$ AND $\gamma(p) = \dfrac{p^2+1.41p+1}{p}$
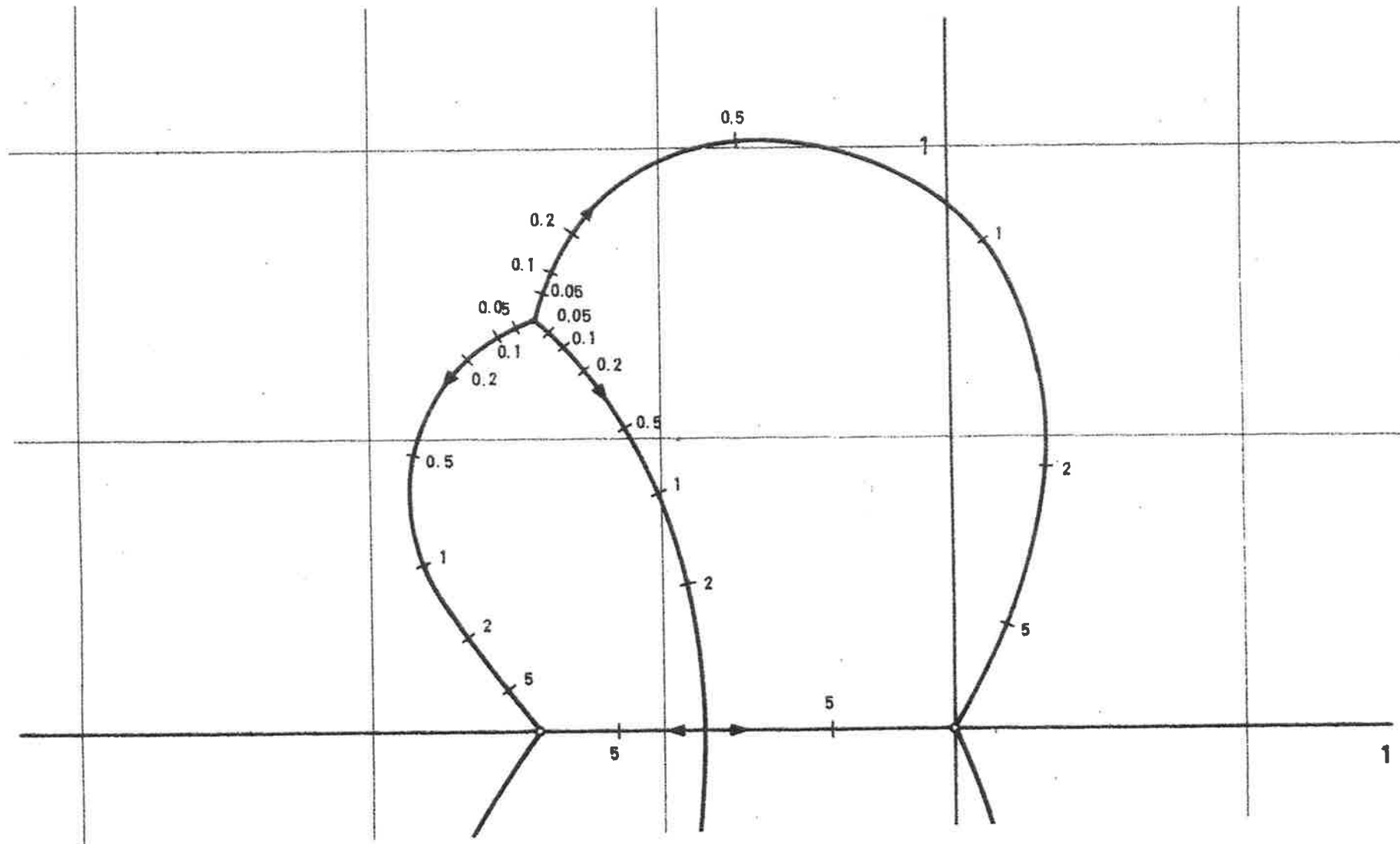
PLATE 8.22

