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Rikte, Sten

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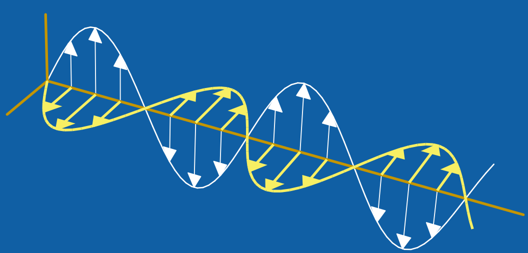
Sten Rikte

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
Sweden



Sten Rikte

Department of Electroscience
Electromagnetic Theory
Lund Institute of Technology
P.O. Box 118
SE-221 00 Lund
Sweden

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Abstract

As a preparatory study of electromagnetic pulse propagation in open waveguides, the modes of propagation of pulses in open slab waveguides are investigated systematically. Core and cladding both consist of simple (linear, homogeneous, isotropic), dispersive materials modeled by temporal convolution with physically sound susceptibility kernels. Under these circumstances, pulses cannot propagate along the guide unless the sum of the (first) initial derivatives of the electric and magnetic susceptibility kernels of the medium in the core is less than the corresponding sum for the medium in the cladding. Only a finite number of pulse modes can be excited, and relevant temporal Volterra integral equations of the second kind for these modes are derived.

1 Introduction

Propagation of time-harmonic electromagnetic waves in closed and open dielectric waveguides is a technically important and well-known subject that is discussed in many textbooks on electromagnetism, see, *e.g.*, [2, 3, 5, 6, 12, 14]. In particular, open circular dielectric waveguides, *e.g.*, optical fibers, are of great importance.

Pulse propagation in waveguides has been less attended to. However, Kristensson [10] managed to analyze the modes of propagation in closed empty guides using wave splitting technique, and later his results were extended to closed guides with isotropic fillings [1]. The purpose of the present paper is to analyze the modes of propagation of electromagnetic pulses in open slab waveguides, which is, of course, an highly artificial model problem. However, the work can be seen as a preparatory study of pulse propagation in open circular waveguides, a subject that would be of great technical interest.

In section 2, the basic field equations for relevant for pulse propagation in simple (linear, homogeneous, isotropic), dispersive materials are presented. Modes of propagation of pulses in the slab waveguide are analyzed in full extent in section 3. A brief discussion of the conditions for propagation concludes the paper in section 4.

2 Basic equations for simple, dispersive media

2.1 Notation

The following notation is used: position is denoted by $\boldsymbol{r} = (x, y, z)$, time by t, electric and magnetic field vectors by $\boldsymbol{E}(\boldsymbol{r}, t)$ and $\boldsymbol{H}(\boldsymbol{r}, t)$, respectively, and corresponding flux densities by $\boldsymbol{D}(\boldsymbol{r}, t)$ and $\boldsymbol{B}(\boldsymbol{r}, t)$. Each field vector is written in the form $\boldsymbol{E}(\boldsymbol{r}, t) = \boldsymbol{u}_x E_x(\boldsymbol{r}, t) + \boldsymbol{u}_y E_y(\boldsymbol{r}, t) + \boldsymbol{u}_z E_z(\boldsymbol{r}, t) = (E_x(\boldsymbol{r}, t), E_y(\boldsymbol{r}, t), E_z(\boldsymbol{r}, t))$, where $\boldsymbol{u}_x$, $\boldsymbol{u}_y$, and $\boldsymbol{u}_z$ are the basis vectors in the Cartesian frame. The dynamics of the fields is modeled by the macroscopic Maxwell equations: $\nabla \times \boldsymbol{E}(\boldsymbol{r}, t) = -\partial_t \boldsymbol{B}(\boldsymbol{r}, t)$ and $\nabla \times \boldsymbol{H}(\boldsymbol{r}, t) = \boldsymbol{J}(\boldsymbol{r}, t) + \partial_t \boldsymbol{D}(\boldsymbol{r}, t)$, where $\boldsymbol{J}(\boldsymbol{r}, t)$ is the current density. For brevity, the independent variables $(\boldsymbol{r}, t)$ are often suppressed. The speed of light in vacuum and the intrinsic impedance of vacuum are denoted by c_0 and η_0, respectively. At a few occasions, the temporal Heaviside step $H(t)$ appear. Finally, the positive square root is intended wherever the square-root operator $\surd$ appears.

2.2 Constitutive relations

The constitutive relations of a simple, causal, time-invariant, and continuous material can be written in the form $c_0\eta_0\boldsymbol{D} = \varepsilon\boldsymbol{E}$ and $c_0\boldsymbol{B} = \mu\eta_0\boldsymbol{H}$, where the *relative* permittivity and permeability operators of the medium are $\varepsilon = 1 + \chi^e(t)*$ and $\mu = 1 + \chi^m(t)*$, respectively, and the asterisk $(*)$ denotes temporal convolution [7]:

$$[\varepsilon\boldsymbol{E}]\,(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) + (\chi^e * \boldsymbol{E})(\boldsymbol{r},t) = \boldsymbol{E}(\boldsymbol{r},t) + \int_{-\infty}^{\infty} \chi^e(t-t')\boldsymbol{E}(\boldsymbol{r},t')\,dt'.$$

The integral kernels $\chi^e(t)$ and $\chi^m(t)$ are the susceptibility kernels of the medium. Owing to causality, these functions vanish for $t<0$, and, for $t>0$, they are assumed to be twice continuously differentiable. Furthermore, the continuity condition [4]

$$\chi^e(+0) = \chi^m(+0) = 0. \tag{2.1}$$

is imposed on the isotropic medium. Condition (2.1) is met by, for instance, the well-known Lorentz model (the resonance model),

$$\chi^e(t) = \frac{\omega_p^2}{\sqrt{\omega_0^2 - \left(\frac{\nu}{2}\right)^2}} \exp\left(-\frac{\nu t}{2}\right) \sin\left(\sqrt{\omega_0^2 - \left(\frac{\nu}{2}\right)^2}\, t\right) H(t), \tag{2.2}$$

which applies to bound electrons in insulators, and, by the Drude model,

$$\chi^e(t) = \frac{\omega_p^2}{\nu}\left(1 - \exp\left(-\nu t\right)\right) H(t),$$

which applies to free electrons in conductors (set $\omega_0 = 0$ in the Lorentz model), and by any linear combination of these models. On the other hand, the Debye model (the relaxation model) for polar liquids and Ohm's law for conductors violate the condition (2.1). In practise, the imposed condition (2.1) is not a severe restriction: by introducing short rise times, any susceptibility kernel can be approximated by an integral kernel, for which the condition (2.1) holds, without much changing the value of the convolution integral. Models that violate (2.1) have been described as "unphysical" in a major textbook on classical electrodynamics [6].

Substituting the constitutive relations into the Maxwell equations gives a linear system of first-order hyperbolic integro-differential equations in the electric and magnetic field vectors only:

$$\begin{cases} \nabla\times\boldsymbol{E} = \mu\eta_0\boldsymbol{H}, \\ \nabla\times\eta_0\boldsymbol{H} = \eta_0\boldsymbol{J} + c_0^{-1}\partial_t\varepsilon\boldsymbol{E}. \end{cases} \tag{2.3}$$

2.3 The complex electromagnetic field vector

It it economical, however not necessary, to introduce a complex time-dependent field, cf. Stratton [13]. Any (real) time-dependent electromagnetic field $(\boldsymbol{E}, \boldsymbol{H})$ in a simple medium can be represented uniquely by the complex field vector

$$\boldsymbol{Q} = \frac{1}{2}\left(\boldsymbol{E} - i\mathcal{Z}\eta_0\boldsymbol{H}\right),$$

where the real temporal integral operator $\mathcal{Z} = 1 + Z(t)*$ is the relative intrinsic impedance of the medium. It is also appropriate to introduce a real temporal integral operator $\mathcal{Y} = 1 + Y(t)*$ defined by $\mathcal{Y}\mathcal{Z} = 1$. This operator is referred to as the relative intrinsic admittance of the medium.

