Dynamic Optimization of Transportation Networks with Delays

Heyden, Martin

2020

Document Version:
Publisher's PDF, also known as Version of record

Link to publication

Citation for published version (APA):
Heyden, M. (2020). *Dynamic Optimization of Transportation Networks with Delays*. [Licentiate Thesis, Department of Automatic Control]. Department of Automatic Control, Lund Institute of Technology, Lund University. Advance online publication.

Total number of authors:
1

Dynamic Optimization of Transportation Networks with Delays

Martin Heyden

LUND UNIVERSITY

Department of Automatic Control

Licentiate Thesis TFRT-3275
ISSN 0280–5316

Department of Automatic Control
Lund University
Box 118
SE-221 00 LUND
Sweden

Printed in Sweden by Media-Tryck.
Lund 2020

Abstract

The topic of this thesis is the optimal control of transportation networks. The problem studied is a dynamical extension of a classical problem in economics, in which the objective is to distribute goods to maximize welfare, whilst satisfying constraints on production and consumption. The main contribution is to show that for a class of welfare functions and dynamics, the optimal control is highly structured, and can be implemented in a way that scales gracefully with network size.

More specifically, it is shown that if the underlying transportation network is structured by a string graph with delays on the edges, an LQ optimal controller can be found by explicitly constructing the solution to a Riccati equation. Next the problem is studied from a user perspective. A method to compensate the users in the network, so that that their choices of levels are also the social optimum is derived. Finally the results are extended to handle directed tree graphs, more general cost functions, and variable production in the network.

In all cases the optimal control can be found by sweeping through the graph once, calculating aggregate utilities and levels. This gives a serial implementation, that is suitable for systems were the is no need for high sample times, such as district heating systems and transportation networks.

Acknowledgments

This research would not have been possible without the guidance and support from my supervisors, Anders Rantzer and Richard Pates. Thank you for for your enthusiasm for my work, and helping me grow as a researchers. I am also thankful for the proofreading of this thesis.

Thanks to Mika, Eva, Monika, Cecilia and Ingrid for good company during the lunch break and for allowing me to never worry about administrative matters. Thanks to Anders Nilsson, Anders Blomdell and Pontus Andersson for their help with all technical issues. Thanks to Leif for his valuable feedback on the typesetting of this thesis. I am also grateful for the great spirit and working environment of the department, which is due to all coworkers and guests, both past and present, at the department. With a special mention to the floorball players and the gym goers.

Thanks to my friends and family for their support and encouragement. It is great to know that you always are there for me. Last but by no means least, thanks to Christina, for listening to all my musings and for all the encouragement and support.

Financial Support

This work was supported by the Swedish Foundation for Strategic Research through the project SSF RIT15-0091 SoPhy.
The author is a member of the LCCC Linnaeus Center and the ELLIIT Excellence Center at Lund University.

Contents

1

Introduction

Optimal resource utilization is important both to maximize the output of an economy, and reduce environmental impact. In this thesis, a transportation problem is studied, with the goal of optimizing the usage of a resource. This will be done mainly from a control perspective, where the goal is to design a controller that can handle large-scale systems.

Many control problems have natural interpretations as dynamical welfare maximization problems. On the other hand, many welfare maximization problems can be solved using methods from control. Examples include transportation networks and district heating systems, where the goal is to distribute the available commodity (goods, heat) optimally among a set of agents. The notion of optimality is typically based on the notion of welfare (or utility) of the individual agents. This thesis will consider problems where the objective is to maximize the sums of the utilities of all the agents in a network (further discussion can be found in Sections 2.4 and 2.1).

Due to globalization and digitalization, these systems, and many others, are growing in size. This leads to new challenges when it comes to controlling them. A major issue is how to handle communication. For example, in current district heating networks it is infeasible for every house to communicate with all the others. This makes it very difficult to implement strategies to optimize performance of the overall network, because the individual agents do not have the required information to implement the optimal control law. Furthermore, it is natural for these systems to change operating point, or change structurally through the addition or removal of agents. To handle this requires scalable control design methods. Finally one must also consider privacy aspects. For example individual agents might not want to (nor be allowed to) share information with the rest of the network. These issues are discussed in Section 2.2, and an overview of previous results in the literature is given in Section 2.3.

In this thesis the classical welfare maximization problem is extended by introducing a transportation network and transportation delays (see Figure 1.1 for a graphical illustration when the transportation network is

Figure 1.1 An illustration of the type of problems considered in this thesis.

a string). The delays and topology introduce dynamical constraints, and one must now consider how the system schedules resource allocation over time when maximizing welfare. The idea is that this captures dynamical aspects that are important for applications that are absent in static problem formulations. This is further described in Section 3.2.

One of the main contributions of this thesis is to derive closed form expressions for the optimal control of such networks when LQ costs are considered. Such controllers typically do not maintain the structure of the network. However, we show that in this case the controller is structured, and can be implemented efficiently using local communication. The communication is implemented by a sweep through the network. Furthermore, the control law is easily updated as agents are added and removed form the network. In the case of string graphs it is also shown how to connect this solution to a market equilibrium, allowing for a price based controller implementation. The results are summarized in Section 3.3, and can also be found in the three papers at the end of the thesis.

2

Background

In this chapter we will give some background to the work presented in the thesis. The chapter starts with some examples of large scale systems. Then follows motivation for why there is a need to develop new methods for design and implementation of controllers for such systems and a brief survey of previous work in the area. Finally some economic background relevant to the problems studied will be given.

2.1 Examples of Large Scale Systems

Traffic and Transportation

The demand on the traffic system increases as more and more people move into urban areas and the transportation of goods increases. A poorly regulated traffic system can lead to congestion, which leads to a severe decrease in performance. There are many tools available to try to increase the throughput of a road network. Such as traffic lights [Nilsson and Como, 2018], variable speed limits and ramp meters [Hegyi et al., 2005].

Transportation of goods around the globe can also be seen as a large transportation system with many different goods, destinations and means of transportation. If the efficiency of such systems can be improved, it would lead to big benefits both in terms of monetary and environmental aspects. For an interesting efficiency measure for such systems, see [Terelius and Johansson, 2015].

District Heating Networks

In todays cities there are many buildings requiring either heating or cooling. The cooling usually results in the excess heat not being utilized. Another example where this can be seen is in the placement of big data centers in cold areas, where the cost of land and cooling is low.

Ideally the heating and cooling of all buildings in a city should be part of the same system so that excess heat can be utilized. Recent initiatives in doing so includes ectogrid by EON [Ectogrid, 2020], which is a system for combining heat pumps and cooling machines. On a similar theme, Bahnhof plans to build a data center in central Stockholm, with the aim that all the excess heat can be utilized [Elementica, 2020].

A city-wide district heating system will naturally be huge in size, containing thousands of buildings, rendering them impossible to control by centralized strategies. Small improvements to the performance of such systems could have big economical and environmental benefits.

Power Networks

A higher penetration of renewable energy production in the grid is very important to reduce the emission of greenhouse gases. However, it gives new challenges for the grid operators. The grid operators need to control the power balance and the voltage of the network. These are heavily coupled as an excess of power increase the voltage. Traditional methods of generating electricity contains a synchronous machines with large inertia, which can be used to easily regulate the power output. It is natural to equip a nuclear plant with a synchronous generator, however it would lead to lower efficiency to do so for a wind farm or a solar panel system. Thus new methods to regulate the power networks must be developed. Failing to do so will, for example, limit how large a percentage of the energy production can be from wind power, while still maintaining grid stability [Mc Garrigle et al., 2013].

The performance requirements will also be increased as with more and more renewables added to the grid there will be a lot more producers to control. Furthermore it is much harder to estimate the possible power output of a renewable compared to a coal or water power plant. This will lead to that the controller must be faster to react to changes power output.

2.2 Need for Structured Controllers

In this section we will motivate the need for structured controllers for large-scale systems. First we will illustrate some of the different control strategies for controlling interconnected systems. We will then discuss some of the issues that appear. That is the communication requirements for each subsystem, how well the controller can handle if components are added to or removed from the network, and also some privacy and coordination aspects.

Different Strategies

In this section we will use the system in Figure 2.1 to illustrate different methods for controlling interconnected systems. The system consists of four subsystems which each have their own input. Furthermore, the subsystems have a direct effect on each other if they are connected by a link. If a

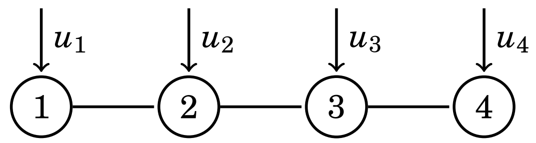

Figure 2.1 A schematic illustration of an interconnected system. The system consists of four parts, indicated by circles. Each subsystem has its own input. Two subsystems affect each other if there is a link between them. Subsystem one then have a direct effect on system two, but not on system three and four.

controller were to be designed for this system using standard methods, such as H_2 or H_∞ control, the controller would generally be dense, i.e. have the following sparsity pattern

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

To calculate any of the inputs u_i it is necessary to know the state of the entire system. For a small number of systems this is no problem, but as more subsystems are added the resulting communication demands will be too great to implement in practice.

One could instead try to design one controller for each subsystem. This would give the following sparsity pattern

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}.$$

This would be very simple to implement, as each subsystem could calculate its input without knowing anything about the other systems. Thus the effort to calculate the input u_i for each input is independent of the network size. However, a new difficulty arises, as it will in general be difficult to ensure stability and design the controllers so that they improve the performance for the entire system. Especially if the controllers are designed with only the local system in mind. Also, the performance is generally expected to be be worse, as less information is available for each decision.

One could also try to design controllers that only use neighboring information. This would give the following sparsity pattern

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} * & * & 0 & 0 \\ * & * & * & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix},$$

This again scales nicely with the size of the network, as the number of other subsystems that need to be communicated with is constant. However the issue of ensuring stability, and optimizing performance when each controller only consider a small part of the system remains. One approach to solve this issue is to design all the controllers together. However, such design might also be infeasible if the network is large, or if it often changes in size.

Another alternative is to try to aggregate information through the graph. For example using the following pattern:

$$\begin{bmatrix} * & * & 0 & 0 \\ 0 & * & * & 0 \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & 0 & 0 \\ 0 & 0 & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

This can be implemented using local communication of states and inputs. For example

$$u_3[t] = c_1 x_3[t] + c_2 u_4[t].$$

Furthermore the scheme allows a simple way for information to propagate through the graph. The obvious downside is that the the time needed to implement the communication will scale with the size of the network. To counteract this one could instead consider a feedback law on the form

$$u_3[t] = c_1 x_3[t] + c_2 u_4[t-1].$$

Communication

As already hinted previously, it would be impossible for for many large scale applications to let every part of the network have global information. Both power networks and district heating networks could consist of thousands of subsystems and if every subsystem were to communicate with each other it would lead to millions of communication channels. This would not be feasible to implement. Alternatively, each subsystem could communicated with some coordinator. Then each subsystem would send its state to the coordinator and receive all other states from the coordinator. This limits the number of communication channels to be proportional the the number of subsystems. However, it still has some issues. Firstly, the size of the messages sent by the coordinator could be very large, leading to increased latency. Secondly, if there is some failure in the coordinator the entire systems breaks down. If there are distributed controllers all over the system, it is likely that the network can handle that a controller breaks.

One way to overcome this issue is to design controllers in each subsystem that only require communication with a subset of the the other subsystems. Typical choices are to only require local information or only communication with neighbors, as illustrated previously. However, any subset of subsystems is in principle possible as long as it doesn't grow with the number of subsystems.

Network grows or shrinks

It is natural for a system to change its operating conditions, for example due to a change in weather for a district heating system or a power grid with a high penetration of renewable energy sources. In some cases the network could also change due to the addition or removal of components to the system. This could for example happen in a power system where different producers are added or removed due to maintenance or shifting weather conditions.

It is thus important that a controller for large scale systems can overcome these issues. A change in operating condition could be overcome by designing a controller that is robust to variations in operating conditions. However, performance will generally be better if the controller is designed with the exact operating conditions in mind. On the other hand, scalable methods for control design will generally give worse performance compared to centralized methods. This results in a choice between having an optimal design for a model that does not take the change in operating conditions into account, or a suboptimal design for a model that does take the changes into account.

When components are added to a system then one must update the controller design, by at the very least designing how to control the new subsystems. To be able to update the controllers efficiently as the system changes in size requires new design methods that are inherently scalable. This could for example be methods that only uses local information, or neighboring information for the design of the controller. However, this will naturally lead to worse performance. A centralized controller synthesis that considers the entire system when designing the controllers for each system will scale poorly with the size of the system. So a centralized design procedure for a distributed controller would be infeasible if the networks is expected to change often.

Privacy and Coordination

Many large scale systems are built by subsystems owned by different entities. Examples include power grids, where solar panels, wind turbines, and other power plants or owned by different persons or companies. And district heating networks where each building might have a different owner. There are also many example where this is not the case, such as when controlling a factory, where each subsystem has the same owner, who only cares about the total performance of the system.

The different owners might want to keep both their performance metric and current states private. Decentralized synthesis and implementation overcomes this issues, by only using local information. However, if the performance of the system is improved by sharing some information with the rest of the network, then the willingness to share information might increase. This gives rise to a trade-off between privacy and performance.

The owners might also care more about their own cost than the cost of the entire systems. It can not be expected for the different players in the systems to make altruistic decisions. Instead there must be some mechanism that makes the individual users choice align with the social optimum. This can be achieved by adding a cost to the users that depends on the choice they make. For example by having road tolls in transportation networks, a price on the heat level in a district heating system, or on the watt usage in a power network.

2.3 Overview of Structured Control

The issues discussed in the previous section has lead to an effort in designing structured controller, that can handle large scale systems. In this section we give an overview of previous work. We make a distinction between when the structure is enforced and when it follows from the plant.

Enforced Structure

Early work on distributed control includes the study of team games [Marschak, 1955; Radner, 1962]. A team consists of member who has access to different information and attempts to make decisions that are optimal for the entire team. The initial results were mainly static.

For LQ control with full information it is well known that the optimal controller is linear and memoryless. However, an important counterexample was given in [Witsenhausen, 1968], where it is shown that this does not need to hold when the controller has information constraints. However, in [Ho et al., 1972] it was shown that for a partially nested information constraint, the optimal LQ controller is linear. Other early examples were the optimal controller was still linear includes [Sandell and Athans, 1974]. Another negative result is [Blondel and Tsitsiklis, 1997], where it is shown that some decentralized control problems are NP hard.

Multilevel or Hierarchical control is a strategy relying on splitting the control problems into different layers. The lowest layer typically is decentralized control of the lowest components of a system, and the higher layers tries to coordinate the lower layers. This approach has for example been applied to power systems [Mansour and El Abiad, 1977; Schweppe, 1978] and manufacturing systems [Jones and McLean, 1986; Boukas et al., 2003].