ROOTLOCUS WITH RESPECT TO THE CROSSCOUPLING COEFFICIENT $\gamma$ FOR THE CHARACTERISTIC
EQUATION OF A SYSTEM WITH S = 3 e = -1 ç = 0 AND $Y_i(P) = \dfrac{P^2 + 1.41P + 1}{P}$
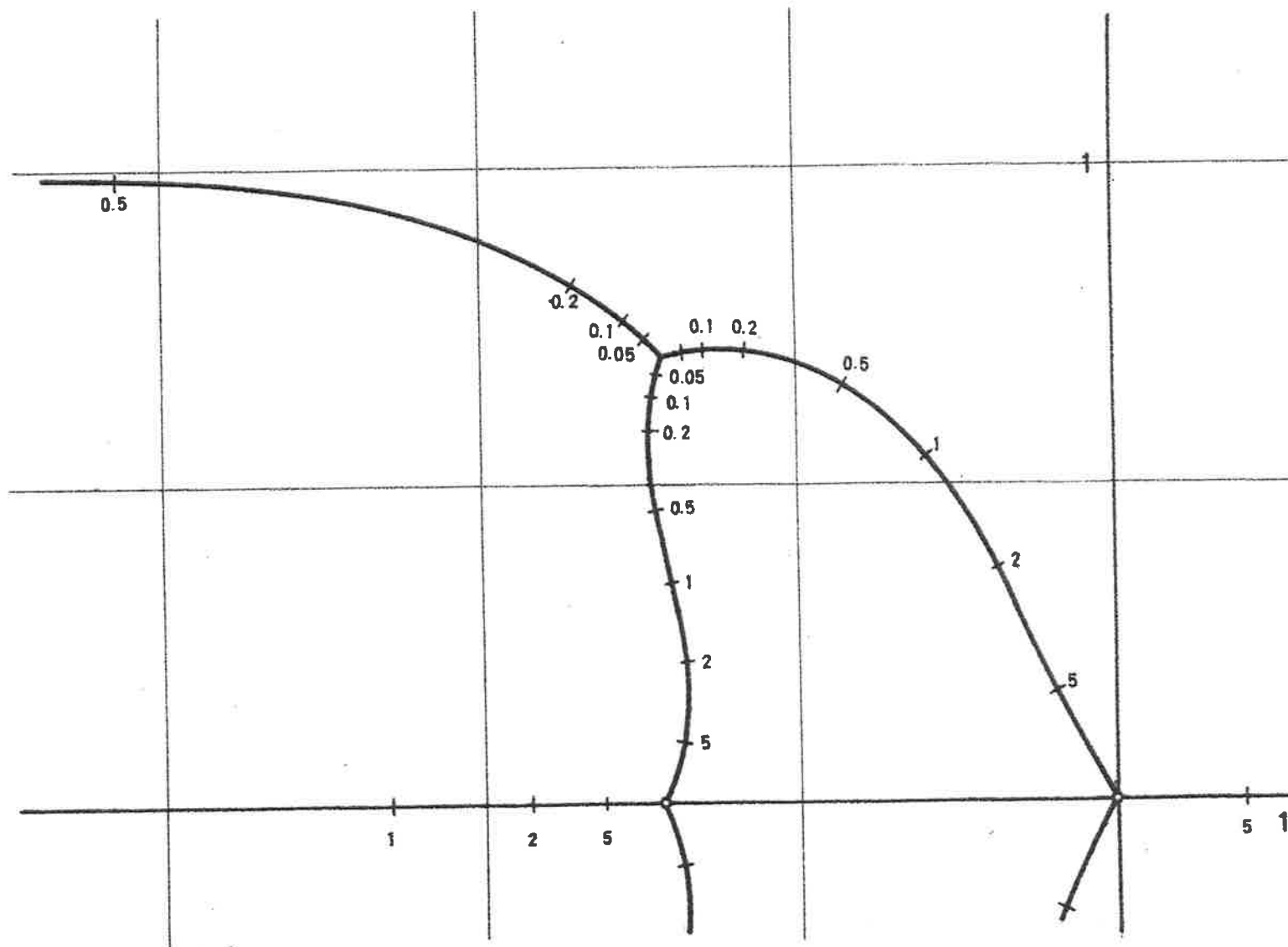
PLATE 8.23

ROOTLOCUS WITH RESPECT TO THE COUPLING COEFFICIENT $\gamma$ FOR THE CHARACTERISTIC EQUATION OF A SYSTEM WITH $S=3$, $e=1$, $\varrho=0$ AND $Y_1(p) = \dfrac{p^2 + 1.41P + 1}{P}$