The introduction of the complex electromagnetic field vector reduces the system of integro-differential equations (2.3) to the first-order dispersive wave equation

$$\nabla \times \boldsymbol{Q} = -ic_0^{-1}\partial_t\mathcal{N}\,\boldsymbol{Q} - i\eta_0\mathcal{Z}\boldsymbol{J}/2, \tag{2.4}$$

where the real temporal integral operator $\mathcal{N} = 1 + N(t)*$ is referred to as the index of refraction of the medium. It is understood that $\mathcal{Y}$, $\mathcal{N}$, and $\mathcal{Z}$ are intrinsic operators of the medium, that is, independent of the field vectors $\boldsymbol{E}$ and $\boldsymbol{H}$.

2.4 The intrinsic operators of the medium

The decoupling of (2.3) in accordance with (2.4) leads to conditions on the relative intrinsic admittance $\mathcal{Y}$ and the index of refraction $\mathcal{N}$ in terms of the susceptibility operators: $\mathcal{N} = \mu\mathcal{Y}$ and $\mathcal{N}\mathcal{Y} = \varepsilon$. Combining these equations gives $\mathcal{N}\mathcal{N} = \mu\varepsilon$. Consequently, the refractive kernel $N(t)$ satisfies the Volterra integral equation of the second kind

$$2N(t) + (N * N)(t) = \chi^e(t) + \chi^m(t) + (\chi^e * \chi^m)(t). \tag{2.5}$$

Volterra integral equations of the second kind are uniquely solvable in the space of continuous functions in each compact time-interval and the solutions depend continuously on data [9]. Consequently, the refractive kernel inherits causality and smoothness properties from the susceptibility kernels.

Similarly, the admittance and impedance kernels satisfy Volterra integral equation of the second kind:

$$\begin{aligned} Y(t) + (Y * \chi^m)(t) &= N(t) - \chi^m(t), \\ Z(t) + (Z * N)(t) &= \chi^m(t) - N(t). \end{aligned}$$

The admittance and impedance kernels inherit causality and regularity from the susceptibility kernels. Notice also that the continuity condition (2.1) implies that

$$N(+0) = Y(+0) = Z(+0) = 0. \tag{2.6}$$

In the non-magnetic case, $\mu = 1$, one obtains $\mathcal{N} = \mu\mathcal{Y} = \mathcal{Y}$.

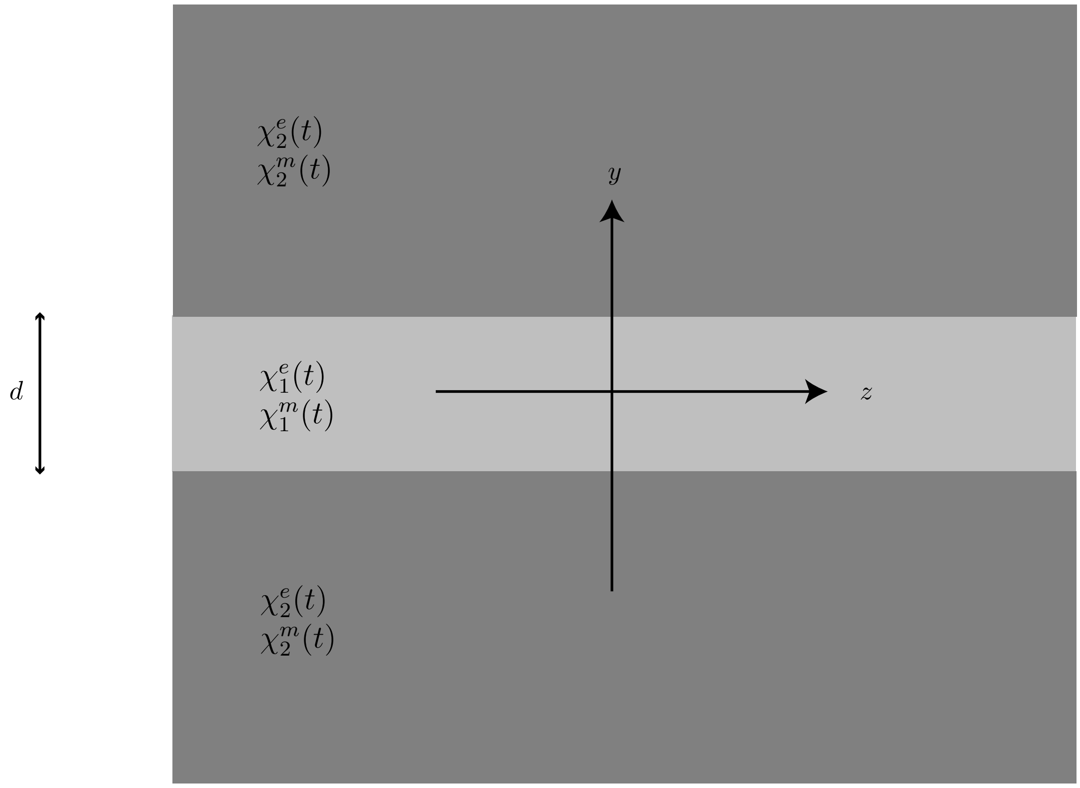

Figure 1: A slab waveguide. The electric and magnetic susceptibility kernels of the core, $|y| < d/2$, are $\chi_1^e(t)$ and $\chi_1^m(t)$, whereas the corresponding properties of the cladding, $|y| > d/2$, are $\chi_2^e(t)$ and $\chi_2^m(t)$. Pulses propagate in the z-direction only.

3 Modes in open slab waveguides

The general idea of an open waveguide is that it should support pulse modes that

1. travel along the guide only, *i.e.*, not in transverse directions, and
2. are confined mainly to the core in order to carry finite energy.

3.1 Basic modes of propagation

The geometry of the slab waveguide is depicted in Figure 1. The coordinate system has been located symmetrically with the y-axis perpendicular to the planes ($y = \pm d/2$) that separate the two materials. Equations (2.3) hold in both regions with

$$\varepsilon = \begin{cases} \varepsilon_1 = 1 + \chi_1^e(t)\, * & (|y| < d/2), \\ \varepsilon_2 = 1 + \chi_2^e(t)\, * & (|y| > d/2) \end{cases}$$

and

$$\mu = \begin{cases} \mu_1 = 1 + \chi_1^m(t)\, * & (|y| < d/2), \\ \mu_2 = 1 + \chi_2^m(t)\, * & (|y| > d/2). \end{cases}$$

Due to the special geometry, all fields are assumed to be independent of, say, the x-coordinate, and, consequently, by (2.3), two kind of solutions can be identified, namely, transverse electric (TE) solutions, for which $E_x(y,z,t)$, $H_y(y,z,t)$, and $H_z(y,z,t)$ are the only non-vanishing components, and transverse magnetic (TM) solutions, for which $H_x(y,z,t)$, $E_y(y,z,t)$, and $E_z(y,z,t)$ are the only non-vanishing components. Furthermore, since the longitudinal fields can be decomposed uniquely in odd and even parts with respect to the transverse coordinate y, four fundamental types of modes can be identified, namely,

- Even longitudinal TE solutions: $H_z(y,z,t)$ is even w.r.t. y, whereas $E_x(y,z,t)$ and $H_y(y,z,t)$ are odd w.r.t. y,
- Odd longitudinal TE solutions: $H_z(y,z,t)$ is odd w.r.t. y, whereas $E_x(y,z,t)$ and $H_y(y,z,t)$ are even w.r.t. y,
- Even longitudinal TM solutions: $E_z(y,z,t)$ is even w.r.t. y, whereas $H_x(y,z,t)$ and $E_y(y,z,t)$ are odd w.r.t. y,
- Odd longitudinal TM solutions: $E_z(y,z,t)$ is odd w.r.t. y, whereas $H_x(y,z,t)$ and $E_y(y,z,t)$ are even w.r.t. y.