One way of simplifying the control of large scale systems is to find the optimal equilibrium for the system, and then try to stabilize the system around this equilibrium. This method has for example been applied in internet congestion control [Low et al., 2002] and power systems [Kundur, 1994]. If the operating conditions changes often, and thus also the optimal equilibrium, then the transient of the system can be an important part of the performance of the system. Then another approach is most likely desirable.

Mean field control is another approach. Each subsystem is typically only affected by the average of all other subsystems, and thus only needs to know the average to make its decisions. Recent work include control of charging for electrical vehicles [Parise et al., 2014], and demand management of electric loads [Grammatico et al., 2015]. A benefit of the mean field approach is that it easily allows to control agents, where each agent make their own egoistic choice.

One simple method to enforce structure is via decentralized or distributed PI controllers. Decentralized PI control would be an elegant solution. However, there exists a class of systems where such a controller can never reject a constant disturbance and constant measurement noise [Andreasson et al., 2014]. It is also shown that letting the integral part depend on neighboring nodes extends the class of systems for which the steady state error is zero. However, distributed PI control has been shown to be able to stabilize district heating networks [De Persis et al., 2014].

One method is to formulate the controller design as an optimal control problem, and enforce a structure on the controller. If the resulting problem is convex, it can easily be solved. An important contributions was given in [Rotkowitz and Lall, 2006], where a sufficient conditions for when this is possible was given by defining the notation of quadratic invariance. It was later shown that it is also a necessary condition [Lessard and Lall, 2011]. To just name a few examples, quadric invariance led the way to an optimal controller for a decentralized two player problem [Lessard and Lall, 2012] and a characterization of distributed controllers subject to delays constraints [Matni, 2014].

An alternative to enforcing structure on the plant is to instead have sparsity promoting terms in the optimization problem, see for example [Fardad et al., 2011] and [Lin et al., 2013]. This method should in theory allow for better understanding of the trade off between sparsity and performance.

The convex optimization approach often leads to centralized synthesis, which could be problematic if the size of the network often changes. In [Langbort et al., 2004] distributed synthesis algorithms are derived for H_∞ control over arbitrary graphs. More recent work includes include system level synthesis [Anderson et al., 2019] where the synthesis can in some cases be of order $\mathcal{O}(1)$.

Achievable Performance It is in general unknown how controller constraints affect the achievable performance of the control system. However there exists results for specific cases, and some of those are presented here.

In [Pates et al., 2017] it is shown that for a platoon of vehicles an accordion like motion will emerge for any controller using only local measurement.

It is also generally unknown how the communication constraints affects the achievable performance. In [Langbort and Delvenne, 2010] it is shown that for a class of LTI discrete systems a controller without communication is at least twice as bad as the optimal controller in the worst case.

Robustness analysis of sparsely interconnected systems using IQC is considered in [Andersen et al., 2014]. A graphical test for robust stability test of interconnected systems was derived in [Kao et al., 2009].

Structure from the plant

An alternative to enforcing the structure is to find problems where the optimal unconstrained controller can be implemented in a structured way. This has the obvious downside that it is not applicable to most systems, and generally requires simple models. However, when the method is possible it allows for the usage of results from the general theory of control, such as robustness and performance guarantees. Furthermore, controllers resulting in this way are often easy to understand. The downside of results of this nature is that they will often not generalize well.

In [Madjidian and Mirkin, 2014] it is shown that the optimal control of a set of wind turbines can be optimally controlled using a combination of a distributed control and a rank one coordination term. This means that the centralized coordinator only need to send out one measurement instead of measurements for every subsystem.

For infinite dimensional spatially invariant systems it is shown that the optimal controller for quadratic objectives has an inherent degree of decentralization [Bamieh et al., 2002].

For H_∞ control it is harder to know if there exists a structured optimal controller as the optimal controller is not unique. For systems with symmetric and Hurwitz state matrix, it is shown in [Lidström and Rantzer, 2016] that there exists a structured H_∞ controller that is suitable for distributed implementation.

2.4 Two Classical Problems in Economics

The economical aspects of control systems are often not considered. However, there could be an increasing need for doing so when they are applied to systems with multiple entities. Mainly to ensure that the systems are fair for all its users, but also to understand how economic decisions within the systems affects the performance of the control systems.

In this section we will give a brief introduction to two classical economics problems. These problems are static, and the models studied in this thesis will be shown to be a natural dynamic extension of these problems.

Competitive Market Equilibrium

A classical problem in economics is to find the equilibrium in a market. We will limit ourself to study only one good. Then any equilbrium is called a partial equilibrium. Often every agent in a system will consider multiple goods at the same time, and then an equilibrium is called a general equilibrium.

Here a simple problem is considered. A set of n agents are each given an initial endowment w_i of some good. The agents can then buy and sell goods from each other. It is assumed that the price p is the same between all agents. Let the amount of the good for each agent after the trade be x_i and that each agent value that level according to $U_i(x_i)$. Then the payoff of each agent is given by

$$U_i(x_i) - p(x_i - w_i). \tag{2.1}$$

For the market to be in equilibrium the total demand must be equal to the total supply:

$$\sum_i x_i = \sum_i w_i. \tag{2.2}$$

The supply is fixed, but the demand depends on the price, so a prices can only be an equilibrium price if the corresponding choices of x_i satisfy. (2.2).

In general the agents choice of x_i could depend on how it affects the prices p. Often each agent assumes that the prices are outside of their control. Such a market is called a competitive market. Then each agent will chose the x that maximizes (2.1). Every competitive equilibrium will be Pareto efficient, i.e. satisfy that there is no way to make any individual better off without making someone else worse off.

Welfare Maximization

The welfare maximization problem instead aims to maximize the total utility. That is to choose the amount of quantity each individual has so that the welfare for the entire system is maximized. This can be formulated as an optimization problem,

$$\begin{aligned} \text{maximize} \quad & W(U_1(x), \dots, U_n(x)) \\ \text{subject to} \quad & \sum_i x_i = \sum_i w_i. \end{aligned} \tag{2.3}$$

We let $U_i(x)$ only depend on x_i and the welfare function W is given by the the sum of the individual utilities. Then the optimal welfare distribution is also Pareto efficient.

Actually every welfare maxima is a competitive equilibrium [Varian, 2003]. We illustrate this point via an example. Consider the welfare maximization problem

$$\begin{aligned} \text{maximize} \quad & \sum_i U_i(x_i) \\ \text{subject to} \quad & \sum_i x_i = \sum_i w_i \end{aligned} \tag{2.4}$$

where U_i is concave. The Lagrangian of the system is given by

$$\mathcal{L}(x, \lambda) = \sum_i U_i(x_i) + \lambda(\sum_i w_i - \sum_i x_i)$$

There exists x^* and λ^* so that x^* is the maximizer of (2.4) and

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_i}(x_i^*, \lambda^*) &= U_i'(x_i^*) - \lambda^* = 0 \\ \frac{\partial \mathcal{L}}{\partial \lambda}(x_i^*, \lambda^*) &= \sum_i w_i - \sum_i x_i^* = 0 \end{aligned}$$

Then if we pick the price p to be $p = \lambda^*$ the first equation implies that x_i^* is the maximizer of the node utility (2.1) and the second implies that supply is equal to demand, that is (2.2) holds. We have thus constructed a competitive equilibrium based on the welfare maximizer.

3

Contributions

The main part of this thesis is the three publications presented in Section 3.1. We give a slightly different motivation for the problem, compared to the one given in the papers in Section 3.2. The results will be briefly summarized in Section 3.3 and some possible directions for future work will be given in Section 3.4.

3.1 Publications

This Licentiate thesis is based on the following three papers where the contribution of the author has been noted.

Heyden, M., R. Pates, and A. Rantzer (2018). “A structured linear quadratic controller for transportation problems”. In: *2018 European Control Conference (ECC)*. IEEE, pp. 1654–1659.

The model was suggest by A. Rantzer. M. Heyden noticed the interesting controller structure, derived the results and wrote the manuscript. A. Rantzer and R. Pates helped revise the manuscript.

Heyden, M., R. Pates, and A. Rantzer (2020). “Price based linear quadratic control under transportation delay”. *Accepted to IFAC World Congress 2020*.

The idea to use prices as coordination is due to A. Rantzer. M. Heyden derived the results and wrote the manuscript. A. Rantzer and R. Pates helped revise the manuscript.

Heyden, M., R. Pates, and A. Rantzer (2020). “Optimal transportation on directed tree graphs”. *Manuscript prepared for journal submission*.

M. Heyden derived the results and wrote the manuscript. A. Rantzer and R. Pates helped revise the manuscript.

3.2 The Problem Studied

Consider the problem in (2.3). A natural extension is to only allow transportation between some of the individuals. Then the problem would be

$$\begin{aligned} \text{maximize} \quad & W\Big(U_1(x_1), \dots, U_n(x_n)\Big) \\ \text{subject to} \quad & x_i = w_i + \sum_j u_{ij} \\ & u_{ij} = -u_{ji}. \end{aligned}$$

In the above the sum is over nodes j that are connected to i and u_{ij} is the transportation from node j to node i. It must hold that $u_{ij} = -u_{ji}$. However, the optimal x will not change as long as the corresponding graph is connected, as then goods can be transported between all the nodes in the system. It is natural to also introduce transportation delays. Then the problem becomes dynamic, and we must consider the optimization over a time horizon T. If we also allow for the goods to be decaying with a rate $1 - \alpha$, where $0 < \alpha \leq 1$ and $\alpha = 1$ corresponds to no decay, the welfare maximization problem can be stated as

$$\begin{aligned} \text{maximize} \quad & \sum_{t=0}^{T} W\Big(U_1(x_1), \dots, U_n(x_n)\Big) \\ \text{subject to} \quad & x_i[t+1] = \alpha(x_i[t] + \sum_j u_{ij}[t-1]) - \sum_h u_{hi}[t] \\ & x_i[0] = w_i. \end{aligned} \tag{3.1}$$

In the above the sum over j are the nodes that send goods to node i. And the sum over h are the nodes to which i sends goods. Here it could in principle hold that $u_{ij} \neq u_{ji}$. However, it would be a suboptimal choice as long as the utility functions are increasing. Throughout this thesis we will assume that the welfare function is on the form

$$W\Big(U_1(x_1), \dots, U_n(x_n)\Big) = \sum_i U_i(x_i).$$

The problem is solved by a two step approach. First, the optimal equilibrium of the system is found. That is finding the x_i^* and u_{ij}^* that are the maximizer of

$$\begin{aligned} \underset{x,u}{\text{maximize}} \quad & \sum_i U_i(x_i) \\ \text{subject to} \quad & x_i = \alpha(x_i + \sum_j u_{ij}) - \sum_h u_{hi}. \end{aligned}$$

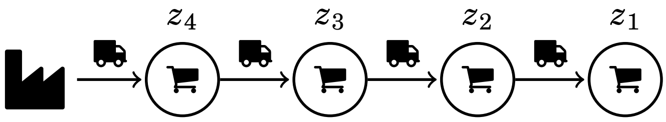

Figure 3.1 An illustration of the type of problems considered in this thesis. Here the transportation network is that of a string graph. There is a factory that produces a good, which is then transported to the different stores in the network. The goal is to optimize the inventory level of each store.

The problem can be solved using standard methods from convex optimization. The folows u_{ij} will in general not be unique. However if the underlying graph is a directed tree, i.e. does not contain any directed cycles, then the solution is unique. And if one want a solution with as few transportation links as possible, then the network should be a directed tree.

The contribution of this thesis lies in the second step, where the dynamical aspect of the problem is considered. We define z_i to be the shift from the equilibrium level according to

$$x_i = x_i^* + z_i$$

The system is studied relative to the equilibrium and the problem is to find the optimal time dependent flows.

In Paper I we study this problem for a string graph. See fig. 3.1 for an illustration. We also assume that the utilities relative to the equilibrium is given by

$$U_i(x_i^* + z_i) - U_i(x_i^*) = -q_i z_i^2.$$

In Paper II the dynamical versions of (2.1) and (2.2) are considered. It is assumed that each agent has to pay for its change in level at every time point, as this is consistent with the payment for the static problem. Then each node considers the following problem

$$\underset{z_i}{\text{maximize}} \quad \sum_{t=0}^{T} U_i(z_i[t]) - p_i[t](z_i[t] - z_i[t-1]). \tag{3.2}$$

Note that the price is allowed to be both time varying and different for different nodes. We also extend the utility functions to contain a linear term,

$$U_i(z) = U_i(x_i + z_i) - U_i(x_i) = b_i z_i - q_i z_i^2.$$

In Paper III the case for directed tree graphs is studied. Furthermore, it is shown how to handle variable production and general concave utility functions.

3.3 Results

In Paper I it is shown that the optimal unconstrained controller is highly structured, with the following diagonal structure pattern

$$\begin{bmatrix} u_{n-1,n}[t] \\ u_{n-2,n-1}[t] \\ u_{n-3,n-2}[t] \\ \vdots \\ u_{32}[t] \\ u_{21}[t] \end{bmatrix} = \begin{bmatrix} \star & \blacklozenge & \blacklozenge & \cdots & \blacklozenge & \blacklozenge & \blacklozenge \\ 0 & \star & \blacklozenge & \cdots & \blacklozenge & \blacklozenge & \blacklozenge \\ 0 & 0 & \star & \cdots & \blacklozenge & \blacklozenge & \blacklozenge \\ \vdots & \vdots & \ddots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \star & \blacklozenge & \blacklozenge \\ 0 & 0 & \cdots & 0 & 0 & \star & \blacklozenge \end{bmatrix} \begin{bmatrix} z_n[t] \\ z_{n-1}[t] + u_{n-1,n}[t-1] \\ z_{n-2}[t] + u_{n-2,n-1}[t-1] \\ \vdots \\ z_2[t] + u_{32}[t-1] \\ z_1[t] + u_{21}[t-1] \end{bmatrix}$$

In the above each $\blacklozenge$ indicates that the elements are the same on that row. Thus to calculate the optimal input only the local level and the aggregate downstream level is needed. Furthermore, it is shown that there exists a distributed method for calculating the parameters needed to implement the controller using local communication. This method consists of a sweep through the graph, starting at the most downstream node, and then going upstream.

In Paper II a each node considers the cost function

$$\sum_{t=0}^{T} U_i(z_i[t]) + p_i z_i[t].$$

It is shown that this problem is equivalent to a natural dynamic extension of (3.2) Expressions for prices $p_i[t]$ are given so that the market equilibrium is also a welfare maximizer. The outflow from each node i is shown to be given by

$$u_{i-1}[t] = \alpha(z_i[t] + u_i[t-1]) - \frac{1}{q_i}\big(p_i[t+1] - b_i\big).$$

In the above p_i is the price, and b_i and q_i are problem data. It is also shown that the prices have a distributed update rule, similar to the the one for the parameters in Paper I.