PLATE 8.24

ROOTLOCUS WITH RESPECT TO THE COUPLING COEFFICIENT $\gamma$ FOR THE CHARACTERISTIC EQUATION OF A SYSTEM WITH $S=5$, $e=0$ $\varrho=0$ AND $Y_i(P) = \dfrac{P^2 + 1.41P + 1}{P}$
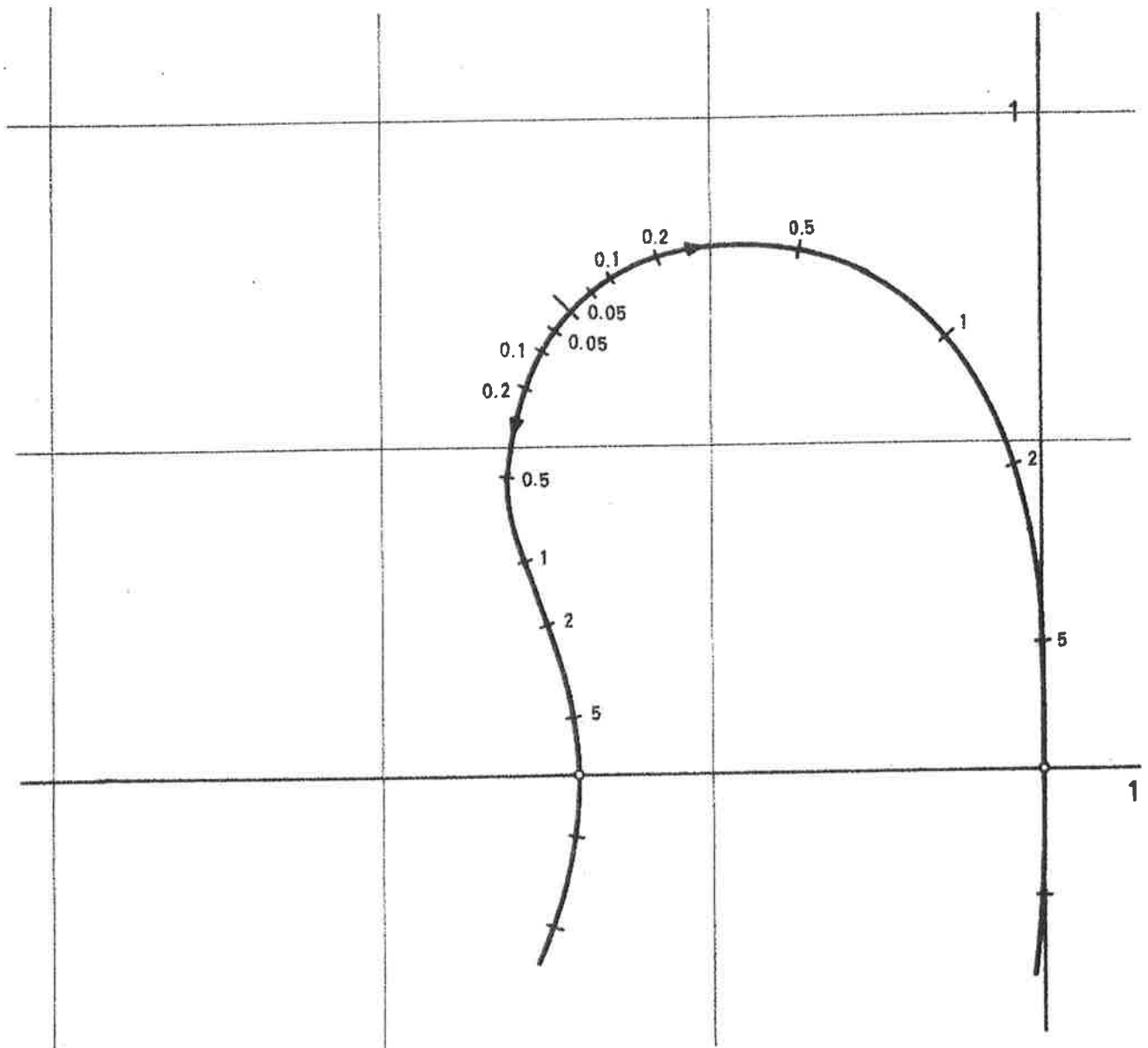
PLATE . 8.25