This is, of course, in concordance with the time-harmonic case.

By definition, pulses are propagated along the z-axis only, and owing to the absence of optical response in permittivity and permeability operators of the constituents [11], wave-fronts travel with the vacuum speed c_0. The aim is to look for up-going or down-going modes of propagation with the z-dependencies

$$\exp\left(\mp z c_0^{-1}\partial_t \mathcal{N}_z\right), \tag{3.1}$$

respectively, where the real temporal integral operator

$$\mathcal{N}_z = 1 + N_z(t)* \qquad \text{(for all } y\text{)},$$

is referred to as the longitudinal refractive index. The integral kernel $N_z(t)$ is supposed to inherit causality and regularity from the susceptibility kernels; in particular

$$N_z(+0) = 0. \tag{3.2}$$

The propagator (3.1) can, therefore, be factored as

$$\delta\left(t \mp z c_0^{-1}\right) * \exp\left(\mp z c_0^{-1} N_z'(t)*\right) = \delta\left(t \mp z c_0^{-1}\right) * \left(1 + P^{\mp}(z,t)*\right),$$

where the kernels $P^{\mp}(z,t)$, for fixed z, satisfy the temporal Volterra integral equations of the second kind [8]

$$tP^{\mp}(z,t) = \mp t z c_0^{-1} N_z'(t) \mp \left(t z c_0^{-1} N_z' * P^{\mp}\right)(z,t) \tag{3.3}$$

in terms of the kernel $N_z'(t)$. In particular, $P^{\mp}(0,t) = 0$, and, by differentiation of both members of (3.3), $P^{\mp}(z,+0) = \mp z c_0^{-1} N_z'(+0)$. The short-hand notation

$$\partial_z = \mp c_0^{-1}\partial_t \mathcal{N}_z = \mp \mathcal{K}_z \tag{3.4}$$

is used frequently below, depending on whether the mode is up-going or down-going.

3.2 Propagators in transverse directions

Propagators are well-known concepts from normal incidence scattering problems, see, *e.g.*, Karlsson and Rikte [8]. These operators take TEM pulses in the plane $y = y_1$ to a plane $y = y_2$, and if the medium is homogeneous, the propagator depends on the distance $y_2 - y_1$ rather than on y_1 and y_2. It is natural to employ the propagator concept also in slab waveguide problems, although pulses do not propagate in transverse directions in this case.

It is appropriate to introduce the intrinsic operators

$$\mathcal{N} = \begin{cases} \mathcal{N}_1 = 1 + N_1(t)\,* & (|y| < d/2), \\ \mathcal{N}_2 = 1 + N_2(t)\,* & (|y| > d/2), \end{cases}$$

$$\mathcal{Y} = \begin{cases} \mathcal{Y}_1 = 1 + Y_1(t)\,* & (|y| < d/2), \\ \mathcal{Y}_2 = 1 + Y_2(t)\,* & (|y| > d/2), \end{cases}$$

and

$$\mathcal{Z} = \begin{cases} \mathcal{Z}_1 = 1 + Z_1(t)\,* & (|y| < d/2), \\ \mathcal{Z}_2 = 1 + Z_2(t)\,* & (|y| > d/2). \end{cases}$$

The dynamics (2.4) of the complex field now be written as

$$\partial_y \begin{pmatrix} Q_z \\ Q_x \end{pmatrix} = \begin{pmatrix} 0 & \left(ic_0^{-1}\partial_t\mathcal{N}\right)^{-1}\left(\left(c_0^{-1}\partial_t\mathcal{N}\right)^2 - \partial_z^2\right) \\ ic_0^{-1}\partial_t\mathcal{N} & 0 \end{pmatrix} \begin{pmatrix} Q_z \\ Q_x \end{pmatrix},$$

where

$$Q_y = -\left(ic_0^{-1}\partial_t\mathcal{N}\right)^{-1}\partial_z Q_x.$$

The propagator for the field vector (Q_z, Q_x) a distance y in the inner or in the outer region is thus, formally,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cosh\left(y\sqrt{\mathcal{K}^2 - \partial_z^2}\right)$$
$$+ \begin{pmatrix} 0 & (i\mathcal{K})^{-1}\sqrt{\mathcal{K}^2 - \partial_z^2} \\ i\mathcal{K}\left(\sqrt{\mathcal{K}^2 - \partial_z^2}\right)^{-1} & 0 \end{pmatrix} \sinh\left(y\sqrt{\mathcal{K}^2 - \partial_z^2}\right),$$

or

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos\left(y\sqrt{\partial_z^2 - \mathcal{K}^2}\right)$$
$$+\, i \begin{pmatrix} 0 & \mathcal{K}^{-1}\sqrt{\partial_z^2 - \mathcal{K}^2} \\ \mathcal{K}\left(\sqrt{\partial_z^2 - \mathcal{K}^2}\right)^{-1} & 0 \end{pmatrix} \sin\left(y\sqrt{\partial_z^2 - \mathcal{K}^2}\right),$$

where

$$\mathcal{K} = c_0^{-1}\partial_t \mathcal{N} = \begin{cases} \mathcal{K}_1 = c_0^{-1}\partial_t \mathcal{N}_1 & (|y| < d/2), \\ \mathcal{K}_2 = c_0^{-1}\partial_t \mathcal{N}_2 & (|y| > d/2), \end{cases}$$

and the positive square-root has been chosen[1]. This propagator can be written as

$$\mathcal{W}^+ \exp\left(-y\sqrt{\mathcal{K}^2 - \partial_z^2}\right) + \mathcal{W}^- \exp\left(y\sqrt{\mathcal{K}^2 - \partial_z^2}\right)$$

where the operators [2]

$$\mathcal{W}^\pm = \frac{1}{2}\begin{pmatrix} 1 & \mp (i\mathcal{K})^{-1}\sqrt{\mathcal{K}^2 - \partial_z^2} \\ \mp i\mathcal{K}\left(\sqrt{\mathcal{K}^2 - \partial_z^2}\right)^{-1} & 1 \end{pmatrix}$$

are orthogonal projections:

$$\begin{cases} \mathcal{W}^\mp \cdot \mathcal{W}^\mp = \mathcal{W}^\mp, \\ \mathcal{W}^\pm \cdot \mathcal{W}^\mp = \mathcal{W}^\mp \cdot \mathcal{W}^\pm = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathcal{W}^+ + \mathcal{W}^- = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases}$$

Consequently, the operators

$$\mathcal{W}^\pm \exp\left(\mp y\sqrt{\mathcal{K}^2 - \partial_z^2}\right)$$

represent independent solutions.