In Paper III it is shown that the optimal production only depends on the total level in the graph. Furthermore the optimal transportation is only dependent on the aggregate level in the two subsets of the graph. These aggregates can be efficiently calculated by iterating through the graph and only using local communication. The structure for the optimal controller for

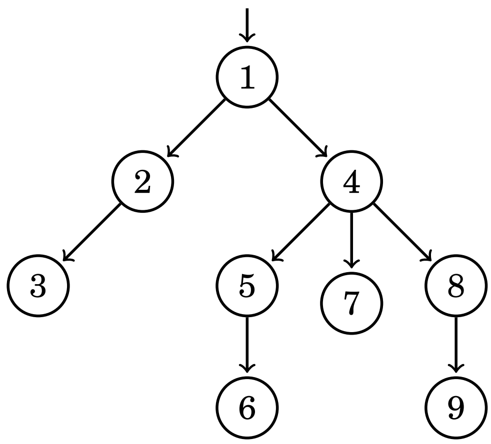

Figure 3.2 An example of a directed tree. The structure of the optimal controller can be seen in (3.3)

the graph in Figure 3.2 is

$$\begin{bmatrix} u_{10}[t] \\ u_{21}[t] \\ u_{32}[t] \\ u_{41}[t] \\ u_{54}[t] \\ u_{65}[t] \\ u_{74}[t] \\ u_{84}[t] \\ u_{98}[t] \end{bmatrix} = \begin{bmatrix} * & * & * & * & * & * & * & * & * \\ \star & \blacklozenge & \blacklozenge & \star & \star & \star & \star & \star & \star \\ 0 & \star & \blacklozenge & 0 & 0 & 0 & 0 & 0 & 0 \\ \star & \star & \star & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge & \blacklozenge \\ 0 & 0 & 0 & 0 & \star & \blacklozenge & 0 & 0 & 0 \\ 0 & 0 & 0 & \star & \star & \star & \blacklozenge & \star & \star \\ 0 & 0 & 0 & \star & \star & \star & \star & \blacklozenge & \blacklozenge \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \star & \blacklozenge \end{bmatrix} \begin{bmatrix} v_{10}[t] + z_1[t] \\ v_{21}[t] + z_2[t] \\ v_{32}[t] + z_3[t] \\ v_{41}[t] + z_4[t] \\ v_{54}[t] + z_5[t] \\ v_{65}[t] + z_6[t] \\ v_{74}[t] + z_7[t] \\ v_{84}[t] + z_8[t] \\ v_{98}[t] + z_9[t] \end{bmatrix} \tag{3.3}$$

In the above $\star$ and $\blacklozenge$ indicates that the elements on that rows are the same.

In summary the main contribution of the thesis lies in showing that the optimal LQ problem is highly structured and can be implemented by using local communication. It is also shown how to design prices so that the welfare maximum is also a market equilibrium.

3.4 Future Work

It is expected that the results presented here could be extended to allow for multiple commodities. It should also be straightforward to extend the results in Paper II to directed trees.

The results presented here could be used to try to analyze the properties of the solution. One could for example consider how a nodes position in the network affect its utility.

From a structured control point of view it would be interesting to add a transportation penalty, and impose the structure of the controller to be the same as presented in the thesis.

Bibliography

Andersen, M. S., S. K. Pakazad, A. Hansson, and A. Rantzer (2014). "Robust stability analysis of sparsely interconnected uncertain systems". *IEEE Transactions on Automatic Control* **59**:8, pp. 2151–2156.

Anderson, J., J. C. Doyle, S. H. Low, and N. Matni (2019). "System level synthesis". *Annual Reviews in Control*.

Andreasson, M., D. V. Dimarogonas, H. Sandberg, and K. H. Johansson (2014). "Distributed pi-control with applications to power systems frequency control". In: *2014 American Control Conference*. IEEE, pp. 3183–3188.

Bamieh, B., F. Paganini, and M. A. Dahleh (2002). "Distributed control of spatially invariant systems". *IEEE Transactions on automatic control* **47**:7, pp. 1091–1107.

Blondel, V. and J. N. Tsitsiklis (1997). "Np-hardness of some linear control design problems". *SIAM journal on control and optimization* **35**:6, pp. 2118–2127.

Boukas, E. et al. (2003). "Hierarchical control of production and maintenance rates in manufacturing systems". *Journal of quality in maintenance engineering* **9**:1, pp. 66–82.

De Persis, C., T. N. Jensen, R. Ortega, and R. Wisniewski (2014). "Output regulation of large-scale hydraulic networks". *IEEE Transactions on Control Systems Technology* **22**:1, pp. 238–245.

Ectogrid (2020). *The ectogrid^{tm} technology*. URL: `http://ectogrid.com/business-customer/` (visited on 2020-02-20).

Elementica (2020). *Turning waste heat into revenue*. URL: `https://www.elementica.se/technology/` (visited on 2020-02-20).

Fardad, M., F. Lin, and M. R. Jovanović (2011). "Sparsity-promoting optimal control for a class of distributed systems". In: *Proceedings of the 2011 American Control Conference*. IEEE, pp. 2050–2055.

Grammatico, S., B. Gentile, F. Parise, and J. Lygeros (2015). "A mean field control approach for demand side management of large populations of thermostatically controlled loads". In: *2015 European Control Conference (ECC)*. IEEE, pp. 3548–3553.

Hegyi, A., B. De Schutter, and H. Hellendoorn (2005). "Model predictive control for optimal coordination of ramp metering and variable speed limits". *Transportation Research Part C: Emerging Technologies* **13**:3, pp. 185–209.

Ho, Y.-C. et al. (1972). "Team decision theory and information structures in optimal control problems–part i". *IEEE Transactions on Automatic control* **17**:1, pp. 15–22.

Jones, A. T. and C. R. McLean (1986). "A proposed hierarchical control model for automated manufacturing systems". *Journal of manufacturing systems* **5**:1, pp. 15–25.

Kao, C.-Y., U. Jönsson, and H. Fujioka (2009). "Characterization of robust stability of a class of interconnected systems". *Automatica* **45**:1, pp. 217–224.

Kundur, P. (1994). *Power System Stability and Control*. McGraw-Hill Professional. ISBN: 007035958X.

Langbort, C., R. S. Chandra, and R. D'Andrea (2004). "Distributed control design for systems interconnected over an arbitrary graph". *IEEE Transactions on Automatic Control* **49**:9, pp. 1502–1519.

Langbort, C. and J.-C. Delvenne (2010). "Distributed design methods for linear quadratic control and their limitations". *IEEE Transactions on Automatic Control* **55**:9, pp. 2085–2093.

Lessard, L. and S. Lall (2011). "Quadratic invariance is necessary and sufficient for convexity". In: *Proceedings of the 2011 American Control Conference*. IEEE, pp. 5360–5362.

Lessard, L. and S. Lall (2012). "Optimal controller synthesis for the decentralized two-player problem with output feedback". In: *2012 American Control Conference (ACC)*. IEEE, pp. 6314–6321.

Lidström, C. and A. Rantzer (2016). "Optimal H_∞ state feedback for systems with symmetric and hurwitz state matrix". In: *2016 American Control Conference (ACC)*. IEEE, pp. 3366–3371.

Lin, F., M. Fardad, and M. R. Jovanović (2013). "Design of optimal sparse feedback gains via the alternating direction method of multipliers". *IEEE Transactions on Automatic Control* **58**:9, pp. 2426–2431. DOI: 10.1109/TAC.2013.2257618.

Low, S. H., F. Paganini, and J. C. Doyle (2002). "Internet congestion control". *IEEE control systems magazine* **22**:1, pp. 28–43.

Madjidian, D. and L. Mirkin (2014). "Distributed control with low-rank coordination". *IEEE Transactions on Control of Network Systems* **1**:1, pp. 53–63.

Mansour, M. O. and A. H. El Abiad (1977). "Hierarchical control of power systems". *IFAC Proceedings Volumes* **10**:1, pp. 130–134.

Marschak, J. (1955). "Elements for a theory of teams". *Management science* **1**:2, pp. 127–137.

Matni, N. (2014). "Distributed control subject to delays satisfying an $\mathcal{H}_\infty$ norm bound". In: *53rd IEEE Conference on Decision and Control*. IEEE, pp. 4006–4013.

Mc Garrigle, E. V., J. P. Deane, and P. G. Leahy (2013). "How much wind energy will be curtailed on the 2020 irish power system?" *Renewable Energy* **55**, pp. 544–553.

Nilsson, G. and G. Como (2018). "Evaluation of decentralized feedback traffic light control with dynamic cycle length". *IFAC-PapersOnLine* **51**:9, pp. 464–469.

Parise, F., M. Colombino, S. Grammatico, and J. Lygeros (2014). "Mean field constrained charging policy for large populations of plug-in electric vehicles". In: *53rd IEEE Conference on Decision and Control*. IEEE, pp. 5101–5106.

Pates, R., C. Lidström, and A. Rantzer (2017). "Control using local distance measurements cannot prevent incoherence in platoons". In: *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*. IEEE, pp. 3461–3466.

Radner, R. (1962). "Team decision problems". *The Annals of Mathematical Statistics* **33**:3, pp. 857–881.

Rotkowitz, M. and S. Lall (2006). "A characterization of convex problems in decentralized control". *IEEE Transactions on Automatic Control* **51**:2, pp. 274–286.

Sandell, N. and M. Athans (1974). "Solution of some nonclassical LQG stochastic decision problems". *IEEE Transactions on Automatic Control* **19**:2, pp. 108–116.

Schweppe, F. C. (1978). "Power systems2000': hierarchical control strategies". *IEEE spectrum* **15**:7, pp. 42–47.

Terelius, H. and K. H. Johansson (2015). "An efficiency measure for road transportation networks with application to two case studies". In: *2015 54th IEEE Conference on Decision and Control (CDC)*. IEEE, pp. 5149–5155.

Varian, H. R. (2003). *Intermediate Microeconomics:A Modern Approach*. 6th ed. W. W. Norton & Company. ISBN: 0-393-97830-3.

Witsenhausen, H. S. (1968). “A counterexample in stochastic optimum control”. *SIAM Journal on Control* **6**:1, pp. 131–147.

Paper I

A Structured Linear Quadratic Controller for Transportation Problems

Martin Heyden Richard Pates Anders Rantzer

Abstract

We study a linear quadratic control problem for transportation optimization on a directed line graph. We show that the solution to the Riccati equation associated with this problem is highly structured. The feedback law is almost upper triangular, and the synthesis of the feedback law is given by a recursion, making it scalable. The structure of the feedback law also allows for an efficient realization of the controller using a local communication scheme.

1. Introduction

In this paper we study a transportation problem on a line graph. The problem can be formulated as an infinite horizon Linear Quadratic (LQ) problem:

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \mathbb{E}\left(\sum_{t=0}^{\infty} x[t]^T Q x[t]\right) \\ \text{subject to} \quad & x[t+1] = \sqrt{\alpha} A x[t] + Bu[t] + w. \end{aligned} \tag{1}$$

In the above, A, B, Q are compatibly dimensioned matrices. The constant α a scalar, and w a vector of normally distributed zero mean random variables. Our main contribution, which is presented formally in Section 3, is to show that when A, B, Q have a particular structure, an optimal control u can be obtained from the formula

$$u_k = \beta_k(g_{k+1} + r_{k+1}) - (1-\beta_k)\sum_{i=1}^{k} g_i + r_i.$$

Here g_k and r_k can be interpreted as local measurements for node k. We give a closed form expression of β_k, which is iteratively calculated. Furthermore, when the system is extended to larger size, the β_k's need not be recalculated. Interestingly this means that the resulting controller is inherently structured, and exhibits a closed form solution that is easily updated as the graph shrinks or grows. Moreover, for the transportation problem, the control loop has a natural scalable interpretation that relies on a simple and local communication scheme. These important observations will be highlighted in Section 4.

The described properties are interesting for large scale system since classical methods such as LQ- and $\mathcal{H}_\infty$-control often becomes infeasible as the feedback matrices are generally dense. This leads to requirements on each actuator to have global information. Furthermore, if there were to be a small change to the system, the entire control synthesis would normally need to be recalculated.

At its heart, this work is another contribution to the field of structured optimal control. Early work include studies on team game problems. In those problems, a set of agents have different information and work toward a common goal. See for example [Radner, 1962], [Ho et al., 1972].

More recently, attempts to formalize the role of structure have been made. In [Rotkowitz and Lall, 2006], it is shown that subject to satisfying a quadratic constraint, the Youla parameterization inherits the structure of the control, allowing for efficient computation of optimal controllers. [Lamperski and Lessard, 2015] presents a class of decentralized controllers for the LQ problems, where the controller and plant satisfy the same delay and

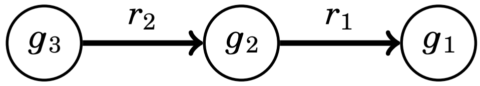

Figure 1. Illustration of the states for delayed mass transfer on a linegraph of three nodes.. The delays are implemented using states at the links, r_i that corresponds to the mass in transit. The value g_k is the mass in each node.

sparsity constraints. In [Shah and Parrilo, 2013], a poset-causal constraint on the controller is added to the $\mathcal{H}_2$ problem. This constraint is similar to the structure of the controller that we attain, by solving a unconstrained problem.

Examples where the structure is not imposed on the controller, but rather a consequence of the plant include [Bamieh et al., 2002], where it is shown that for spatially invariant systems, the optimal controller is localized in space. In [Madjidian and Mirkin, 2014], an optimal control problem for coordination is solved. The solution is structured, containing a diagonal part and a rank one part.

Our controller allows for a structured controller that solves the unconstrained problem. Furthermore, the controller can be efficiently calculated via a closed form iterative expression. Our work relies on a classical Riccati based method.

Notation

We let $\mathbf{0}$ denote a column vector of zeros, and $\mathbf{1}$ a column vector of ones. The first basis vector is written as $\mathbf{e}_1 = [1, 0, \ldots, 0]^T$. For these three type of vectors, the size is always clear from context. Furthermore, we let $\mathbb{E}$ denote expectation and $\mathbb{R}$ the rational numbers.

2. Motivating Problem

Consider transportation of goods with unit delay on a line graph. Such dynamics can be described by the following difference equations,

$$\begin{aligned} g_k[t+1] &= g_k[t] - u_{k-1}[t] + r_k[t] + w_k \\ r_k[t+1] &= u_k[t]. \end{aligned} \tag{2}$$

Here g_k is the amount of goods in node k and r_k is the goods in transit, about to be received at node k. w_k is zero mean white noise. The input u_k is the amount of goods that is sent from node $k+1$ to node k. See Figure 1 for an illustration of the three node case.