ROOTLOCUS WITH RESPECT TO THE CROSSCOUPLING
COEFFICENT $\gamma$ FOR THE CHARACTERISTIC EQUATION
OF A SYSTEM WITH $S = 3$ $e = -1$
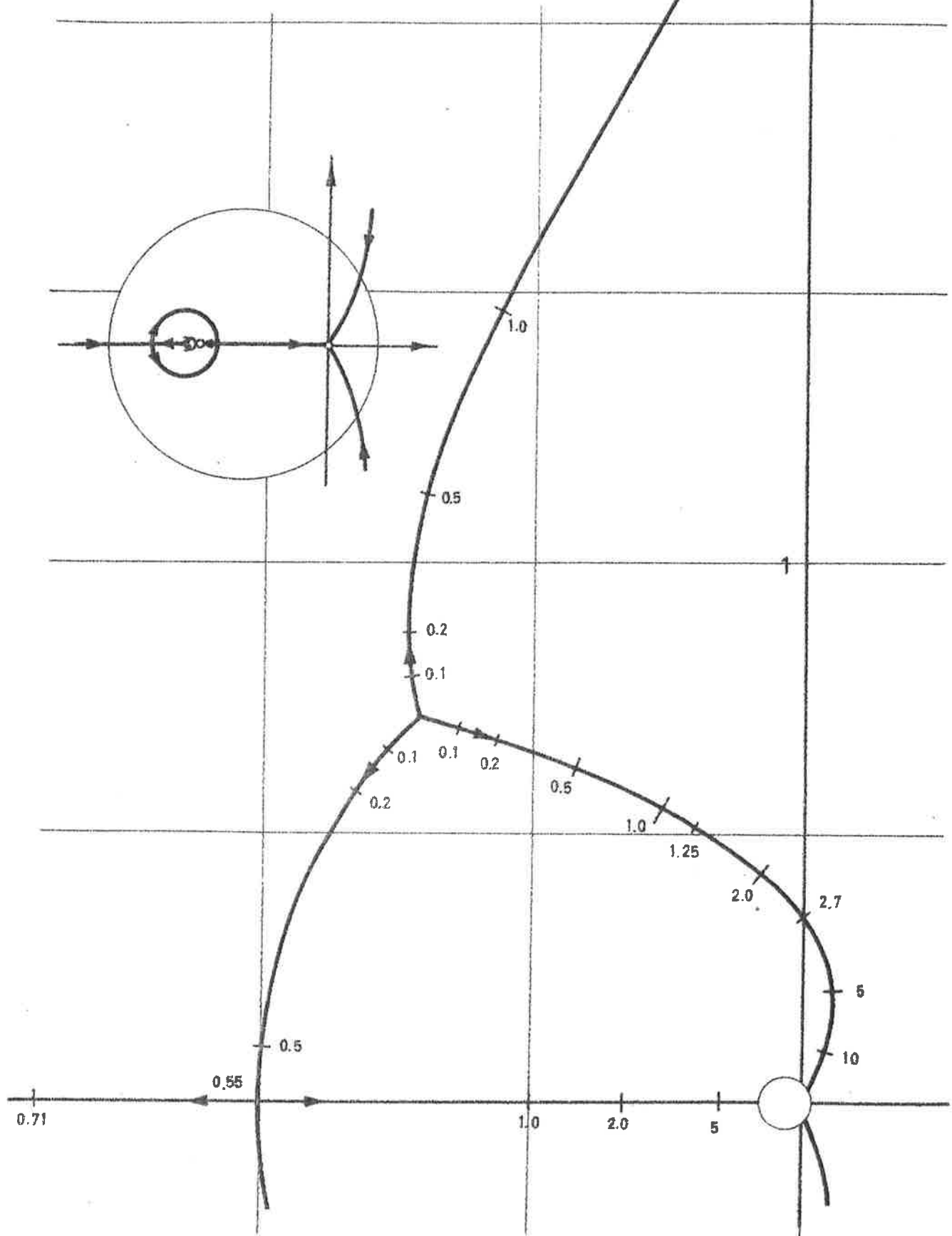AND $Y_1(p) = \dfrac{(p^2 + 1.41p + 1)(P + 0.05)}{P(P + 1.46)}$

PLATE 8.41

ROOTLOCUS WITH RESPECT TO THE CROSSCOUPLING
COEFFICIENT $\varrho$ FOR THE CHARACTERISTIC EQUATION
OF A SYSTEM WITH

$S = 3, e = 0, \gamma = 0, \sigma^* = 0.71p$

$$Y(p) = \frac{p^2 + 1.41p + 1}{p}$$