3.3 Condition for existence of pulse modes

For a specific mode, (3.4) applies, and

$$\begin{aligned} \mathcal{K}^2 - \partial_z^2 =& \mathcal{K}^2 - \mathcal{K}_z^2 = c_0^{-2}\partial_t^2\left(2 + (N + N_z)(t)*\right)(N - N_z)(t)* \\ =& c_0^{-2}\left(2 + (N + N_z)(t)*\right)\left((N - N_z)'(+0) + (N - N_z)''(t)*\right), \end{aligned}$$

where the initial conditions (2.6) and (3.2) have been used. The positive square-root of this operator is well defined if only $N'(+0) \neq N_z'(+0)$. Thus,

$$N_1'(+0) \neq N_z'(+0) \neq N_2'(+0). \tag{3.5}$$

From the proceeding sections follows that pulse modes with finite energy cannot propagate unless

$$N_1'(+0) \leq N_z'(+0) \leq N_2'(+0). \tag{3.6}$$

[1]The negative square-root could equally well have been chosen.
[2]These operators are known as spectral projections.

Combining this with (3.5) gives

$$N_1'(+0) < N_z'(+0) < N_2'(+0), \tag{3.7}$$

and, in particular,

$$N_1'(+0) < N_2'(+0), \tag{3.8}$$

which is the condition for existence of propagating finite energy pulse modes in the slab. Observe that the inequality (3.8) is equivalent to

$$(\chi_1^e)'(+0) + (\chi_1^m)'(+0) < (\chi_2^e)'(+0) + (\chi_2^m)'(+0).$$

3.4 Square-root, exponential, cosine, and sine operators

One of the relevant square-root operators introduced above is of the form

$$d/2\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2} = \frac{d}{2c_0}\sqrt{2(N_2'(+0) - N_z'(+0)} + U_2(t)*,$$

where

$$\begin{aligned} \frac{d}{2c_0} 2\sqrt{2\left(N_2'(+0) - N_z'(+0)\right)} U_2(t) + \left(U_2 * U_2\right)(t) = {} & [2(N_2''(t) - N_z''(t)) \\ & + \left(N_2'(+0) - N_z'(+0)\right)\left(N_2(t) + N_z(t)\right) \\ & + \left(\left(N_2'' - N_z''\right) * \left(N_2 + N_z\right)\right)(t)] \left(\frac{d}{2c_0}\right)^2 . \end{aligned} \tag{3.9}$$

Since $N_z(t) = (H * N_z')(t)$ combined with

$$N_z'(t) = N_z'(+0)H(t) + (H * N_z'')(t), \tag{3.10}$$

gives

$$N_z(t) = N_z'(+0)tH(t) + ((tH) * N_z'')(t),$$

equation (3.9) is a temporal Volterra integral equation of the second kind in the unknown kernels $U_2(t)$ and $N_z''(t)$ for a fixed value of $N_z'(+0)$.

Observe that $N_2''(t)$ and for that matter $N_1''(t)$ are known quantities since differentiation of both members of equation (2.5) gives the temporal Volterra integral equation of the second kind

$$2N''(t) + N'(+0)N(t) + (N * N'')(t) = (\chi^e)''(t) + (\chi^m)''(t) + ((\chi^e)' * (\chi^m)')(t),$$

where $N(t)$ in the left member and the function in the right member are known, and $2N'(+0) = (\chi^e)'(+0) + (\chi^m)'(+0)$ is known as well.

The corresponding exponentials are

$$\begin{aligned}\exp\left(\mp y\sqrt{\mathcal{K}^2-\mathcal{K}_z^2}\right) &= \exp\left(\mp y c_0^{-1}\sqrt{2(N_2'(+0)-N_z'(+0)}\right)\exp\left(\mp y2/dU_2(t)*\right)\\ &= \exp\left(\mp y c_0^{-1}\sqrt{2(N_2'(+0)-N_z'(+0)}\right)\left(1+V_2^{\mp}(y,t)*\right),\end{aligned}$$

where the kernels $V_2^{\mp}(y,t)$ for fixed y satisfy the temporal Volterra integral equations of the second kind

$$tV_2^{\mp}(y,t) = \mp ty2/dU_2(t) \mp \left(ty2/dU_2 * V_2^{\mp}\right)(y,t) \tag{3.11}$$

in terms of the kernel $U_2(t)$. In view of (3.7), only the upper solution is finite.

The other square-root operator of interest is of the form

$$d/2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2} = \frac{d}{2c_0}\sqrt{2(N_z'(+0)-N_1'(+0)} + U_1(t)*,$$

where

$$\begin{aligned}\frac{d}{2c_0}2\sqrt{2\left(N_z'(+0)-N_1'(+0)\right)}U_1(t) + \left(U_1 * U_1\right)(t) = {} & [2(N_z''(t)-N_1''(t))\\ & + \left(N_z'(+0)-N_1'(+0)\right)\left(N_z(t)+N_1(t)\right)\\ & + \left(\left(N_z''-N_1''\right)*\left(N_z+N_1\right)\right)(t)]\left(\frac{d}{2c_0}\right)^2.\end{aligned} \tag{3.12}$$

This is a temporal Volterra integral equation of the second kind in the kernels $U_1(t)$ and $N_z''(t)$ for a fixed value of $N_z'(+0)$.

The sine and cosine operators

$$\begin{cases}\sin\left(y\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right) = \dfrac{\exp\left(iy\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)-\exp\left(-iy\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)}{2i}\\ \cos\left(y\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right) = \dfrac{\exp\left(iy\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)+\exp\left(-iy\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)}{2}\end{cases}$$

are perhaps most easily obtained from the complex exponentials

$$\begin{aligned}\exp\left(\mp iy\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right) &= \exp\left(\mp iy c_0^{-1}\sqrt{2(N_z'(+0)-N_1'(+0)}\right)\exp\left(\mp iy2/dU_1(t)*\right)\\ &= \exp\left(\mp iy c_0^{-1}\sqrt{2(N_z'(+0)-N_1'(+0)}\right)\left(1+V_1^{\mp}(y,t)*\right),\end{aligned}$$

where the complex conjugate kernels $V_1^{\mp}(y,t)$ for fixed y satisfy the temporal Volterra integral equations of the second kind

$$tV_1^{\mp}(y,t) = \mp ity2/dU_1(t) \mp \left(ity2/dU_1 * V_1^{\mp}\right)(y,t) \tag{3.13}$$

in terms of the kernel $U_1(t)$.

3.5 TM modes

For transverse magnetic modes,

$$\begin{pmatrix} Q_z \\ Q_x \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -i\mathcal{Z} \end{pmatrix}\begin{pmatrix} E_z \\ \eta_0 H_x \end{pmatrix}$$

and, consequently,

$$\begin{pmatrix} E_z \\ \eta_0 H_x \end{pmatrix} = 2\begin{pmatrix} 1 & 0 \\ 0 & i\mathcal{Y} \end{pmatrix}\begin{pmatrix} Q_z \\ Q_x \end{pmatrix}.$$

The propagator for the vector $(E_z, \eta_0 H_x)$ is therefore

$$\begin{pmatrix} \cos\left(y\sqrt{\partial_z^2-\mathcal{K}^2}\right) & \mathcal{Z}\mathcal{K}^{-1}\sqrt{\partial_z^2-\mathcal{K}^2}\sin\left(y\sqrt{\partial_z^2-\mathcal{K}^2}\right) \\ -\mathcal{Y}\mathcal{K}\left(\sqrt{\partial_z^2-\mathcal{K}^2}\right)^{-1}\sin\left(y\sqrt{\partial_z^2-\mathcal{K}^2}\right) & \cos\left(y\sqrt{\partial_z^2-\mathcal{K}^2}\right) \end{pmatrix}$$

or, equivalently, the sum of the independent propagators

$$\frac{1}{2}\begin{pmatrix} 1 & \pm\mathcal{Z}\mathcal{K}^{-1}\sqrt{\mathcal{K}^2-\partial_z^2} \\ \pm\mathcal{Y}\mathcal{K}\left(\sqrt{\mathcal{K}^2-\partial_z^2}\right)^{-1} & 1 \end{pmatrix}\exp\left(\mp y\sqrt{\mathcal{K}^2-\partial_z^2}\right). \tag{3.14}$$

In the outer region, the propagators (3.14) apply. In order to obtain finite-energy solutions in each transverse plane, one has to demand that

$$E_z(d/2,z,t) = \mathcal{Z}_2\mathcal{K}_2^{-1}\sqrt{\mathcal{K}_2^2-\partial_z^2}\,\eta_0 H_x(d/2,z,t), \tag{3.15}$$

in view of the results presented in section 3.4.