REMARK 1
We do not restrict the input u_k to be positive. We will instead work around a nominal flow, a negative input will correspond to sending less goods compared to the nominal flow. □

Now, for N nodes, let the state space $x \in \mathbb{R}^{2N-1}$ be described by

$$x = [g_N, r_{N-1}, g_{N-1}, \dots, r_1, g_1], \tag{3}$$

and input space $u \in \mathbb{R}^{N-1}$ by

$$u = [u_{N-1}, \dots, u_1]. \tag{4}$$

Starting with $N = 2$ the system can be described by $x[t+1] = A_2 x[t] + B_2 u[t]$, with

$$A_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}. \tag{5}$$

Now, given that we have a state space description for $k-1$ nodes, we can find one for k nodes by adding one node, one delay state, and one input according to (3) and (4). This gives the following recursion

$$A_k = \begin{bmatrix} 1 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{e}_1 & A_{k-1} \end{bmatrix}, \quad B_k = \begin{bmatrix} -1 & \mathbf{0}^T \\ 1 & \mathbf{0}^T \\ \mathbf{0} & B_{k-1} \end{bmatrix}. \tag{6}$$

For a graph of N nodes we let $A = A_N$ and $B = B_N$. We can then write the dynamics for the N node system as $x[t+1] = Ax[t] + Bu[t]$. If there were a decay of goods with decay rate $\sqrt{\alpha}$, the dynamics would be $x[t+1] = \sqrt{\alpha}Ax[t] + Bu[t]$.

Note that the problem is not symmetric, and the underlying graph is directed. We say that the links are in the direction from the sender to the receiver. We also define downstream as in the direction of the links, and upstreams as the opposite direction.

We can let $g = 0$ correspond to the optimal inventory level. This will not change the dynamics. Then it is reasonable to penalize deviation from this inventory levels. Let $Q = Q_N$ be defined as

$$Q_N = \text{diag}(q_N, 0, q_{N-1}, 0, \dots, q_1). \tag{7}$$

Then $x^T Q x$ will describe the total penalty, where we allow for different nodes to have different weighting.

Now assume that we can never reach the optimal inventory levels everywhere, due to there not being enough goods. Then there will be no need to penalize goods in transit, as they are already implicitly punished by not being available in any node.

3. Sparse Controller for a LQ problem

In this section we aim to solve two optimal control problems subject to the dynamics and cost function in the previous section. The first problem is the infinite horizon LQ problem, given in (1). The second problem is a discounted infinite horizon LQ problem. Let the discount factor α take values $0 < \alpha < 1$. The problem is formulated as

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \mathbb{E}\left(\sum_{t=0}^{\infty} \alpha^t x[t]^T Q x[t] \right) \\ \text{subject to} \quad & x[t+1] = Ax[t] + Bu[t] + w. \end{aligned} \tag{8}$$

REMARK 2
The reader has by now noticed that there is no penalty on the input. This is not a coincidence, and will indeed be necessary for the results that will be presented. □

We now aim to solve problems (1) and (8). This is done using a Riccati based approach.

The difference Riccati equation appears when solving finite horizon LQ problems, see for example [Bertsekas, 2012]. If the iteration of the difference equation converges to a fix-point, then that fix-point solves the algebraic Riccati equation. This equation can then be used to solve the infinite horizon problem. Some of the available convergence and uniqueness results can be found in [Bitmead and Gevers, 1991]. These do however require a penalty on the input given by a positive definite matrix. Work on Riccati equation with singular input penalty includes [Ntogramatzidis and Ferrante, 2015]. We will use a simple proof to show that the feedback law given by the solution to the Riccati equation is indeed optimal.

It is easy to show that for both problem formulations in (1) and (8), the corresponding difference Riccati equation is

$$X_{j+1} = \alpha A^T X_j A - \alpha A^T X_j B(B^T X_j B)^{-1} B^T X_j A + Q.$$

Note that the index j denotes the iteration number, instead of the size of the system. Any fix-point satisfies the algebraic Riccati equation,

$$\alpha A^T X A - X + Q = \alpha A^T X B(B^T X B)^{-1} B^T X A. \tag{9}$$

We show that, for these matrices, there exist at least one positive definite solution of (9) by explicitly constructing it. The proposed solution is highly structured. Next, we show that the solution can be used to construct the optimal feedback law.

THEOREM 1
Given $A = A_N, B = B_N$ as in (5)-(6) and $Q = Q_N$ as in (7), recursively define γ_k as

$$\gamma_{k+1} = \alpha \frac{q_{k+1}\gamma_k}{q_{k+1} + \gamma_k}, \quad \gamma_1 = \alpha q_1.$$

Also define $\tilde{X} = \tilde{X}_N$ by the recursion:

$$\tilde{X}_{k+1} = \begin{bmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & \gamma_k & \gamma_k \mathbf{1}^T \\ \mathbf{0} & \gamma_k \mathbf{1} & \gamma_k \mathbf{1}\mathbf{1}^T + \tilde{X}_k \end{bmatrix}, \quad \tilde{X}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \gamma_1 & \gamma_1 \\ 0 & \gamma_1 & \gamma_1 \end{bmatrix}.$$

Then one positive definite solution to the Riccati equation (9) is given by

$$X = \frac{1}{1-\alpha}\gamma_N \mathbf{1}\mathbf{1}^T + Q + \tilde{X}, \tag{10}$$

□

For the proof see the appendix.

The corresponding feedback matrix for (8) $K = -(B^T X B)^{-1} B^T X A$ for X in (10) is given by

$$\begin{aligned} K_{k+1} &= \begin{bmatrix} \frac{q_{k+1}}{q_{k+1}+\gamma_k} & -\frac{\gamma_k}{q_{k+1}+\gamma_k} & -\frac{\gamma_k}{q_{k+1}+\gamma_k}\mathbf{1}^T \\ 0 & K_k \mathbf{e}_1 & K_k \end{bmatrix}, \\ K_2 &= \begin{bmatrix} \frac{q_2}{q_2+\gamma_1} & -\frac{\gamma_1}{q_2+\gamma_1} & -\frac{\gamma_1}{q_2+\gamma_1} \end{bmatrix}, \end{aligned} \tag{11}$$

This gives the input $u = Kx$,

$$\begin{aligned} u_k &= \frac{q_{k+1}}{q_{k+1}+\gamma_k}(g_{k+1} + r_{k+1}) - \frac{\gamma_k}{q_{k+1}+\gamma_k}\sum_{i=1}^{k} g_i + r_i \\ u_{N-1} &= \frac{q_N}{q_N+\gamma_{N-1}} g_N - \frac{\gamma_{N-1}}{q_N+\gamma_{N-1}}\sum_{i=1}^{N-1} g_i + r_i. \end{aligned} \tag{12}$$

The corresponding version for (1) is given by $K = -(B^T X B)^{-1} B^T X \sqrt{\alpha} A$, which gives

$$\begin{aligned} u_k &= \sqrt{\alpha}\frac{q_{k+1}}{q_{k+1}+\gamma_k}(g_{k+1} + r_{k+1}) - \sqrt{\alpha}\frac{\gamma_k}{q_{k+1}+\gamma_k}\sum_{i=1}^{k} g_i + r_i \\ u_{N-1} &= \sqrt{\alpha}\frac{q_N}{q_N+\gamma_{N-1}} g_N - \sqrt{\alpha}\frac{\gamma_{N-1}}{q_N+\gamma_{N-1}}\sum_{i=1}^{N-1} g_i + r_i. \end{aligned} \tag{13}$$

THEOREM 2
The feedback law in (13) is optimal for (1) and the feedback law in (8) is optimal for (12).

Proof We prove the theorem for (1). The closed loop system ($\sqrt{\alpha}A + BK$) is asymptotically stable (see Lemma 1 in appendix). Furthermore, by Lemma 2 (also in appendix) we have that only stabilizing controllers can be optimal.

Let X be the solution to the algebraic Riccati equation (9). Let U_s be the set of input sequences so that $x \to 0$ as $t \to \infty$. Then $\forall u \in U_s$ and subject to the system dynamics,

$$\lim_{T\to\infty} \sum_{t=0}^{T-1} x[t]^T Q x[t] + x^T[N] X x[N] = \lim_{T\to\infty} \sum_{t=0}^{T-1} x[t]^T Q x[t].$$

We know that $u = Kx$ minimizes the LHS, and thus also minimizes the RHS, which is the infinite horizon problem. □

REMARK 3
Let $\Gamma_k = \sum_{i=k}^{N-1} \gamma_i$. Then $\tilde{X}_N$ can be written as

$$\tilde{X}_N = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & \Gamma_{N-1} & \Gamma_{N-1} & \dots & \Gamma_{N-1} & \Gamma_{N-1} & \dots & \Gamma_{N-1} & \Gamma_{N-1} \\ 0 & \Gamma_{N-1} & \Gamma_{N-1} & \dots & \Gamma_{N-1} & \Gamma_{N-1} & \dots & \Gamma_{N-1} & \Gamma_{N-1} \\ \vdots & \vdots & \vdots & \ddots & & & & & \\ 0 & \Gamma_{N-1} & \Gamma_{N-1} & & \Gamma_k & \Gamma_k & \dots & \Gamma_k & \Gamma_k \\ 0 & \Gamma_{N-1} & \Gamma_{N-1} & & \Gamma_k & \Gamma_k & \dots & \Gamma_k & \Gamma_k \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \ddots & & \\ 0 & \Gamma_{N-1} & \Gamma_{N-1} & & \Gamma_k & \Gamma_k & & \Gamma_1 & \Gamma_1 \\ 0 & \Gamma_{N-1} & \Gamma_{N-1} & & \Gamma_k & \Gamma_k & & \Gamma_1 & \Gamma_1 \end{bmatrix}.$$

In this representation it is clear that X is highly structured. In fact, it only has $N-1$ degrees of freedom. □

3.1 Change of Variables

We also present the main points of the theorem in a new set of variables. In these coordinates the cost to go is tridiagonal, and the calculation of each input relies on only two states. Take $z = Sx$ with $S = S_N$ defined recursively,

$$S_k = \begin{bmatrix} 1 & 1 & \mathbf{1}^T \\ 0 & 1 & \mathbf{1}^T \\ 0 & 0 & S_o \end{bmatrix} \quad S_k^{-1} = \begin{bmatrix} 1 & -1 & \mathbf{0}^T \\ 0 & 1 & -\mathbf{e}_1^T \\ 0 & 0 & S_{k-1}^{-1} \end{bmatrix}, \tag{14}$$

starting at

$$S_2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad S_2^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

Let $[z_{2N-1}, \ldots, z_1] = z = Sx$. We can relate the new variables to the nodes by noting that $z_{2k} = \sum_{i=1}^{k} g_i + x_i = f_k$. Here we have defined f_k, which is the amount of goods downstream of node $k+1$. In this representation the cost to go matrix becomes tridiagonal,

$$x^T X x = z(S^{-1})^T X S^{-1} z = z^T (X_N^* + \mathbf{e}_1^t \mathbf{e}_1 \frac{1}{1-\alpha} \gamma_N) z.$$

With X_N^* defined by the recursion:

$$X_k^* = \begin{bmatrix} q_k & -q_k & 0 \\ -q_k & q_k + \gamma_{k-1} & 0 \\ 0 & 0 & X_{k-1}^* \end{bmatrix}, \quad X_2^* = \begin{bmatrix} q_2 & -q_2 & 0 \\ -q_2 & q_2 + \gamma_1 & 0 \\ 0 & 0 & q_1 \end{bmatrix}$$

The input $u = K^* z = K S^{-1} z$ relies on only two elements per input,

$$u_k = \frac{q_{k+1}}{q_{k+1} + \gamma_k} z_{2k+2} - z_{2k} = \frac{q_{k+1}}{q_{k+1} + \gamma_k} f_{k+1} - f_k$$
$$u_{N-1} = \frac{q_N}{q_N + \gamma_{N-1}} (f_{N-1} + g_N) - f_{N-1}.$$

4. Two Important observations

We now highlight two important properties of the results in the previous section. The feedback synthesis is scalable in one direction, and the implementation allows for a simple and efficient communication scheme.

4.1 Scalable Synthesis

The proposed method for solving the Riccati equation does so exactly, and its time-complexity is linear in the number of nodes.

Furthermore, the solution for a problem of size N, can be used to construct the solution for a problem of size $N+1$. This follows from the recursive nature of the calculation of γ_N and that the feedback law is unchanged in the old nodes when a new node is added. The only calculations that are required to implement the new feedback law is to calculate γ_N. This can be done using γ_{N-1} which was already calculated. Furthermore, the solution for size $N-1$ can be recovered from the solution for N. If the node furthest upstreams were to be removed, there would not be any effect on any of the remaining links. Hence, it is very computationally efficient to add and remove nodes upstream.

In general, when adding a node, only the nodes upstream of the new node needs their γ's to be recalculated, while the nodes downstream can keep theirs.

4.2 Distributed Implementation

It is reasonable to assume that node $k+1$ decides the value of u_k. Then g_{k+1} and r_{k+1} are local measurements. To implement the feedback, each node need in addition to the local information access to the sum $f_k = \sum_{i=1}^{k} g_i + x_i$, which is the sum of goods downstream of node $k+1$. f_k can be calculated by recursion through the graph:

- Receive f_{k-1}.
- Calculate $f_k = f_{k-1} + g_k + r_k$.
- Send f_k upstream.

The main benefit of this scheme is that the number of communication channels is proportional to the number of nodes. If each node were to communicate with every other node, the number of communication channels would instead be proportional to the square of the number of nodes.

One downside is that node k can not send its information until it received information from node $k-1$. Thus, the latency of the communication is proportional to the number of nodes. It is also vulnerable to faulty communication channels as it becomes impossible to calculate the output for every node upstreams of the faulty communication channel.

5. Application to Transportation

So far we have assumed that there is an underlying flow that allows for the implementation of the feedback law. Now we give an example with the dynamics considered and where there exists a natural net flow.

Consider inventory control for a set of stores. Then there is some transportation between the stores to keep the inventory level at an optimal level. We assume the topology of the stores and transportation takes the form of a directed line graph. This does not require that the stores are geographically distributed as a line.

