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H-infinity Optimal Control for Systems with a Bottleneck Frequency

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***Abstract*— We characterize a class of systems for which the H-infinity optimal control problem can be simplified in a way that enables sparse solutions and efficient computation. For a subclass of the systems, an optimal controller can be explicitly expressed in terms of the matrices of the system's state-space representation. In many applications, the controller given by this formula, which is static, can be implemented in a decentralized or distributed fashion. Examples are temperature dynamics in buildings, water irrigation and electrical networks.**

***Index Terms*— Distributed control, H infinity control, Linear systems, Network analysis and Control, Optimal control.**

I. Introduction

SINCE the problem of H_∞ control was first formulated in 1981 [1], several solution techniques have been proposed. Youla-Kucera parametrization [2], Riccati-based approaches [3] and the optimization-based approach that uses linear matrix inequalities [4] are by now well known and conventional methods to this problem. However, they are based on a centralized view of the control problem, i.e., that a single process element is to be controlled by a single control element, and sparsity is generally not a trait of the controllers derived. Furthermore, these conventional methods to H_∞ control need to be performed numerically, and in order to achieve optimality, the computational procedure needs to be iterated.

For systems with a large number of process and control elements, the complexity in controller design stems, among other things, from the requirements on the structure of the controller. Enforcing a controller to have a certain structure or sparsity pattern could greatly complicate the synthesis or even make it intractable, see, e.g., [5], [6]. Although there exist procedures for some information structures [7], [8] and convexity has been shown for certain decentralized control problems [9], [10], the computational procedure they demand need not be efficient. However, in [11]–[13] methods for efficient computation of distributed controllers for so called positively dominated systems are presented. Local stability conditions are demonstrated in [14], [15] that are independent of the system's network and size. Similarly, [16] presents a scalable stability criterion for certain interconnected systems.

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Scalability of traditional distributed optimal control methods are further addressed in [17]. Moreover, regularization is used to enforce sparsity in, e.g., [18], [19]. Finally, in [20], [21], control problems for so called spatially invariant systems are investigated. Given their solution, the controller has a degree of spatial localization similar to the plant. Similarly, the approach in [22] renders controllers that adopt and preserve the distributed spatial structure of the system and in [23], in certain cases, the closed-loop system exhibits an interconnection structure that inherits the passivity properties of its components.

In this paper, we characterize a class of systems for which it is possible to construct a H_∞ optimal controller by only considering the behaviour of the system at a single frequency, that we term the *bottleneck frequency* of the system. This property simplifies the synthesis procedure and can also enable sparse solutions, suitable for the control of large-scale systems. In most applications considered, the bottleneck frequency is the zero frequency. For example, this is the case for positively dominated systems. Further, we identify a subclass of the systems for which the optimal controller can be given analytically, in a simple form, expressed in terms of the matrices of the system's state-space representation. If the matrices are sparse, the derived static controller is often sparse as well. For applications of large-scale systems, such as temperature dynamics in buildings [24] and water irrigation networks [25], the controllers we derive are decentralized or distributed, which is illustrated through examples. The previous work [26] covered some preliminary results on the explicit controllers, however, only for systems with certain type of symmetry in their state-space representation. The present work covers asymmetric systems as well.

A. Outline

Section II illustrates our main contributions through an example. Thereafter, in Section III, the class of systems that exhibit a bottleneck frequency are formally characterized. In Section IV, analytical solutions to the H_∞ optimal control problem are obtained for a subclass of the systems and Section V covers applications with large-scale systems.

B. Notation and preliminaries

The real and complex numbers are denoted $\mathbb{R}$ and $\mathbb{C}$, respectively. Moreover, $\mathbb{R}^{n\times m}$ and $\mathbb{C}^{n\times m}$ are the spaces of n-by-m real-valued and complex-valued matrices. For vectors, only the length is specified. The identity matrix is written as

I. If a scalar, vector or matrix x belongs to a set X, we write $x \in X$. The transpose of $M \in \mathbb{R}^{m\times n}$ is written M^T while the conjugate transpose of $M \in \mathbb{C}^{m\times n}$ is written M^*. The l_2 norm of a vector $v \in \mathbb{C}^n$ is denoted $|v|$. The l_2-induced matrix norm is denoted $\|M\|$ for $M \in \mathbb{C}^{n\times m}$. $M \in \mathbb{R}^{n\times n}$ is said to be Hurwitz if all its eigenvalues have negative real part. Further, for $M \in \mathbb{C}^{n\times n}$, positive and negative (semi)definiteness are denoted $M \succ (\succeq) 0$ and $M \prec (\preceq) 0$, respectively.