3.5.1 Even longitudinal TM modes

For even longitudinal TM solutions, $H_x(0,z,t) = 0$. The solution in the inner region is given by

$$\begin{cases} E_z(y,z,t) = \cos\left(y\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)f(z,t) \\ \eta_0 H_x(y,z,t) = -\mathcal{Y}_1\mathcal{K}_1\left(\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)^{-1}\sin\left(y\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)f(z,t) \end{cases} \quad (|y| < d/2),$$

where $f(z,t) = E_z(0,z,t)$ is a real function. Combination with the condition for finite outer solutions (3.15) gives the Volterra integral equation of the second kind

$$-\mathcal{Z}_1\mathcal{K}_1^{-1}\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\cot\left(d/2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)f = \mathcal{Z}_2\mathcal{K}_2^{-1}\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}f \tag{3.16}$$

or, equivalently, since f is arbitrary, the temporal integral operator identity

$$\begin{aligned}&\left(\varepsilon_1\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}+i\varepsilon_2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\exp\left(id\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\\&=\varepsilon_1\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}-i\varepsilon_2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2},\end{aligned} \tag{3.17}$$

which determines the possible longitudinal refractive indices, $\mathcal{N}_z$, *i.e.*, the modes of propagation. Condition (3.15) shows that, for a fixed mode, the outer solution is

$$\begin{cases}E_z(y,z,t)=\exp\left(-\left(|y|-d/2\right)\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)E_z(d/2,z,t)\\ \eta_0 H_x(y,z,t)=\operatorname{sgn}(y)\exp\left(-\left(|y|-d/2\right)\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\eta_0 H_x(d/2,z,t)\end{cases}\qquad (|y|>d/2).$$

To obtain the different modes of propagation, one has to solve equation (3.17) (or equation (3.16)). The variety of modes arise from the possible values of the initial derivative of the longitudinal refractive kernel, $N_z'(+0)$. Since the principal parts of both members of equation (3.16) must agree, one obtains

$$-\sqrt{\lambda_z-\lambda_1}\cot\left(\sqrt{\lambda_z-\lambda_1}\right)=\sqrt{\lambda_2-\lambda_z}, \tag{3.18}$$

where the introduced dimensionless numbers are defined by

$$\lambda_i=\frac{d^2}{c_0^2 2}N_i'(+0)\qquad (i=1,2,z). \tag{3.19}$$

Equation (3.18) shows that the inequalities (3.6) hold: for if $N_z'(+0)>N_2'(+0)$, then the right-hand side of equation (3.18) becomes purely imaginary, which can never be the case for the left-hand side, and, moreover, if $N_1'(+0)>N_z'(+0)$, then equation (3.18) reads

$$0>-\sqrt{\lambda_1-\lambda_z}\coth\left(\sqrt{\lambda_1-\lambda_z}\right)=\sqrt{\lambda_2-\lambda_z}>0,$$

which, since the members are of opposite signs, leads to a contradiction. Furthermore, the equation

$$-\sqrt{\lambda_z-\lambda_1}\cot\left(\sqrt{\lambda_z-\lambda_1}\right)=\sqrt{\lambda_2-\lambda_z}\qquad (\lambda_1\le\lambda_z\le\lambda_2), \tag{3.20}$$

has at most a finite number of zeros for given values of λ_1, λ_2. This is so because the right-hand side is positive and strictly decreasing, whereas the left-hand side (extended to all $\lambda_z>\lambda_1$) is strictly increasing from the value -1 to the value $+\infty$ in the interval $0\le\lambda_z-\lambda_1\le\pi^2$ and strictly increasing from the value $-\infty$ to the value $+\infty$ in the intervals $(n\pi)^2\le\lambda_z-\lambda_1\le((n+1)\pi)^2$ for $n=1,2,3,\cdots$. If $\lambda_2-\lambda_1<(\pi/2)^2$, then there is no solution. The situation when $\lambda_1=1$ and $\lambda_2=12$ is depicted in Figure 2.

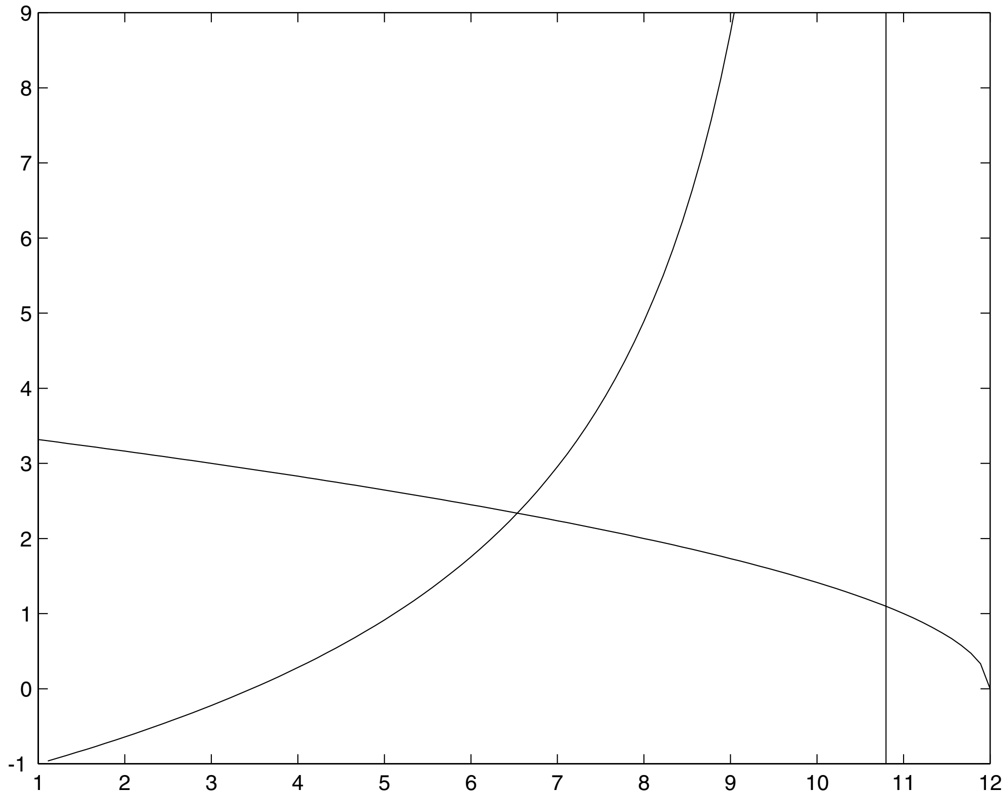

Figure 2: The functions $-\sqrt{\lambda_z - \lambda_1}\cot\left(\sqrt{\lambda_z - \lambda_1}\right)$ and $\sqrt{\lambda_2 - \lambda_z}$ restricted to $\lambda_1 \leq \lambda_z \leq \lambda_2$ in case $\lambda_1 = 1$ and $\lambda_2 = 12$. There is precisely one point of intersection, namely, approximately at $\lambda_z = 6.5$; the other visible "solution" is due to a curiosity of MATLAB.