Let the amount of goods in node k be denoted $\hat{g}_k$. The transportation is in the direction of the graph and has a delay of one time unit. Let the nodes be numbered in increasing order as we go upstream. We denote the goods in transit from node k as $\hat{r}_{k-1}$. Then the incoming goods to node k is $\hat{r}_k$. The amount of goods sent downstream in the graph by node k is denoted $\hat{u}_{k-1}$. There are also external influences $\hat{w}_k \in \mathcal{N}(\bar{w}_k, \sigma_k)$ for each node, which

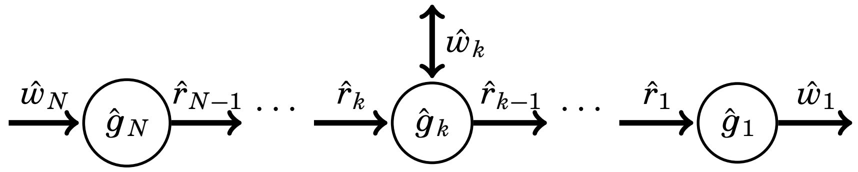

Figure 2. Illustration of the inventory control problem. Each node k corresponds to a store with inventory level $\hat{g}_k$. Each store is affected by an external net production $\hat{w}_k$. To balance the inventory level over the stores there is transportation between the stores. $\hat{r}_k$ is the goods in transit from store $k+1$ to store k.

corresponds to consumption and external transportation. See Figure 2 for an illustration. The dynamics of edges and nodes are given by

$$\begin{aligned} \hat{g}_k[t+1] &= \hat{g}_k[t] + (\hat{r}_k[t] - \hat{u}_{k-1}[t]) + \hat{w}_k[t] \\ \hat{r}_k[t+1] &= \hat{u}_k[t]. \end{aligned} \tag{15}$$

Each node k have a utility function describing how much it values having an inventory level of $\hat{g}_k$ goods,

$$U_k(\hat{g}_k) = q_k \hat{g}_k (a_k - \hat{g}_k). \tag{16}$$

The parameters q_k and a_k should both be positive. These utility functions have the property that the benefit of having access to more goods is decreasing with the amount of goods, that is $\partial^2 U/\partial^2(\hat{g}) < 0$. Furthermore, when $\hat{g}_k > a_k/2$ we have that $\partial U/\partial(\hat{g}) < 0$. The intended working area is $0 < \hat{g} < a_k/2$.

We value higher inventory levels more the earlier we get them. Thus the following pay off function is chosen

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \mathbb{E} \sum_{t=0}^{\infty} \alpha^t \sum_{k=1}^{N} U_k(\hat{g}[t]) \\ \text{Subject to} \quad & \text{dynamics in (15).} \end{aligned}$$

We assume that there is a underlying flow in the graph, which could for example have been found using static optimization. However, due to the variable external influences, we want to apply feedback around this static flow. Then the transportation is happening independent of our choice of u, and we can assume that it has already been paid for. Thus we do not put any penalty on the input u.

Further, assume that the expected production and consumption are equal. The problem can be transformed to a problem of the form of (8) by controlling around the nominal flow. To do so, we must change variables

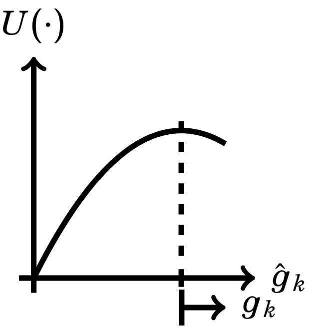

Figure 3. Plot of utilities in (16) and the relationship between g and $\hat{g}$. g can be interpreted as the negative demand for each node.

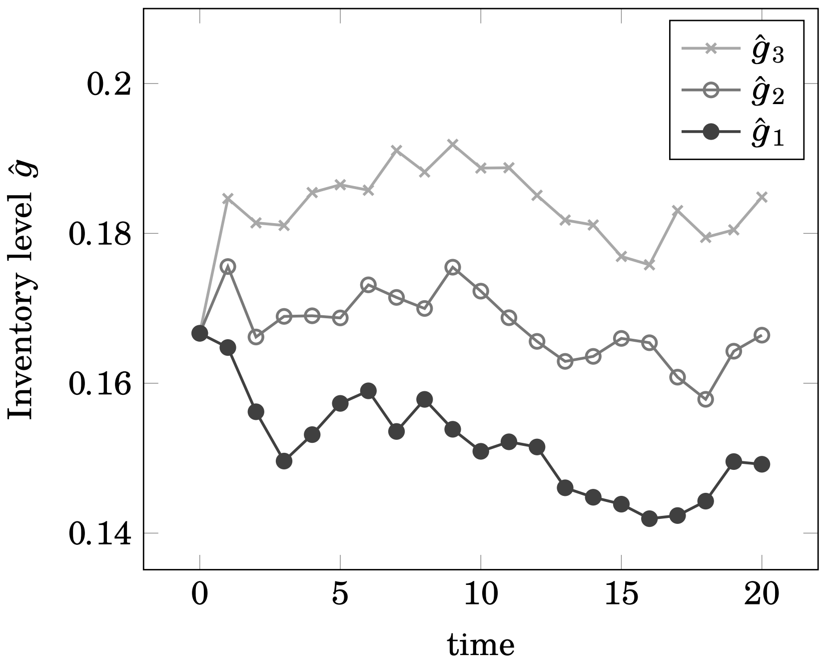

Figure 4. Simulation of store dynamics in (15). The goal is to keep optimal inventory level in the three stores. Due to a discount factor and a net flow through the graph, the levels are higher in the stores upstreams.

so that the pay-off function is quadratic. We do this by letting $g = \hat{g} - a_k/2$. The utility function and the change of variables are depicted in Figure 3. The new variable g can be interpreted as the negative demand for each node. Also, note that g is negative in the intended working area.

The input and flows will be controlled around the nominal flow $\bar{u} = \bar{r}$, so that $\hat{u} = \bar{u} + u$, $\hat{r} = \bar{r} + r$. For the details, see Lemma 3 in Appendix. Note that for $\hat{u}$ to be non-negative, we need $u \geq -\bar{u}$.

For a simulation of the system, see Figure 4. A discount factor of $\alpha = 0.95$ and utilities $U(\hat{g}_i) = \hat{g}_i(1 - \hat{g}_i)$ were used. The noise had variance $\bar{w} = 0.0.0025$ for all i.

6. Conclusions

We have presented a recursive solution to a class of optimal control problems. This solution is easily extended as the system grows. The structure of the feedback law allows for an efficient implementation using a local communication scheme. We have showed that the optimal control problem can be used to solve an inventory control problem.

It is expected that the results presented here will generalize to tree graphs and periodic B matrices. This is subject to future work.

Appendix

Proof of Theorem 1: The theorem is trivially to show for $N = 2$. Now assume that the theorem holds for $N-1$. Let $A_o = A_{N-1}$, $B_o = B_{N-1}$, $Q_o = Q_{N-1}$ and $\tilde{X}_o = \tilde{X}_{N-1}$ denote the matrices for the system of size $N-1$. Then the relation between the old and the new system matrices are given by

$$A = \begin{bmatrix} 1 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{e}_1 & A_o \end{bmatrix}, \; B = \begin{bmatrix} -1 & \mathbf{0}^T \\ 1 & \mathbf{0}^T \\ \mathbf{0} & B_o \end{bmatrix}, \; Q = \begin{bmatrix} q_N & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & Q_o \end{bmatrix},$$

$$\tilde{X} = \begin{bmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & \gamma_{N-1} & \gamma_{N-1}\mathbf{1}^T \\ \mathbf{0} & \gamma_{N-1}\mathbf{1} & \gamma_{N-1}\mathbf{1}\mathbf{1}^T \end{bmatrix} + \begin{bmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & 0 & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \tilde{X}_o \end{bmatrix}.$$

We start with the RHS of (9). Standard calculations and noting especially that $\mathbf{e}_1 = A_o\mathbf{e}_1$ and $\mathbf{e}_1\tilde{X}_o = 0$ gives

$$(B^TXB)^{-1} = \begin{bmatrix} (q_N + \gamma_{N-1})^{-1} & \mathbf{0}^T \\ \mathbf{0} & (B_o^TX_oB_o)^{-1} \end{bmatrix}$$

$$B^TXA = \begin{bmatrix} -q_N & \gamma_{N-1} & \gamma_{N-1}\mathbf{1}^T \\ \mathbf{0} & B_o^TX_oA\mathbf{e}_1 & B_o^TX_oA_o \end{bmatrix}$$

Define $K_o = -(B_o^TX_oB_o)^{-1}B_o^TX_oA_o$. Corresponding definition for the system of size N gives

$$-K = (B^TXB)^{-1}B^TXA = \begin{bmatrix} -\dfrac{q_N}{q_N + \gamma_{N-1}} & \dfrac{\gamma_{N-1}}{q_N + \gamma_{N-1}} & \dfrac{\gamma_{N-1}}{q_N + \gamma_{N-1}}\mathbf{1}^T \\ 0 & -K_o\mathbf{e}_1 & -K_o \end{bmatrix}.$$

Let, for the system of size $N-1$,

$$\Xi_o = A_o^TX_oB_o(B_o^TX_oB_o)^{-1}B_o^TX_oA_o.$$

Then, for the system of size N

$$\Xi = A^T XB(B^T XB)^{-1}B^T XA$$
$$= \begin{bmatrix} \frac{q_N^2}{q_N+\gamma_{N-1}} & -\frac{q_N\gamma_{N-1}}{q_N+\gamma_{N-1}} & -\frac{q_N\gamma_{N-1}}{q_N+\gamma_{N-1}}\mathbf{1}^T \\ -\frac{q_N\gamma_{N-1}}{q_N+\gamma_{N-1}} & \frac{\gamma_{N-1}^2}{q_N+\gamma_{N-1}} + \mathbf{e}_1^T\Xi_o\mathbf{e}_1 & \frac{\gamma_{N-1}^2}{q_N+\gamma_{N-1}}\mathbf{1}^T + \mathbf{e}_1^T\Xi_o \\ -\frac{q_N\gamma_{N-1}}{q_N+\gamma_{N-1}}\mathbf{1} & \frac{\gamma_{N-1}^2}{q_N+\gamma_{N-1}}\mathbf{1} + \Xi_o\mathbf{e}_1 & \frac{\gamma_{N-1}^2}{q_N+\gamma_{N-1}}\mathbf{1}\mathbf{1}^T + \Xi_o \end{bmatrix}.$$

For the LHS of (9) we have, $A^T\mathbf{1}\mathbf{1}^T A = \mathbf{1}\mathbf{1}^T$,

$$A^T(\tilde{X}+Q)A =$$
$$\begin{bmatrix} 0 & 0 & \mathbf{0}^T \\ 0 & \gamma_{N-1} & \gamma_{N-1}\mathbf{1}^T \\ \mathbf{0} & \gamma_{N-1}\mathbf{1} & \gamma_{N-1}\mathbf{1}\mathbf{1}^T \end{bmatrix} + \begin{bmatrix} q_N & 0 & \mathbf{0}^T \\ 0 & \mathbf{e}_1^T A_o^T(\tilde{X}_o+Q_o)A_o\mathbf{e}_1 & \mathbf{e}_1^T A_o^T(\tilde{X}_o+Q_o)A_o \\ \mathbf{0} & A_o^T(\tilde{X}_o+Q_o)A_o\mathbf{e}_1 & A_o^T(\tilde{X}_o+Q_o)A_o \end{bmatrix}$$

The induction base can be rewritten as

$$-\gamma_{N-1}\mathbf{1}\mathbf{1}^T + \alpha A_o^T(\tilde{X}_o+Q_o)A_o - \tilde{X}_o = \alpha\Xi_o.$$

While the Riccati equation itself can be rewritten as

$$-\gamma_N\mathbf{1}\mathbf{1}^T + \alpha A^T(\tilde{X}+Q)A - \tilde{X} = \alpha\Xi.$$

For element (2,2), (2,3), (3,2), and (3,3) of the Riccati equation, we would like to show that

$$-\gamma_N\mathbf{1}\mathbf{1}^T + \alpha A_o^T(\tilde{X}_o+Q_o)A_o - \tilde{X}_o + (\alpha-1)\gamma_{N-1}\mathbf{1}\mathbf{1}^T = \alpha\left(\frac{\gamma_{N-1}^2}{q_N+\gamma_{N-1}} + \Xi_o\right).$$

Applying the induction base gives

$$\gamma_N - \gamma_{N-1} + (\alpha-1)\gamma_{N-1} = \alpha\frac{\gamma_{N-1}^2}{q_N+\gamma_{N-1}}$$

Which is easy to show being true. For element (1,1) we need to show that,

$$-\gamma_N + \alpha q_N - \alpha\frac{q_N^2}{q_N+\gamma_{N-1}},$$

equals zero. It can be rewritten as

$$-\gamma_N + \gamma_N + \alpha\frac{q_N^2}{q_N+\gamma_{N-1}} - \alpha\frac{q_N^2}{q_N+\gamma_{N-1}} = 0.$$

Finally, for the remaining elements of the Riccati equation, we have that

$$-\gamma_N = -\alpha \frac{q_N \gamma_{N-1}}{q_N + \gamma_{N-1}} = -\gamma_N.$$

We have that $X > 0$ since $\mathbf{1}\mathbf{1}^T > 0$, $Q \geq 0$ and $\tilde{X}_N \geq 0$. The last inequality follows from that $\tilde{X}_N = \sum_k B_k^T B_k$, with

$$B = [0, \ldots, 0, \sqrt{\gamma_k}, \ldots, \sqrt{\gamma_k}]. \qquad \square$$

Lemma 1

Given $A = A_N$, $B = B_N$ in (6), $K = K_N$ in (11), and an arbitrary constant p, $pA + BK$ has one eigenvalue with value p and $2N - 2$ eigenvalues with value zero. $\square$

Proof Let $\beta_k = \frac{q_{k+1}}{q_{k+1}+\gamma_k}$. Then $pA_k + B_k K_k$ can be written recursively, given $A_o,\ B_o,\ K_o$ of the system of size $k-1$, as

$$pA_k + B_k K_k = \begin{bmatrix} p - \beta_{k-1} & 1 - \beta_{k-1} & (1-\beta_{k-1})\mathbf{1}^T \\ \beta_{k-1} & \beta_{k-1} - 1 & (\beta_{k-1} - 1)\mathbf{1}^T \\ 0 & pA_o + B_o K_o \mathbf{e}_1 & pA_o + B_o K_o \end{bmatrix},$$

with

$$pA_2 + B_2 K_2 = \begin{bmatrix} p + \beta_1 & 1 - \beta_1 & 1 - \beta_1 \\ \beta_1 & \beta_1 - 1 & \beta_1 - 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Using the change of variables defined in (14), with $S_o = S_{k-1}$, and that $\mathbf{1}^T(pA + BK) = p\mathbf{1}^T$, we have that

$$S(pA + BK)S^{-1} =$$
$$\begin{bmatrix} p & 0 & 0 \\ \beta_{k-1} & 0 & 0 \\ 0 & S_o(pA_o + B_o K_o)\mathbf{e}_1 & \begin{matrix} S_o(pA_o + B_o K_o)\mathbf{e}_1(-\mathbf{e}_1^T) + \\ S_o(pA_o + B_o K_o)S_o^{-1} \end{matrix} \end{bmatrix}.$$

Note that $S_o^{-1}\mathbf{e}_1 = \mathbf{e}_1$. The lower right element of $S(pA + BK)S^{-1}$ can the be written as

$$S_o(pA + BK)S_o^{-1}(-\mathbf{e}_1\mathbf{e}_1^T + I).$$

Assume that $S_o(pA + BK)S_o^{-1}$ is lower diagonal, and that the only non zero diagonal element is element (1,1). Then $S_o(pA + BK)S_o^{-1}(-\mathbf{e}_1\mathbf{e}_1^T + I)$ is strictly lower diagonal. Then $(pA + BK)$ has one eigenvalue of value p and the other eigenvalues have value 0. Note also that $S(pA + BK)S^{-1}$

satisfies the assumption of being lower diagonal with element (1,1) being the only non zero diagonal element.