The space $L_\infty(j\mathbb{R})$ is the space of complex matrix functions that are essentially bounded on $j\mathbb{R}$. The rational subspace of $L_\infty(j\mathbb{R})$ is denoted RL_∞ and consists of all proper and real rational transfer matrices with no poles on the imaginary axis. The space H_∞ is a subspace of $L_\infty(j\mathbb{R})$ with functions that are analytic and bounded in the open right-half-plane. The H_∞ norm is denoted $\|\cdot\|_\infty$ and defined as $\|G\|_\infty := \sup_{\mathrm{Re}(s)>0} \|G(s)\|$ for $G \in H_\infty$. The real rational subspace of H_∞ is denoted by RH_∞ and consists of all proper and real rational stable transfer matrices. Moreover, for a stable transfer matrix G, it holds that $\|G\|_\infty = \sup_{\omega\in\mathbb{R}} \|G(j\omega)\|$. For further details, see [2]. Finally, the Laplace transform of a time domain signal $y(t)$ is denoted $\hat{y}(s)$.

II. Our main contributions illustrated through an example

In this section, the main contributions of this note will be illustrated through the following example, variations of which have previously been used in [26]–[28]. Consider a system composed of subsystems $i = 1, \dots, N$ with

$$\dot{x}_i(t) = -a_i x_i(t) + w_i(t) + \sum_{(i,j)\in\mathcal{E}} u_{ij}(t) - u_{ji}(t), \quad t \geq 0,$$

where x_i is the state of subsystem i, w_i is a disturbance and the control inputs u_{ij} are to be designed. Furthermore, (i,j) is in the set $\mathcal{E}$ if and only if subsystems i and j are connected, and $a_i > 0$ for all i. The objective is to design each u_{ij} based on state feedback so as to minimize the cost

$$\int_0^\infty \left(\sum_{i=1}^N x_i(t)^2 + \sum_{(i,j)\in\mathcal{E}} u_{ij}(t)^2 + u_{ji}(t)^2 \right) \mathrm{d}t,$$

where $x_i(0) = 0$ for all i, over the set of disturbances for which $\int_0^\infty \left(\sum_{i=1}^N w_i^2(t) \right) \mathrm{d}t = 1$. Equivalently, the system can be written in the form

$$\dot{x} = Ax + Bu + Iw \tag{1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$ and $w \in \mathbb{R}^l$ are vectors comprised of the states x_i, control inputs u_{ij} and disturbances w_i, respectively. It follows that the matrix A is diagonal with $A_{ii} = -a_i$ and $B_{ij} = 1$, $B_{ji} = -1$ if $(i,j) \in \mathcal{E}$, otherwise the entries of B are equal to zero. The considered problem can now be written as

$$\inf_{K\in\mathbb{R}^{m\times n}} \sup_{\omega\in\mathbb{R}} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (j\omega I - A - BK)^{-1} \right\|, \tag{2}$$

where K is such that $A + BK$ is Hurwitz. Note that, in the case of state feedback, it is well known that optimality can be achieved with a static controller. Therefore, in this example, we can restrict the problem to control laws of the form $u = Kx$ where $K \in \mathbb{R}^{m\times n}$.

The considered system belongs to a class for which (2) can be simplified to a problem at a single frequency $\omega = \omega_0$, the so called *bottleneck frequency* of the system. Further, it is possible to solve the simplified problem explicitly by means of a simple least squares argument. These claims will now be illustrated, and formally stated in the following sections.

To begin, notice that the following always holds

$$(2) \geq \inf_{K\in\mathbb{R}^{m\times n}} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (j\omega_0 I - A - BK)^{-1} \right\|, \tag{3}$$

where $\omega_0 \in \mathbb{R}$. For the particular system treated, it can be shown that it holds with equality for $\omega_0 = 0$. Now, consider

$$\begin{aligned} \text{minimize} \quad & |\bar{x}|^2 + |\bar{u}|^2 \\ \text{subject to} \quad & 0 = A\bar{x} + B\bar{u} + \bar{w}, \end{aligned} \tag{4}$$

where $\bar{x}$, $\bar{u}$ and $\bar{w}$ are vectors of appropriate dimensions and $\bar{w}$ is given. The optimization problem (4) is a standard least-squares type problem with solution

$$\begin{bmatrix} \bar{x}_* \\ \bar{u}_* \end{bmatrix} = \begin{bmatrix} -A \\ -B^T \end{bmatrix} (A^2 + BB^T)^{-1} \bar{w}, \tag{5}$$

see Lemma 1, where the matrix $A^2 + BB^T$ is invertible by assumption. Moreover, the solution (5) shows that $\bar{u}_* = B^T A^{-1} \bar{x}_*$. It is straightforward to verify that

$$\inf_{K\in\mathbb{R}^{m\times n}} \left\| \begin{bmatrix} I \\ K \end{bmatrix} (A + BK)^{-1} \right\|, \tag{6}$$

i.e., the right-hand side of the inequality in (3) evaluated at $\omega_0 = 0$, is lower-bounded by the supremum of the squareroot of (4) over $|\bar{w}| = 1$. From this, it follows that

$$(6) \geq \|(A^2 + BB^T)^{-1}\|^{\frac{1}{2}}.$$

Finally, for systems (1) with A symmetric and Hurwitz, as in this example, the controller gain $K = B^T A^{-1}$ is always stabilizing and achieves this lower bound. Hence, it is a solution to (2).