For a fixed solution λ_z of (3.20), the temporal integral operator identity (3.17) becomes

$$\begin{aligned}
&\left((1+\chi_1^e(t)*)\left(\sqrt{\lambda_2-\lambda_z}+U_2(t)*\right)+i\,(1+\chi_2^e(t)*)\left(\sqrt{\lambda_z-\lambda_1}+U_1(t)*\right)\right)\\
&\cdot\exp\left(i2\sqrt{\lambda_z-\lambda_1}\right)\left(1+V_1^+(d,t)*\right)\\
&=(1+\chi_1^e(t)*)\left(\sqrt{\lambda_2-\lambda_z}+U_2(t)*\right)-i\,(1+\chi_2^e(t)*)\left(\sqrt{\lambda_z-\lambda_1}+U_1(t)*\right),
\end{aligned}$$

where the law of exponents for the exponential has been used. Since, due to (3.20),

$$\left(\sqrt{\lambda_2-\lambda_z}+i\sqrt{\lambda_z-\lambda_1}\right)\exp\left(i2\sqrt{\lambda_z-\lambda_1}\right)=\sqrt{\lambda_2-\lambda_z}-i\sqrt{\lambda_z-\lambda_1},$$

this identity reduces to the temporal Volterra integral equation of the second kind

$$\begin{aligned}
&\left(U_2 + \sqrt{\lambda_2 - \lambda_z}\chi_1^e + U_2 * \chi_1^e + iU_1 + i\sqrt{\lambda_z - \lambda_1}\chi_2^e + iU_1 * \chi_2^e\right)(t) \\
&+ \Big(\Big(U_2 + \sqrt{\lambda_2 - \lambda_z}\chi_1^e + U_2 * \chi_1^e + iU_1 + i\sqrt{\lambda_z - \lambda_1}\chi_2^e + \\
&\qquad iU_1 * \chi_2^e\Big)(\cdot) * V_1^+(d,\cdot)\Big)(t) \\
&+ \left(\sqrt{\lambda_2 - \lambda_z} + i\sqrt{\lambda_z - \lambda_1}\right) V_1^+(d,t) \\
&= c\left(U_2 + \sqrt{\lambda_2 - \lambda_z}\chi_1^e + U_2 * \chi_1^e - iU_1 - i\sqrt{\lambda_z - \lambda_1}\chi_2^e - iU_1 * \chi_2^e\right)(t),
\end{aligned} \tag{3.21}$$

where

$$c = \exp\left(-i2\sqrt{\lambda_z - \lambda_1}\right) = \frac{\sqrt{\lambda_2 - \lambda_z} + i\sqrt{\lambda_z - \lambda_1}}{\sqrt{\lambda_2 - \lambda_z} - i\sqrt{\lambda_z - \lambda_1}}.$$

Together with equation (3.9),

$$\begin{aligned}
&2\sqrt{\lambda_2 - \lambda_z}U_2(t) + (U_2 * U_2)(t) = 2\left(d/(2c_0)\right)^2 \left(N_2''(t) - N_z''(t)\right) \\
&+ (\lambda_2 - \lambda_z)\left(N_2(t) + N_z(t)\right)/2 + \left(\left(d/(2c_0)\right)^2 \left(N_2'' - N_z''\right) * \left(N_2 + N_z\right)\right)(t),
\end{aligned} \tag{3.22}$$

and equation (3.12),

$$\begin{aligned}
&2\sqrt{\lambda_z - \lambda_1}U_1(t) + (U_1 * U_1)(t) = 2\left(d/(2c_0)\right)^2 \left(N_z''(t) - N_1''(t)\right) \\
&+ (\lambda_z - \lambda_1)\left(N_z(t) + N_1(t)\right)/2 + \left(\left(d/(2c_0)\right)^2 \left(N_z'' - N_1''\right) * \left(N_z + N_1\right)\right)(t),
\end{aligned} \tag{3.23}$$

where

$$N_z(t) = \lambda_z \left((2c_0)/d\right)^2 tH(t)/2 + ((tH) * N_z'')(t),$$

and the lower equation (3.13),

$$tV_1^+(d,t) = it2U_1(t) + \left(it2U_1(\cdot) * V_1^+(d,\cdot)\right)(t), \tag{3.24}$$

equation (3.21) forms a system of temporal Volterra integral equations of the second kind in four unknown kernels, namely, $N_z''(t)$, $U_2(t)$, $U_1(t)$, and $V_1^+(d,t)$. Such systems of equations are uniquely solvable in the space of continuous functions, and the solution depends continuously on (susceptibility) data. In particular, the initial values are easily obtained from the system of integral equations consisting of (3.21), (3.22), (3.23), and (3.24):

$$\begin{cases}
U_2(+0) + iU_1(+0) + \left(\sqrt{\lambda_2 - \lambda_z} + i\sqrt{\lambda_z - \lambda_1}\right) V_1^+(d,+0) \\
= \left(\sqrt{\lambda_2 - \lambda_z} + i\sqrt{\lambda_z - \lambda_1}\right)\left(U_2(+0) - iU_1(+0)\right) / \left(\sqrt{\lambda_2 - \lambda_z} - i\sqrt{\lambda_z - \lambda_1}\right), \\
\sqrt{\lambda_2 - \lambda_z}U_2(+0) = \left(d/(2c_0)\right)^2 \left(N_2''(+0) - N_z''(+0)\right), \\
\sqrt{\lambda_z - \lambda_1}U_1(+0) = \left(d/(2c_0)\right)^2 \left(N_z''(+0) - N_1''(+0)\right), \\
V_1^+(d,+0) = 2iU_1(+0),
\end{cases} \tag{3.25}$$

where the last condition has been obtained by temporal differentiation of both members of (3.24). This gives

$$N_z''(+0) = \frac{\frac{\lambda_z-\lambda_1}{\lambda_2-\lambda_1} N_2''(+0) + \left(\frac{\lambda_2-\lambda_z}{\lambda_2-\lambda_1} + \sqrt{\lambda_2-\lambda_z}\right) N_1''(+0)}{1+\sqrt{\lambda_2-\lambda_z}}$$

from which the other initial values readily follow.

In order to obtain the fields in an arbitrary point y, some of equations (3.13) and (3.11) can be employed, depending on whether $|y| < d/2$ or $|y| > d/2$. The kernel $N_z'(t)$ is obtained from $N_z''(t)$ and $N_z'(+0)$ in a straightforward manner by integration, see (3.10). Having obtained this kernel, the solution to the propagation problem in the z-direction is obtained by solving the temporal Volterra integral equations of the second kind (3.3) for the propagator kernels $P^{\mp}(z,t)$. Solving these equations determines thoroughly each mode of propagation for arbitrary excitation $f(z=0,t) = E_z(y=0,z=0,t)$.

3.5.2 Odd longitudinal TM modes

For odd longitudinal TM solutions, $E_z(0,z,t) = 0$, and the solution in the inner region becomes

$$\begin{cases} E_z(y,z,t) = \mathcal{Z}_1\mathcal{K}_1^{-1}\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\sin\left(y\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)f(z,t) \\ \eta_0 H_x(y,z,t) = \cos\left(y\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)f(z,t) \end{cases} \qquad (|y| < d/2),$$

where $f(z,t) = \eta_0 H_x(0,z,t)$ is a real function. Insertion in the condition for finite outer solutions (3.15) gives an equation that determines the modes of propagation:

$$\mathcal{Z}_1\mathcal{K}_1^{-1}\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\tan\left(d/2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)f = \mathcal{Z}_2\mathcal{K}_2^{-1}\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}f. \tag{3.26}$$

The condition (3.15) shows that the outer solution is

$$\begin{cases} E_z(y,z,t) = \operatorname{sgn}(y)\exp\left(-\left(|y|-d/2\right)\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)E_z(d/2,z,t) \\ \eta_0 H_x(y,z,t) = \exp\left(-\left(|y|-d/2\right)\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\eta_0 H_x(d/2,z,t) \end{cases} \qquad (|y| > d/2).$$

Since the principal parts of both members of equation (3.26) must agree, one obtains

$$\sqrt{\lambda_z-\lambda_1}\tan\left(\sqrt{\lambda_z-\lambda_1}\right) = \sqrt{\lambda_2-\lambda_z}, \tag{3.27}$$

where the λ's are given by (3.19).