It is easily checked that $S_2(pA_2 + B_2K_2)S_2^{-1}$ satisfies the assumption of being lower diagonal with (1,1) being the only diagonal element. Thus the lemma holds for all $N \geq 2$ by induction. □

LEMMA 2
Given $A = A_N$ and $B = B_N$ in (6) and $Q = Q_N$ in (7). Let $x[t+1] = Ax+Bu$. Then

$$\lim_{T\to\infty} \sum_{t=0}^{T} x[t]^T Qx[t] \tag{17}$$

is bounded, only if $x[T] \to 0, T \to \infty$. □

Proof We prove the lemma by proving that

$$\sum_{t=0}^{N} x[t]^T Qx[t] = 0$$

only if $x[0] = 0$. Assume that there exists a $x[0] \neq 0$ s.t (17) holds. Then at least one $r_k[0] = c \neq 0$. Then $u_{k-1}[0] = c$, which gives that $r_{k-1}[1] = c$. This will eventually lead to $r_1[\zeta] = c$, with $\zeta < N$. This will however give that $g_1[\zeta + 1] = c$, which gives a non zero cost. □

LEMMA 3
Assume that $\sum_{i=1}^{N} \bar{w}_i = 0$ and $\sum_{i=k}^{N} \bar{w}_i = e_k > 0$ for $k \geq 2$. Then there exists $\bar{u}_k = \bar{r}_k = e_k > 0$ such that, for all k and any $\hat{g}_k$

$$\hat{g}_k[t+1] = \hat{g}_k[t] + (\bar{r}_k[t] - \bar{u}_{k-1}[t]) + \hat{w}_k[t] = \hat{g}_k[t] + w \tag{18}$$

with $w_k = \hat{w}_k - \bar{w}_k \in \mathcal{N}(0, \sigma_k)$. Also, let $u_k = \hat{u}_k - \bar{u}_k$, $r_k = \hat{r}_k - \bar{r}_k$, $g_k = \hat{g}_k - a_k/2$. Take $x = [g_N, r_{N-1}, \ldots r_1, g_1]$, $u = [u_N, \ldots, u_1]$. Then the solution to

$$\begin{aligned} \underset{\hat{s}}{\text{maximize}} \quad & \sum_{t=0}^{\infty} \alpha^t \sum_{k=1}^{N} U_k(\hat{g}_k) \\ \text{subject to} \quad & \text{dynamics in (15)}, \end{aligned} \tag{19}$$

can be found as $\hat{u} = \bar{u} + u$, where s is the solution to (8) with A, B, Q as in (6) and (7). □

Proof The change of variables from $\hat{g}_k$ to g_k does not change the dynamics of the system, so

$$g_k[t+1] = g_k[t] + (\hat{r}_{k-1}[t] - \hat{u}_k[t]) + \hat{w}_k[t].$$

Working around the nominal flow with u_k and r_k gives, by using (18),

$$\begin{cases} g_k[t+1] = g_k[t] + (r_k[t] - u_{k-1}[t]) + w_k[t] \\ r_k[t+1] = u_k[t]. \end{cases}$$

These dynamics are described by $x[t+1] = Ax[t] + Bu[t] + w$ with A and B as in (6). For the optimization criterion, note that

$$\max_{\hat{g}_k} q_k \hat{g}_k (a_k - \hat{g}_k) = \max_{g_k} -q_k g_k^2 - 0.25a_k^2$$

and

$$\arg\max_{g_k} -q_k g_k^2 - 0.25a_k^2 = \arg\min_{g_k} q_k g_k^2.$$

We then have that minimizing $x^T Qx$ gives the maximum utility. □

References

Bamieh, B., F. Paganini, and M. A. Dahleh (2002). "Distributed control of spatially invariant systems". *IEEE Transactions on automatic control* **47**:7, pp. 1091–1107.

Bertsekas, D. P. (2012). *Dynamic programming and optimal control*. 4th Edition. Vol. 1. Athena scientific Belmont, MA.

Bitmead, R. R. and M. Gevers (1991). "Riccati difference and differential equations: convergence, monotonicity and stability". *The Riccati Equation*, pp. 263–291.

Ho, Y.-C. et al. (1972). "Team decision theory and information structures in optimal control problems–part i". *IEEE Transactions on Automatic control* **17**:1, pp. 15–22.

Lamperski, A. and L. Lessard (2015). "Optimal decentralized state-feedback control with sparsity and delays". *Automatica* **58**, pp. 143–151.

Madjidian, D. and L. Mirkin (2014). "Distributed control with low-rank coordination". *IEEE Transactions on Control of Network Systems* **1**:1, pp. 53–63.

Ntogramatzidis, L. and A. Ferrante (2015). "The discrete-time generalized algebraic riccati equation: order reduction and solutions structure". *Systems & Control Letters* **75**, pp. 84–93.

Radner, R. (1962). "Team decision problems". *The Annals of Mathematical Statistics* **33**:3, pp. 857–881.

Rotkowitz, M. and S. Lall (2006). "A characterization of convex problems in decentralized control". *IEEE Transactions on Automatic Control* **51**:2, pp. 274–286.

Shah, P. and P. A. Parrilo (2013). "H_2-optimal decentralized control over posets: a state-space solution for state-feedback". *IEEE Transactions on Automatic Control* **58**:12, pp. 3084–3096.

Paper II

Price Based Linear Quadratic Control Under Transportation Delay

Martin Heyden Richard Pates Anders Rantzer

Abstract

We study a simple transportation problem on a string graph where the objective is to control the node levels of some decaying quantity. The problem is considered from two perspectives. The first is to find the social optimum where the flows minimize the total cost. The second is to find prices for the nodes so that the users transportation decisions align with the social optimum. We give an implementation of the optimal feedback law that only requires local states and prices, where the optimizing prices have a distributed update rule. The prices also align the social and user optimum in a budget neutral way and give all nodes better cost than if they were on their own.

1. Introduction

In this work we study optimal transportation of a decaying quantity. This could be a transportation network, as illustrated in Section 2. The objective is to control the node levels by regulating the transportation between the nodes to optimize performance. The challenge is to do this in a manner that scales well with network size, whilst accounting for dynamical effects such as transportation delays.

To capture the essence of the problem, we consider a string network with N nodes in discrete time. We let the transportation delay be one time unit on every link and define the dynamics to be

$$\begin{aligned} x_{2i-1}[t+1] &= \alpha(x_{2i-1}[t] + x_{2i}[t]) - u_{i-1}[t] \\ x_{2i}[t+1] &= u_i[t], \end{aligned} \tag{1}$$

for $1 \leq i \leq N$. The variable x_{2i-1} is the level in node i and x_{2i} is the amount in transportation towards node i. The control input u_{i-1} is the amount leaving node i. At the boundaries we have $u_0 = 0$, $u_N = 0$, and $x_{2N} = 0$. See Figure 1 for an illustration when $N = 3$. The constant $0 < \alpha \leq 1$ is the decay rate.

The nodes are numbered as in Figure 1, where the most downstream node has index one, the second most downstream has index two, and so on. Furthermore, we index the links according to the node which they enter.

We assume that each node i values its level x_{2i-1} according to the quadratic function, $U_i(x_{2i-1}) = b_i x_{2i-1} - 1/2 q_i x_{2i-1}[t]^2$, where $q_i > 0$, $b_i > 0$, and $b_{i+1} = \alpha b_i$. The last assumption will be motivated in the next section. We study the problem from two perspectives. First we consider the social optimum problem where the objective is to maximize a global utility function which is the sum of local utility functions,

$$\begin{aligned} \underset{x,u}{\text{maximize}} \quad & J(x) = \sum_{i=1}^{N} \sum_{t=1}^{T} \left(b_i x_{2i-1}[t] - \frac{1}{2} q_i x_{2i-1}[t]^2 \right) \\ \text{subject to} \quad & \text{Dynamics in (1)} \\ & x_i[0] \text{ given.} \end{aligned} \tag{2}$$

In the above, $x[t] \in \mathbb{R}^{2N-1}$ defined for $1 \leq t \leq T$ and $u[t] \in \mathbb{R}^{N-1}$ defined for $0 \leq t \leq T-1$. This problem is a variation of the standard Linear Quadratic control problem, with the difference that it contains a linear cost term and no input penalty. The infinite horizon version of this problem was solved in [Heyden et al., 2018] for $b_i = 0$ by finding the solution to the algebraic Riccati equation.

As a system grows large, it is infeasible for every node to have global information. To overcome this issue we aim to solve the problem in a distributed way. This will be achieved using prices that only require local

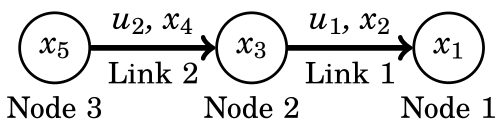

Figure 1. A graphical illustration of the studied problem. There are three nodes whose levels are to be controlled. Each link has a corresponding input u that is the flow entering the link and a state that is the goods currently in transit on the link. We have also indicated the indexing conventions for the nodes and links.

communication to implement the feedback law. These prices will also be used as a synchronization mechanism so that each individual node have optimal level when considering its own utility. These objectives are well aligned with the theme of solving optimization problems using Lagrange multipliers.

Early work on distributed control includes team game problems, where a set of agents works for the same goal, but with different information, see for example [Radner, 1962].

In this paper the structure follows from the plant, as in [Bamieh et al., 2002] where the optimal control is shown to be localized in space for spatially invariant systems. Other recent work includes [Shah and Parrilo, 2013], where the optimal poset-causal controller is found, which is similar to the controller structure obtained in this paper. In [Lamperski and Lessard, 2015] problems where the plant and controller satisfy the same delay and sparsity constraints are considered, and it is shown how to find the optimal LQ state feedback.

Lagrange multipliers used for coordination is well studied, for example as shadow prices in network congestion control [Kelly et al., 1998], [Low et al., 2002] and in distributed MPC [Giselsson et al., 2013]. They have also been suggested in control of power grids [Jafarian et al., 2016], [Jokić et al., 2009]. Normally these problems are either static, or of high complexity. Requiring either solving for all the prices and states at the same time, or solving a Riccati equation. In our specific problem, the prices are not the Lagrange multipliers, but rather a simple linear combination that is much simpler to compute than the Lagrange multipliers are. For pioneering work in using prices for coordination see [Cohen, 1978] which was later used to control water supply networks in [Carpentier and Cohen, 1993].

Preview of Results

To give incentive for the individual nodes to follow the social optimum we will introduce prices and study the problem from a node perspective. Each node i will be presented with a price vector $p_i[t]$ that will affect the nodes utility proportional to their levels,

$$C_i(x_{2i-1}, p_i) = p_i[0]x_{2i-1}[0] + \sum_{t=1}^{T} \Big(b_i x_{2i-1}[t] - \frac{1}{2} q_i x_{2i-1}^2[t] - p_i[t] x_{2i-1}[t] \Big). \quad (3)$$

Typically increasing $x_{2i-1}[t]$ will lead to a trade off between the increased utility from the $b_i x_{2i-1}[t] - 1/2 q_i x_{2i-1}^2[t]$ term, and the decreased utility from the cost $p_i[t]x_{2i-1}[t]$. The utility function in (3) will be further discussed in Section 2. Each node will naturally consider the following problem

$$\begin{array}{ll} \underset{x_{2i-1}}{\text{maximize}} & C_i(x_{2i-1}, p_i) \\ \text{subject to} & p_i \text{ given.} \end{array} \quad (4)$$

We find the solution to social optimum problem by studying the Lagrangian of the problem. The main contribution lies in deriving a set of prices from the Lagrange multipliers that allows for a distributed implementation of the optimal feedback law and aligns social and user optimum. The prices are given by a simple, temporally decoupled, expression

$$p_i[t+1] = \begin{cases} b_i - \gamma_i \sum_{j=1}^{2i} x_j[t] & 0 \le t \le T-i \\ 0 & t > T-i, \end{cases} \quad (5)$$

where γ is defined by the following iteration

$$\gamma_1 = q_1, \quad \gamma_i = \frac{\alpha^2 \gamma_{i-1} q_i}{\alpha^2 \gamma_{i-1} + q_i}, \; i \ge 2. \quad (6)$$

With p as in (5), the optimal inputs are given by

$$u_{i-1}[t] = \alpha(x_{2i-1}[t] + x_{2i}[t]) - \frac{1}{q_i}\big(p_i[t+1] - b_i\big). \quad (7)$$

The combined structure of (5) and (7) allows for a simple implementation of the optimal u using only local communication. The expression for the optimal prices in (5) indicates that the price should increase the more a node values its level from the term b_i, and decrease when more goods is available.

2. Motivation for the problem

How can the dynamics in (1) arise? We will consider a simple model of a generic transportation network for a decaying quantity. This could for example be a district heating network, or an inventory control system for decaying goods.

Each node i in the network has a constant production (or consumption) w_i. Furthermore, the quantity can be transported along the links of the system. The transportation must be positive, which essentially implies that the demand is bigger downstream than upstream. This will be the case if there is a producer at the top of the network. Finally, we make the simplifying assumption that the decay has a homogeneous rate $1-\alpha$ throughout the system. We can write the dynamic for the level ζ_i in each node as

$$\zeta_i[t+1] = \alpha\big(\zeta_i[t] + v_i[t-1]\big) + w_i - v_{i-1}[t].$$

In the above v_{i-1} is the quantity leaving the node and v_i is the quantity arriving to the node. The quantity leaving the node goes immediately into transportation and will take one time unit to arrive.

Let the flows $v[t] = \bar{v}$ be constant. Then as t grows large, each node will have an equilibrium level $\bar{\zeta}_i$ where the inflow equals the outflow,

$$\bar{\zeta}_i = \frac{1}{1-\alpha}(w_i + \alpha\bar{v}_i - \bar{v}_{i-1}).$$

We assume that each node values its level according to a quadratic function $U_i(\zeta_i)$ and the optimal equilibrium is the solution to

$$\begin{aligned} \underset{\bar{v}}{\text{maximize}} \quad & \sum_i U_i(\bar{\zeta}_i) \\ \text{subject to} \quad & \bar{\zeta}_i = \frac{1}{1-\alpha}(w_i + \alpha\bar{v}_i - \bar{v}_{i-1}). \end{aligned} \tag{8}$$

Now we study the system around this equilibrium. We introduce a new state vector $x \in \mathbb{R}^{2N-1}$ where odd indices correspond to node levels and even indices to quantity in transit. Furthermore we define the new input vector $u \in \mathbb{R}^{N-1}$. These variables are defined around the equilibrium levels,

$$\begin{aligned} u_i[t] &= v_i[t] - \bar{v}_i, \\ x_{2i-1}[t] &= \zeta_i[t] - \bar{\zeta}_i \\ x_{2i}[t+1] &= u_i[t] \end{aligned}$$

Then the dynamics for x are given by (1).

The utility relative to the optimum can be written as

$$U_i(\bar{\zeta}_i + x_{2i-1}) - U_i(\bar{\zeta}_i) = \sum_{t=1}^{T}\left(b_i x_{2i-1} - \frac{1}{2}q_i x_{2i-1}^2[t]\right),$$

where $q_i < 0$.

REMARK 1
Since $\bar{\zeta}_i$ solves (8) we must have that

$$b_{i+1} = \alpha b_i. \tag{9}$$

Otherwise the utility could be improved by making a small perturbation ϵ to $\bar{v}_i$ which would increase $\bar{\zeta}_i$ by $\alpha/(1-\alpha)\epsilon$ and decrease $\bar{\zeta}_{i+1}$ by $1/(1-\alpha)\epsilon$.□

Now, assume that operating conditions changes and the optimal equilibrium changes. Finding the optimal transition to this new equilibrium would correspond to solving (2). The lack of penalty on the flows can be motivated by that the cost of changing the transportation is small. For example, if the transportation is done via trucks, then there is typically no additional cost if a truck transport more goods. However, there is still a loss in transporting the quantity in that the quantity being transported is not utilized.

How can the user problem in (3) be motivated? We note that close to the equilibrium $p_i[t] > 0$, $t \geq 1$. We shall later see that that is the case for $t = 0$ as well. Its natural that the users in the transportation network pay for their levels. Now if the new equilibrium level is lower for a node, then that node would expect to be paid if it is to actively send away some of its quantity. Likewise, if a node gets a higher level, that node would be expected to pay for it. This is captured by the term $p_i[0]x_{2i-1}[0]$.

Since the new equilibrium can not be reached immediately the nodes should also pay if they continue having a higher level, or get compensated if it is to low. This is captured by the terms $p_i[t]x_{2i-1}[t]$.

3. Results

We start by giving the solution to (2) in Theorem 1. This result shows that the i-th entry of the optimal control input can computed based only on local measurements of the quantity x, and a local price p_i. Next we show in Proposition 1 how these prices can be used to align the user problem in (4) to the social optimum. The prices have additional appealing properties. Firstly, the node utilities are higher than if they had zero flows and no payments, and secondly, the sum of all payments equal zero. The proofs of the results presented here will be given in Section 5.

THEOREM 1
Define γ_i as in (6), and $p[t]$ by

$$p_i[t+1] = \begin{cases} b_i - \gamma_i \sum_{j=1}^{2i} x_j[t], & 0 \leq t \leq T-i \\ 0 & t > T-i. \end{cases} \tag{10}$$

Then the optimal u for (2) is given by

$$u[t] = \alpha \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & & & \\ & & & \ddots & \ddots & & & \\ & & & 0 & 1 & 1 & 0 \\ & & & & & 0 & 1 \end{bmatrix} x[t] - \begin{bmatrix} 0 & \frac{1}{q_2} & & 0 \\ & & \ddots & \\ 0 & & & \frac{1}{q_N} \end{bmatrix} (p[t+1] - b). \qquad (11)$$

With $p[t] = [p_1[t], \ldots, p_N[t]]^T$ and $b = [b_1, \ldots, b_N]^T$. □

If we write out the expressions for each input we get (7). From the theorem we see that there exists a simple method for calculating the optimal feedback law, using only local states and local prices. Furthermore, $p_i[t+1]$ can be calculated recursively through the graph,

$$p_i[t+1] = \gamma_i \left(-x_{2i-1}[t] - x_{2i}[t] + \frac{1}{\gamma_{i-1}} p_{i-1}[t+1] \right) + (1 - \frac{1}{\alpha \gamma_{i-1}}) b_i,$$

requiring only local communication.

Eq. (7) has a very natural interpretation from the user optimal perspective, as it is the solution to

$$\begin{aligned} \underset{u_{i-1}[t-1]}{\text{minimize}} \quad & b_i x_{2i-1}[t] - \frac{1}{2} q_i x_{2i-1}^2[t] - p_i[t] x_{2i-1}[t] \\ \text{subject to} \quad & x_{2i-1}[t] = \alpha(x_{2i-1}[t-1] + x_{2i}[t-1]) - u_{i-1}[t], \end{aligned}$$

which corresponds to the node optimizing its utility for the next time point.

Remark 2

At first sight it may seem like (11) is non causal as the input at time t depends on prices at time $t+1$. However, from (10) we can see that prices at time $t+1$ depends on state at time t, and the expression is indeed causal. As the prices are associated with the states when the input has taken affect, it is natural that the prices are one time-index ahead of the inputs. □

Remark 3

It might be surprising that some of the prices are zero and thus the corresponding nodes will have optimal levels, $x_i = -b_2/q_2$, as t gets closer to T. This is due to the boundary effects of the system, where the level of a node can be increased without decreasing the value of others. This is done buy putting the deficit in the transportation states. □

Proposition 1

In addition to the definitions in Theorem 1, let $m = \min(T - i, N)$, and

$$p_i[0] = \sum_{t=1}^{T-(i-1)} \alpha^{t-(t_0+\tau)} b_i - \alpha \left(\sum_{j=i}^{m} \gamma_j \sum_{k=1}^{2j} x_k[t] + \sum_{j=N+1}^{T} \gamma_N \alpha^{2(j-N)} \sum_{k=1}^{2N-1} x_k[t] \right)$$

Then the following holds

(i) The optimal x for (2) and (4) are equal.

(ii) The node utility satisfies

$$C_i(x_{2i-1}, p_i) \geq \sum_{t=1}^{T} \Big(b_i \alpha^t x_{2i-1}[0] - q_i (\alpha^t x_{2i-1}[0])^2 \Big).$$

(iii) The sum of payments are equal to zero,

$$\sum_{i=1}^{N} \left(p_i[0] x_{2i-1}[0] - \sum_{t=1}^{T} p_i[t] x_{2i-1}[t] \right) = 0.$$

□

The proposition shows not only that is possible to align the user and social optimum. We also have that each node is never worse off than if they had no in- or outflow. This is what would happen if the node was not part of the transportation network. This is an important property as otherwise the nodes would be reluctant to be part of the network.

Furthermore the payment scheme is budget neutral, i.e. the payments sum to zero. This is significant as if the scheme had a budget deficit, it would be very hard to find someone to supply additional money to the system with nothing in return.

4. Analysis of the Lagrangian

In this section we perform the necessary analysis of the Lagrangian for (2) needed to prove Theorem 1 and Proposition 1. An important part is to construct an alternative user utility based on the Lagrange multipliers, and showing that it is equal to the original one in (4).

Lagrangian

By making the constraint $x_{2i}[t+1] = u_i[t]$ explicit we can write the Lagrangian of (2) as

$$\begin{aligned} L(x, u, \lambda) = J(x) + \sum_{t=0}^{T-1} \Bigg[& \lambda_1[t+1] \Big\{ \alpha\big(x_1[t] + u_1[t-1]\big) - x_1[t+1] \Big\} \\ & + \sum_{i=1}^{N-1} \lambda_i[t+1] \Big\{ \big(\alpha(x_{2i-1}[t] + u_i[t-1]) - u_{i-1}[t]\big) - x_{2i-1}[t+1] \Big\} \\ & + \lambda_N[t+1] \Big\{ \big(\alpha x_{2N-1}[t] - u_{N-1}[t]\big) - x_{2N-1}[t+1] \Big\} \Bigg]. \end{aligned} \tag{12}$$

The Lagrange dual variable has dimensions $\lambda[t] \in \mathbb{R}^N$ for $1 \leq t \leq T$. The dual variables $\lambda_i[t]$ have natural economic interpretation as the marginal change in social utility when $x_{2i-1}[t]$ changes.

Alternative user optimal problem

Based on the Lagrangian we define an alternative user utility function, and show that it is equal to the original in (3). In this formulation the node utility will include a cost based on the level change,

$$\hat{C}_i(x_{2i-1}, \lambda_i) = \sum_{t=1}^{T} b_i x_{2i-1}[t] - \frac{1}{2} q_i x_{2i-1}^2[t] - \lambda_i[t] \underbrace{\left(x_{2i-1}[t] - \alpha x_{2i-1}[t-1]\right)}_{\text{change in level}}. \tag{13}$$

Note that all the terms in $\hat{C}$ are in the Lagrangian L. By letting

$$p_i[t] = \begin{cases} \alpha\lambda_i[1] & t = 0 \\ \lambda_i[t] - \alpha\lambda_i[t+1] & 1 \leq t \leq T-1 \\ \lambda_i[T] & t = T, \end{cases} \tag{14}$$

the node utility can be rewritten as

$$\begin{aligned} \hat{C}_i(x_{2i-1}, \lambda_i) &= \sum_{t=1}^{T-1} b_i x_{2i-1}[t] - \frac{1}{2} q_i x_{2i-1}^2[t] - (\lambda_i[t] - \alpha\lambda_i[t+1]) x_{2i-1}[t] \\ &\qquad + \alpha\lambda_i[1] x_{2i-1}[0] - \lambda_i[T] x_{2i-1}[T] \\ &= \sum_{t=1}^{T} \left(b_i x_{2i-1}[t] - \frac{1}{2} q_i x_{2i-1}^2[t] - p_i[t] x_{2i-1}[t] \right) + p_i[0] x_{2i-1}[0] \\ &= C_i(x_{2i-1}, p_i). \end{aligned}$$

Thus the two different user optimal problems are equal, and we can analyze either one of them. We will use the Lagrangian version for analysis, while the p version will be used for implementation.

Optimality Conditions

The optimization problem in (2) is concave as it is the maximization of a concave cost function under affine constraints. Thus necessary and sufficient optimality conditions are given by the KKT conditions (see [Boyd and Vandenberghe, 2004])

$$\nabla_\lambda L = 0, \quad \nabla_u L = 0, \quad \nabla_x L = 0.$$

$\nabla_\lambda L = 0$ is equal to the dynamics constraint being satisfied.

For a standard LQ problem with a penalty on the input, $\nabla_u L = 0$ gives u as a function of λ. See [Cannon et al., 2008] for a slightly more general MPC case. Here we instead get the following

$$\frac{\partial L}{\partial u_i[t]} = -\lambda_{i+1}[t+1] + \alpha\lambda_i[t+2] = 0, \quad 0 \le t \le T-1 \tag{15a}$$

$$\frac{\partial L}{\partial u_i[T-1]} = -\lambda_{i+1}[T] = 0. \tag{15b}$$

Note that it is due to the lack of penalty on u that $\nabla_u L$ is independent of u.

Next we study $\nabla_x L$. Normally this allows us to solve for λ given x, going backwards in time. Calculating the gradients gives

$$\frac{\partial L}{\partial x_{2i-1}[t]} = b_i - q_i x_{2i-1}[t] + \alpha\lambda_i[t+1] - \lambda_i[t], \quad 1 \le t \le T-1 \tag{16a}$$

$$\frac{\partial L}{\partial x_{2i-1}[T]} = b_i - q_i x_{2i-1}[T] - \lambda_i[T]. \tag{16b}$$

Combining the two optimality conditions, we get the following lemma.

LEMMA 1
The optimal inventory level x satisfies

$$x_{2i-1}[t] = \frac{\alpha q_{i-1}}{q_i} x_{2(i-1)-1}[t+1] \tag{17}$$

for $i \ge 2$ and $t \le T-1$. □

Proof Using (16a) and (15a) gives for $t \le T-2$

$$\begin{aligned} x_{2i-1}[t] &= \frac{\alpha\lambda_i[t+1] - \lambda_i[t] + b_i}{q_i} \\ &= \frac{\alpha q_{i-1}}{q_i}\frac{\alpha\lambda_{i-1}[t+2] - \lambda_{i-1}[t+1] + b_{i-1}}{q_{i-1}} + \frac{b_i - \alpha b_{i-1}}{q_i} \\ &= \frac{\alpha q_{i-1}}{q_i} x_{2(i-1)-1}[t+1] \end{aligned}$$

where we have used that $\alpha b_{i-1} = b_i$. The case for $t = T-1$ follows similarly. □

5. Proof of Theorem 1 and Proposition 1

We are now ready to prove Theorem 1 and Proposition 1.

Proof of Theorem 1 For every input (recall that $x_{2N} = 0$), we have from the dynamics that

$$x_{2(i+1)-1}[t+1] = \alpha(x_{2(i+1)-1}[t] + x_{2(i+1)}[t]) - u_i[t] \Rightarrow$$
$$u_i[t] = \alpha(x_{2(i+1)-1}[t] + x_{2(i+1)}[t]) - x_{2(i+1)-1}[t+1].$$

Using (16a-b) we get for $t \leq T-2$, that the optimal u must satisfy

$$u_{i-1}[t] = \alpha(x_{2i-1}[t] + x_{2i}[t]) + \frac{\alpha\lambda_i[t+2] - \lambda_i[t+1] + b_i}{q_i}$$

and for $t = T-1$,

$$u_{i-1}[T-1] = x_{2i+1}[T-1] + x_{2i}[T-1] + \frac{-\lambda_i[T] + b_i}{q_i}.$$

Thus with the relation between p and λ as defined in (14) we have that the optimal u is given by (11). The expressions for p in (10) follows from Proposition 2 (see the appendix). □

Proof of Proposition 1 As the nodes choices of levels has no effect on the prices, the optimal level from the nodes perspective must satisfy

$$0 = \frac{\partial \hat{C}_i}{\partial x_{2i-1}[t]} = \frac{\partial L}{\partial x_{2i-1}[t]}.$$

This must also hold for the social optimum, thus proving (i).

Furthermore we see that choosing the social optimum inventory levels are better than choosing $x_{2i-1}[t] = \alpha^t x_{2i-1}[0]$, as it is not a minimizer of C. Thus proving (ii).