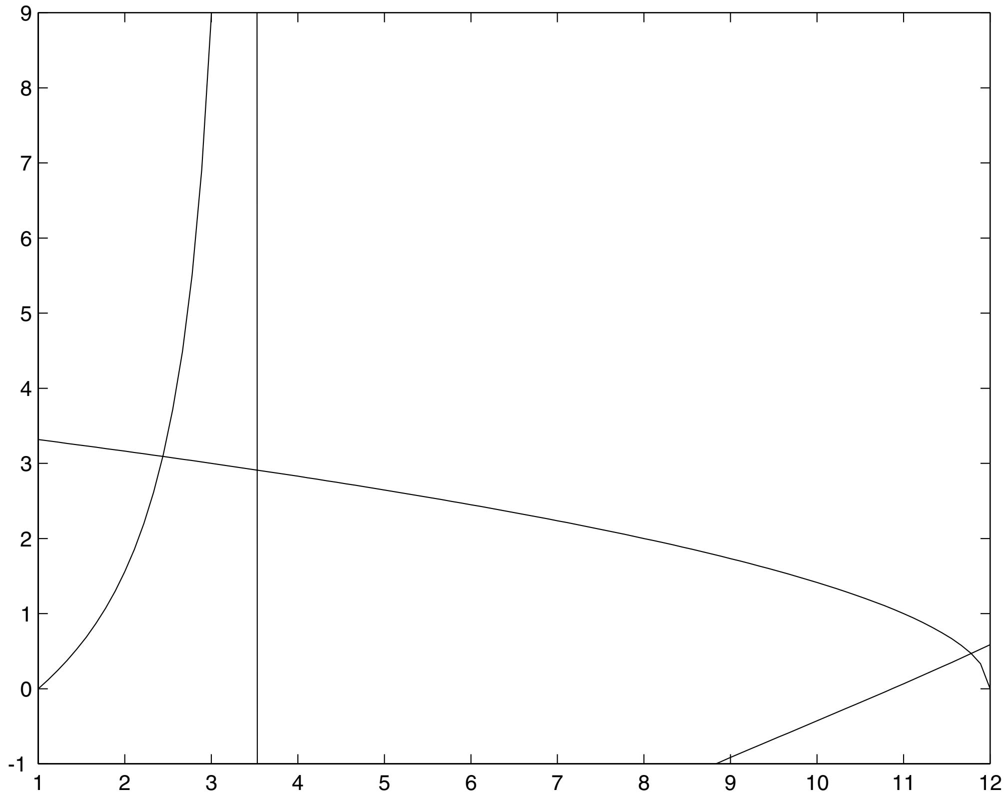

Figure 3: The functions $\sqrt{\lambda_z - \lambda_1}\tan\left(\sqrt{\lambda_z - \lambda_1}\right)$ and $\sqrt{\lambda_2 - \lambda_z}$ restricted to $\lambda_1 \le \lambda_z \le \lambda_2$ in case $\lambda_1 = 1$ and $\lambda_2 = 12$. There are two points of intersection, namely, approximately at $\lambda_z = 2.5$ and at $\lambda_z = 11.8$.

Equation (3.27) shows that the inequalities (3.6) hold: for if $N_z'(+0) > N_2'(+0)$, then the right-hand side of equation (3.27) becomes purely imaginary, which can never be the case for the left-hand side, and, moreover, if $N_1'(+0) > N_z'(+0)$, then equation (3.27) reads

$$0 > -\sqrt{\lambda_1 - \lambda_z}\tanh\left(\sqrt{\lambda_1 - \lambda_z}\right) = \sqrt{\lambda_2 - \lambda_z} > 0,$$

which, since the members are of opposite signs, leads to a contradiction. Furthermore, the equation

$$\sqrt{\lambda_z - \lambda_1}\tan\left(\sqrt{\lambda_z - \lambda_1}\right) = \sqrt{\lambda_2 - \lambda_z} \qquad (\lambda_1 \le \lambda_z \le \lambda_2), \tag{3.28}$$

has at least one zero and at most a finite number of zeros for given values of λ_1, λ_2. This is so because the right-hand side is positive and strictly decreasing, whereas the left-hand side (extended to all $\lambda_z > \lambda_1$) is strictly increasing from the value 0 to the value $+\infty$ in the interval $0 \le \lambda_z - \lambda_1 \le (\pi/2)^2$ and strictly increasing from the value $-\infty$ to the value $+\infty$ in the interval $((n+1/2)\pi)^2 \le \lambda_z - \lambda_1 \le ((n+3/2)\pi)^2$ for $n = 0, 1, 2, 3, \cdots$. The situation when $\lambda_1 = 1$ and $\lambda_2 = 12$ is depicted in Figure 3.

For each possible value of λ_z, the dispersion equation becomes

$$\begin{aligned}&\left(\varepsilon_2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}-i\varepsilon_1\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\exp\left(id\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\\&=\varepsilon_2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}+i\varepsilon_1\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}.\end{aligned}$$

Since, due to (3.28),

$$\left(\sqrt{\lambda_z-\lambda_1}-i\sqrt{\lambda_2-\lambda_z}\right)\exp\left(i2\sqrt{\lambda_z-\lambda_1}\right)=\sqrt{\lambda_z-\lambda_1}+i\sqrt{\lambda_2-\lambda_z},$$

this identity reduces to the temporal Volterra integral equation of the second kind

$$\begin{aligned}&\left(U_1+\sqrt{\lambda_z-\lambda_1}\chi_2^e+U_1*\chi_2^e-iU_2-i\sqrt{\lambda_2-\lambda_z}\chi_1^e-iU_2*\chi_1^e\right)(t)\\&+\Big(\Big(U_1+\sqrt{\lambda_z-\lambda_1}\chi_2^e+U_1*\chi_2^e-iU_2-i\sqrt{\lambda_2-\lambda_z}\chi_1^e\\&\qquad-iU_2*\chi_1^e\Big)(\cdot)*V_1^+(d,\cdot)\Big)(t)\\&+\left(\sqrt{\lambda_z-\lambda_1}-i\sqrt{\lambda_2-\lambda_z}\right)V_1^+(d,t)\\&=c\left(U_1+\sqrt{\lambda_z-\lambda_1}\chi_2^e+U_1*\chi_2^e+iU_2+i\sqrt{\lambda_2-\lambda_z}\chi_1^e+iU_2*\chi_1^e\right)(t),\end{aligned}\tag{3.29}$$

where

$$c=\exp\left(-i2\sqrt{\lambda_z-\lambda_1}\right)=\frac{\sqrt{\lambda_z-\lambda_1}-i\sqrt{\lambda_2-\lambda_z}}{\sqrt{\lambda_z-\lambda_1}+i\sqrt{\lambda_2-\lambda_z}}.$$

Together with equations (3.22), (3.23), and (3.24), (3.29) forms a system of temporal Volterra integral equations of the second kind in unknown kernels: $N_z''(t)$, $U_2(t)$, $U_1(t)$, and $V_1^+(d,t)$. The complete modes are obtained just as in section 3.5.1.