The sum of all the payments are

$$-\sum_{i=1}^{N}\sum_{t=1}^{T}\lambda_i[t]\Big(x_{2i-1}[t] - \alpha x_{2i-1}[t-1]\Big). \tag{18}$$

Using that

$$x_{2i-1}[t] - \alpha x_{2i-1}[t-1] = -u_{i-1}[t-1] + \alpha u_i[t-2],$$

The sum in (18) can be rewritten as

$$\sum_{i=1}^{N-1}\left(\sum_{t=0}^{T-2}\Big(-\lambda_{i+1}[t+1] + \alpha\lambda_i[t+2]\Big)u_i[t] - \lambda_{i+1}[T]u_i[T-1]\right).$$

This is equal to zero, since $\lambda_{i+1}[t+1] = \alpha\lambda_i[t+2]$ and $\lambda_i[T] = 0$ for $i \geq 2$. Thus proving (iii). □

6. Conclusions & Future work

We have considered the social and user optimum for a simple transportation problem on a string graph. By solving the social problem using a Lagrange multiplier approach we gave an implementation of the feedback law in terms of local prices and local states that allows for a distributed implementation. Furthermore, these prices aligned the two problem in a budget neutral way so that the nodes are never worse off than if they had been on their own.

Some of the assumptions in this paper could be relaxed. For example, the transportation delays could be changed to be a multiple of the sample time, allowing for non homogeneous delays. The results can also be extended to poly-trees, and we intend to do so in an upcoming publication.

Appendix

In the appendix we will derive the optimal Lagrange multipliers $\lambda[t_0+\tau]$ in terms of $x[t_0-1]$. We will show that each λ can be found as a sum of the the corresponding node levels in Lemma 2. These node levels can in turn be found by studying a time shifted aggregate level as shown in Lemma 3. This shifted aggregate can then be written in terms of a non shifted aggregate at t_0-1 in Lemma 4.

LEMMA 2
The optimal Lagrange multipliers are given by

$$\lambda_i[t_0] = \sum_{t=t_0}^{T} \alpha^{t-t_0}\big(b_i - q_i x_{2i-1}[t]\big).$$

□

Proof We have from (16) that $\lambda_i[T] = b_i - q_i x_i[T]$ and $\lambda_i[t] = b_i - q_i x_i[t] + \alpha\lambda_i[t+1]$. From that the lemma follows trivially. □

Next we show how each node level can be written in terms of a time shifted level vector.

LEMMA 3
The optimal inventory levels satisfy

$$x_{2i-1}[t_0+k] = \begin{cases} \dfrac{\gamma_{i+k}}{\alpha^k q_i} \displaystyle\sum_{j=1}^{i+k} \dfrac{x_{2j-1}[t_0+k+(i-j)]}{\alpha^{k+i-j}} & i+k \le N \\ \dfrac{\gamma_N}{\alpha^{N-i} q_i} \displaystyle\sum_{j=1}^{N} \dfrac{x_{2j-1}[t_0+k+(i-j)]}{\alpha^{N-j}} & i+k > N. \end{cases} \tag{19}$$

□

Proof We start by showing the lemma for $k = 0$. Using (17) gives

$$\begin{aligned} x_3[t] &= \frac{\alpha q_1}{q_2} x_1[t+1] \Rightarrow \\ (1 + \frac{\alpha^2 q_1}{q_2}) x_3[t] &= \frac{\alpha^2 q_1}{q_2} \left(\frac{x_1[t+1]}{\alpha} + x_3[t] \right) \Rightarrow \\ x_3[t] &= \frac{\alpha^2 \gamma_1}{q_2 + \alpha^2 \gamma_1} \left(\frac{x_1[t+1]}{\alpha} + x_3[t] \right). \end{aligned}$$

Now assume that (19) holds for $i-1$ and $k = 0$. Then using (17) again gives

$$\begin{aligned} x_{2i-1}[t] &= \frac{\alpha q_{i-1}}{q_i} x_{2(i-1)-1}[t+1] \\ &= \frac{\alpha q_{i-1}}{q_i} \frac{\gamma_{i-2}}{q_{i-1} + \gamma_{i-2}} \left(\sum_{j=1}^{i-1} \frac{x_{2j-1}[t + ((i-1) - j)]}{\alpha^{(i-1)-j}} \right) \end{aligned}$$

Which gives that

$$\left(1 + \frac{\alpha^2 \gamma_{i-1}}{q_i} \right) x_{2i-1} = \frac{\alpha^2 \gamma_{i-1}}{q_i} \left(\sum_{j=1}^{i} \frac{x_{2j-1}[t + (i-j)]}{\alpha^{i-j}} \right)$$

From which it follows that the lemma holds for $k = 0$. Now assume that the lemma holds for $k - 1$. Then if $i + k \le N$

$$\begin{aligned} x_{2i-1}[t_0 + k] &= \frac{q_{i+1}}{\alpha q_i} x_{2(i+1)}[t_0 + k - 1] \\ &= \frac{q_{i+1}}{\alpha q_i} \frac{\gamma_{(i+1)+(k-1)}}{\alpha^{k-1} q_{i+1}} \sum_{j=1}^{i+1+k-1} \frac{x_{2j-1}[t_0 + (k-1) + ((i+1) - j)]}{\alpha^{(k-1)+(i+1)-j}} \\ &= \frac{\gamma_{i+k}}{\alpha^k q_i} \sum_{j=1}^{i+k} \frac{x_{2j-1}[t_0 + k + (i-j)]}{\alpha^{k+i-j}} \end{aligned}$$

For $i + k > N$ define $\hat{k}$ and $\hat{t}_0$ so that

$$\begin{aligned} i + \hat{k} &= N \\ \hat{t}_0 + \hat{k} &= t_0 + k. \end{aligned} \tag{20}$$

Then using that $x_{2i-1}[t_0 + k] = x_{2i-1}[\hat{t}_0 + \hat{k}]$ gives the second part. □

Finally, we will show that the time shifted level vector can be written in terms of $x[t_0 - 1]$.

LEMMA 4
The optimal x for (2) satisfies for $i+k \leq N$:

$$\sum_{j=1}^{i+k} \frac{x_{2j-1}[t_0+k+(i-j)]}{\alpha^{k+i-j}} = \alpha \sum_{j=1}^{i+k} \left(x_{2j-1}[t_0-1]+x_{2j}[t_0-1]\right)$$

and for $i+k > N$:

$$\sum_{j=1}^{N} \frac{x_{2j-1}[t_0+k+(i-j)]}{\alpha^{N-j}} = \alpha^{k+i-N+1} \sum_{j=1}^{N} \Big(x_{2j-1}[t_0-1]+x_{2j}[t_0-1]\Big) \quad \square$$

Proof We start with the first equality. Using that

$$\begin{aligned} x_{2j-1}[t+n] &= \alpha^{n+1}(x_{2j-1}[t-1]+x_{2j}[t-1]) \\ &\quad - \sum_{\tau=0}^{n-1} \alpha^{(n-1)-\tau} u_{j-1}[t+\tau] + \sum_{\tau=0}^{n-2} \alpha^{n-1-\tau} u_j[t+\tau], \end{aligned}$$

we have for $i+k \leq N$:

$$\begin{aligned} \sum_{j=1}^{i} \frac{x_{2j-1}[t_0+k+(i-j)]}{\alpha^{k+i-j}} &= \alpha \sum_{j=1}^{n} \Big(x_{2i-1}[t_0-1]+x_{2i}[t_0-1]\Big) \\ + \alpha^{-1-\tau}\Big(\sum_{j=1}^{i} \sum_{\tau=0}^{k+(i-j)-2} u_j[t_0+\tau] &- \sum_{j=1}^{i} \sum_{\tau=0}^{k+(i-j)-1} u_{j-1}[t_0+\tau]\Big) \end{aligned}$$

Now using that $u_0 = 0$ the last row can be rewritten

$$\sum_{j=1}^{i-2} \sum_{\tau=0}^{k+(i-j)-2} u_j[t+\tau] - \sum_{j=2}^{i-1} \sum_{\tau=0}^{k+(i-j)-1} u_{j-1}[t+\tau] = 0$$

For the second equality we use (20) again,

$$\begin{aligned} \sum_{j=1}^{N} \frac{x_{2j-1}[t_0+k+(i-j)]}{\alpha^{N-j}} &= \sum_{j=1}^{N} \frac{x_{2j-1}[\hat{t}_0+\hat{k}+(i-j)]}{\alpha^{\hat{k}+i-j}} \\ &= \alpha \sum_{j=1}^{N} \Big(x_{2j-1}[\hat{t}_0-1]+x_{2j}[\hat{t}_0-1]\Big) \\ &= \alpha^{k-(N-i)+1} \sum_{j=1}^{N} \Big(x_{2j-1}[t_0-1]+x_{2j}[t_0-1]\Big) \end{aligned}$$

Where we have used that $\hat{t}_0 - t_0 = k - \hat{k}$. $\square$

We also need the following lemma, which shows that there exist a boundary effect in the optimal controller that makes some of the states locally optimal.

LEMMA 5
The optimal inventory levels satisfy

$$x_{2i-1}[t] = \frac{b_i}{q_i} \quad \forall t \geq T - (i-2), \; i \geq 2.$$

□

Proof We start by showing the lemma for $i = 2$. As $u_1[T-1]$ only affects $x_3[T]$, the optimal value corresponds to maximizing the local utility, so that $x_3[T] = b_2/q_2$. Thus

$$u_1[T-1] = -\frac{b_2}{q_2} + \alpha(x_3[T-1] + x_4[T-1])$$

and $x_3[T] = b_2/q_2$, independent of all other $u_i[t]$.

Now assume that the lemma holds for all $i \leq n$. Then $u_i[t]$ only needs to consider x_{2i+1} for all $t \geq T - i$. Thus the optimal $u_n[t]$ satisfies

$$u_n[t] = -\frac{b_{n+1}}{q_{n+1}} + \alpha(x_{2(n+1)-1}[t] + x_{2(n+1)}[t])$$

$\forall t \geq T - n$ and

$$x_{n+1}[t] = \frac{b_{i+1}}{q_{i+1}} \quad \forall t \geq T - (n-1).$$

Thus the Lemma holds for all n. □

We are now ready to state the following proposition, which gives expressions for the optimal λ.

PROPOSITION 2
Let $m = \min(T - t_0 - (i-1), N)$ and

$$\Xi(t_0, \tau) = \alpha^{1-\tau}\left(\sum_{j=i+\tau}^{m} \gamma_j \sum_{k=1}^{2j} x_k[t_0 - 1] + \sum_{j=N+1}^{T-t_0+1} \gamma_N \alpha^{2(j-N)} \sum_{k=1}^{2N-1} x_k[t_0 - 1]\right)$$

Then the optimal λ's are given by

$$\lambda_i[t_0 + \tau] = \sum_{t=t_0+\tau}^{T-(i-1)} \alpha^{t-(t_0+\tau)} b_i - \Xi(t_0, \tau)$$

□

Proof From Lemma 2 and 5 we have that

$$\lambda_i[t_0+\tau] = \alpha^{-\tau} \sum_{k=\tau}^{T-t_0-(i-1)} \alpha^k \Big(b_i - q_i x_{2i-1}[t_0+k] \Big)$$

Combining Lemma 3 and 4 gives that

$$\alpha^k q_i x_{2i-1}[t_0+k] = \begin{cases} \alpha\gamma_{i+k} \sum_{j=1}^{i+k} x_{2j-1}[t_0-1] + x_{2j}[t_0-1] & i+k \leq N \\ \alpha\gamma_N \alpha^{2(k+i-N)} \sum_{j=1}^{N} x_{2j-1}[t_0-1] + x_{2j}[t_0-1] & i+k > N \end{cases}$$

Which gives that

$$\alpha^{-\tau} \sum_{k=\tau}^{T-t_0-(i-1)} \alpha^k q_i[t_0+k] = \alpha^{1-\tau} \left(\sum_{j=i+\tau}^{m} \gamma_j \sum_{k=1}^{2j} x_k[t_0-1] + \sum_{j=N+1}^{T-t_0+1} \gamma_N \alpha^{2(j-N)} \sum_{k=1}^{2j} x_k[t_0-1] \right)$$

From Lemma 5 it follows that $\lambda_i[t] = 0, \; t \geq T-(i-1)$. □

References

Bamieh, B., F. Paganini, and M. A. Dahleh (2002). "Distributed control of spatially invariant systems". *IEEE Transactions on automatic control* **47**:7, pp. 1091–1107.

Boyd, S. and L. Vandenberghe (2004). *Convex optimization*. Cambridge university press.

Cannon, M., W. Liao, and B. Kouvaritakis (2008). "Efficient mpc optimization using pontryagin's minimum principle". *International Journal of Robust and Nonlinear Control: IFAC-Affiliated Journal* **18**:8, pp. 831–844.

Carpentier, P. and G. Cohen (1993). "Applied mathematics in water supply network management". *Automatica* **29**:5, pp. 1215–1250.

Cohen, G. (1978). "Optimization by decomposition and coordination: a unified approach". *IEEE Transactions on automatic control* **23**:2, pp. 222–232.

Giselsson, P., M. D. Doan, T. Keviczky, B. De Schutter, and A. Rantzer (2013). "Accelerated gradient methods and dual decomposition in distributed model predictive control". *Automatica* **49**:3, pp. 829–833.

Heyden, M., R. Pates, and A. Rantzer (2018). "A structured linear quadratic controller for transportation problems". In: *2018 European Control Conference (ECC)*. IEEE, pp. 1654–1659.

Jafarian, M., J. Scherpen, and M. Aiello (2016). "A price-based approach for voltage regulation and power loss minimization in power distribution networks". In: *2016 IEEE 55th Conference on Decision and Control (CDC)*. IEEE, pp. 680–685.

Jokić, A., M. Lazar, and P. P. van den Bosch (2009). "Real-time control of power systems using nodal prices". *International Journal of Electrical Power & Energy Systems* **31**:9, pp. 522–530.

Kelly, F. P., A. K. Maulloo, and D. K. Tan (1998). "Rate control for communication networks: shadow prices, proportional fairness and stability". *Journal of the Operational Research society* **49**:3, pp. 237–252.

Lamperski, A. and L. Lessard (2015). "Optimal decentralized state-feedback control with sparsity and delays". *Automatica* **58**, pp. 143–151.

Low, S. H., F. Paganini, and J. C. Doyle (2002). "Internet congestion control". *IEEE control systems magazine* **22**:1, pp. 28–43.

Radner, R. (1962). "Team decision problems". *The Annals of Mathematical Statistics* **33**:3, pp. 857–881.

Shah, P. and P. A. Parrilo (2013). "H_2-optimal decentralized control over posets: a state-space solution for state-feedback". *IEEE Transactions on Automatic Control* **58**:12, pp. 3084–3096.