3.6 TE modes

Since the situation for TE modes is quite analogous to the one in the TM case, only the main equations are given. For transverse electric modes,

$$\begin{pmatrix}Q_z\\Q_x\end{pmatrix}=\frac{1}{2}\begin{pmatrix}-i\mathcal{Z}&0\\0&1\end{pmatrix}\begin{pmatrix}\eta_0H_z\\E_x\end{pmatrix}$$

and, consequently,

$$\begin{pmatrix}\eta_0H_z\\E_x\end{pmatrix}=2\begin{pmatrix}i\mathcal{Y}&0\\0&1\end{pmatrix}\begin{pmatrix}Q_z\\Q_x\end{pmatrix}.$$

The propagator for the vector (η_0H_z,E_x) is therefore

$$\begin{pmatrix}\cos\left(y\sqrt{\partial_z^2-\mathcal{K}^2}\right)&-\mathcal{Y}\mathcal{K}^{-1}\sqrt{\partial_z^2-\mathcal{K}^2}\sin\left(y\sqrt{\partial_z^2-\mathcal{K}^2}\right)\\\mathcal{Z}\mathcal{K}\left(\sqrt{\partial_z^2-\mathcal{K}^2}\right)^{-1}\sin\left(y\sqrt{\partial_z^2-\mathcal{K}^2}\right)&\cos\left(y\sqrt{\partial_z^2-\mathcal{K}^2}\right)\end{pmatrix}$$

or, equivalently, the sum of the independent propagators

$$\frac{1}{2}\begin{pmatrix} 1 & \mp\mathcal{Y}\mathcal{K}^{-1}\sqrt{\mathcal{K}^2-\partial_z^2} \\ \mp\mathcal{Z}\mathcal{K}\left(\sqrt{\mathcal{K}^2-\partial_z^2}\right)^{-1} & 1 \end{pmatrix}\exp\left(\mp y\sqrt{\mathcal{K}^2-\partial_z^2}\right). \tag{3.30}$$

In the outer region, the propagators (3.30) apply. In order to obtain finite-energy solutions in each transverse plane, one has to demand that

$$\eta_0 H_z(d/2,z,t) = -\mathcal{Y}_2\mathcal{K}_2^{-1}\sqrt{\mathcal{K}_2^2-\partial_z^2}E_x(d/2,z,t), \tag{3.31}$$

in view of the results presented in section 3.4.

3.6.1 Even longitudinal TE modes

For even longitudinal TE solutions, $E_x(0,z,t)=0$. The solution in the inner region is given by

$$\begin{cases} \eta_0 H_z(y,z,t) = \cos\left(y\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)f(z,t) \\ E_x(y,z,t) = \mathcal{Z}_1\mathcal{K}_1\left(\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)^{-1}\sin\left(y\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)f(z,t) \end{cases} \quad (|y|<d/2),$$

where $f(z,t)=\eta_0 H_z(0,z,t)$ is a real function. Combination with the condition for finite outer solutions (3.31) gives the Volterra integral equation of the second kind

$$-\mathcal{Y}_1\mathcal{K}_1^{-1}\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\cot\left(d/2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)f = \mathcal{Y}_2\mathcal{K}_2^{-1}\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}f$$

or, equivalently, since f is arbitrary, the temporal integral operator identity

$$\begin{aligned}&\left(\mu_1\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}+i\mu_2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\exp\left(id\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2}\right)\\ &= \mu_1\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}-i\mu_2\sqrt{\mathcal{K}_z^2-\mathcal{K}_1^2},\end{aligned}$$

which determines the possible longitudinal refractive indices, $\mathcal{N}_z$, *i.e.*, the modes of propagation. Condition (3.31) shows that, for a fixed mode, the outer solution is

$$\begin{cases} \eta_0 H_z(y,z,t) = \exp\left(-(|y|-d/2)\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)\eta_0 H_z(d/2,z,t) \\ E_x(y,z,t) = \operatorname{sgn}(y)\exp\left(-(|y|-d/2)\sqrt{\mathcal{K}_2^2-\mathcal{K}_z^2}\right)E_x(d/2,z,t) \end{cases} \quad (|y|>d/2).$$

Clearly, the same λ_z's as for even longitudinal TM modes are obtained. The complete even longitudinal TE modes are obtained analogously to the even longitudinal TM modes.

3.6.2 Odd longitudinal TE modes

For odd longitudinal TE solutions, $\eta_0 H_z(0,z,t) = 0$, and the solution in the inner region becomes

$$\begin{cases} \eta_0 H_z(y,z,t) = -\mathcal{Y}_1 \mathcal{K}_1^{-1} \sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2} \sin\left(y\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right) f(z,t) \\ E_x(y,z,t) = \cos\left(y\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right) f(z,t) \end{cases} \quad (|y| < d/2),$$

where $f(z,t) = E_x(0,z,t)$ is a real function. Insertion in the condition for finite outer solutions (3.31) gives an equation that determines the modes of propagation:

$$\mathcal{Y}_1 \mathcal{K}_1^{-1} \sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2} \tan\left(d/2\sqrt{\mathcal{K}_z^2 - \mathcal{K}_1^2}\right) f = \mathcal{Y}_2 \mathcal{K}_2^{-1} \sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2} f.$$

The condition (3.31) shows that the outer solution is

$$\begin{cases} \eta_0 H_z(y,z,t) = \operatorname{sgn}(y) \exp\left(-\left(|y| - d/2\right)\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right) \eta_0 H_z(d/2,z,t) \\ E_x(y,z,t) = \exp\left(-\left(|y| - d/2\right)\sqrt{\mathcal{K}_2^2 - \mathcal{K}_z^2}\right) E_x(d/2,z,t) \end{cases} \quad (|y| > d/2).$$

Clearly, the same λ_z's as for odd longitudinal TM modes are obtained. The complete odd longitudinal TE modes are obtained analogously to odd longitudinal TM modes.

4 Discussion

The main result presented in this paper is the condition (3.8) for existence of propagating pulse modes in the slab waveguide:

$$N_1'(+0) < N_2'(+0). \tag{4.1}$$

The corresponding condition for propagating time-harmonic modes of angular frequency ω is (in the lossless case) well known, see, *e.g.*, Cheng [2]:

$$n_2(\omega) < n_1(\omega), \tag{4.2}$$

where the index of refraction $n_i(\omega)$ is the Fourier transform of the distribution $\delta(t) + N_i(t)$, where $\delta(t)$ is the Dirac delta function ($i = 1, 2$). A natural question to ask is whether these two expressions are connected in some way.

To answer this question, let us restrict ourselves to non-magnetic media and to single-resonance Lorentz models (2.2) (in the lossless case $\upsilon = 0$) in both core (subscript 1) and cladding (subscript 2), and denote the natural frequencies by $\omega_{0,i}$ and the plasma frequencies by $\omega_{p,i}$ $(i = 1, 2)$. If now $\omega >> \max(\omega_{0,1}, \omega_{0,2})$, then

$$n_i(\omega) = \sqrt{1 - \frac{\omega_{p,i}^2}{\omega^2}} \qquad (i = 1, 2)$$

and, consequently, (4.2) becomes $\omega_{p,1}^2 < \omega_{p,2}^2$. This is precisely condition (4.1).

This simple example indicates that condition (4.1) has to be fulfilled in order that high-frequency components propagate. For electric and magnetic susceptibility kernels that are linear combinations of Lorentz kernels, condition (4.1) becomes

$$\sum_i \omega_{p,1,i}^2 < \sum_j \omega_{p,2,j}^2,$$

where i and j run over the number of (electric and magnetic) Lorentz processes in the core (subscript 1) and in the cladding (subscript 2), respectively, and $\omega_{p,1,i}$ and $\omega_{p,2,j}$ are the corresponding plasma frequencies.

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