In this example, the optimal control law $u = B^T A^{-1} x$ is equivalent to

$$u_{ij} = -x_i/a_i + x_j/a_j.$$

Notice that each control input does only depend on the states of the subsystems it directly affects, i.e., it is built on nearest neighbour information, although this was not enforced by the synthesis procedure through, e.g., structural constraints or regularization. Instead, it is simply the sparsity of the matrices of the system's state-space representation that results in the sparsity of the control law.

This aforementioned property of the optimal control law is related to that exhibited by spatially invariant and spatially distributed systems. However, the systems we can consider are not restricted to such systems. Moreover, notice that the control law scales well even if N is large and that even if the set $\mathcal{E}$ changes, i.e., a link is added or removed, it will not change the already existing control inputs, which makes it suitable for systems of large scale.

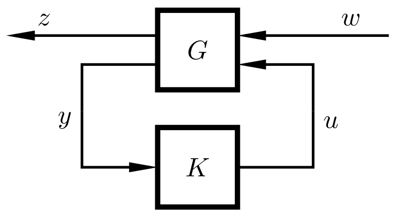

Fig. 1. Block diagram of system $\boldsymbol{G}$ and controller $\boldsymbol{K}$. Signals $\boldsymbol{z}$, $\boldsymbol{w}$, $\boldsymbol{y}$ and $\boldsymbol{u}$ are the regulated output, disturbance, measurement and control input of the system.

III. Simplified Synthesis Procedure for Systems with a Bottleneck Frequency

In this section, we characterize systems that have a bottleneck frequency. As previously mentioned, and illustrated in Section II, the H_∞ optimal control problem can be decomposed in a very particular way for systems of this class. To begin, we will give a formal definition of the H_∞ optimal control problem.

Consider the system

$$\begin{bmatrix}\hat{z}(s)\\ \hat{y}(s)\end{bmatrix} = G(s)\begin{bmatrix}\hat{w}\\ \hat{u}\end{bmatrix} := \begin{bmatrix}G_{zw}(s) & G_{zu}(s)\\ G_{yw}(s) & G_{yu}(s)\end{bmatrix}\begin{bmatrix}\hat{w}(s)\\ \hat{u}(s)\end{bmatrix} \tag{7}$$

where $G(s) \in RL_\infty$. The signals $\hat{z}$, $\hat{w}$, $\hat{u}$ and $\hat{y}$ have dimensions k, l, m and n, respectively. The system $G(s)$ is to be controlled through $\hat{u}(s) = K(s)\hat{y}(s)$, $K(s) \in RL_\infty$, as depicted in Figure 1. The H_∞ *optimal control problem* is

$$\inf_{K\in RL_\infty} \|F_l(G,K)\|_\infty \tag{8}$$

where K is an internally stabilizing controller and

$$F_l(G,K) := G_{zw} + G_{zu}K(I - G_{yu}K)^{-1}G_{yw}$$

is the lower linear fractional transformation of $G(s)$ with $K(s)$, and $K(s)$ is such that $F_l(G(s),K(s)) \in RH_\infty$. It is assumed that $G(s)$ is such that there exists a stabilizing $K(s) \in RL_\infty$ for which $I - G_{yu}(s)K(s)$ is invertible.

For (7) that exhibits a bottleneck frequency ω_0, the simplified problem

$$\min_{C\in\mathbb{R}^{m\times n}} \|F_l(G(j\omega_0),C)\|, \tag{9}$$

describes a solution $K_0(s)$ to (8) through $K_0(j\omega_0) = C$ with the constraints that $K_0(s) \in RL_\infty$ need be stabilizing and

$$\omega_0 \in \arg\max_{\omega\in\mathbb{R}} \|F_l(G(j\omega),K_0(j\omega))\|.$$

This result is formally presented in the following theorem.

Theorem 1: Given $G(s) \in RL_\infty$, suppose there exist $\omega_0 \in \mathbb{R}$ and stabilizing $K_0(s) \in RL_\infty$ such that

$$K_0(j\omega_0) \in \arg\min_{C\in\mathbb{C}^{m\times n}} \|F_l(G(j\omega_0),C)\|, \tag{10}$$

$$\omega_0 \in \arg\max_{\omega\in\mathbb{R}} \|F_l(G(j\omega),K_0(j\omega))\|. \tag{11}$$

Then, K_0 minimizes $\|F_l(G,K)\|_\infty$ over $K \in RL_\infty$.

Remark 1: Given the assumptions in Theorem 1, the point (K_0,ω_0) is a saddle-point. This is a property that is also of interest in the dynamic game approach to H_∞ control, see [29].

Proof: There exist $\omega_0 \in \mathbb{R}$ and stabilizing $K_0(s) \in RL_\infty$ such that

$$\begin{aligned}\inf_{K\in RL_\infty} \|F_l(G(j\omega_0),K(j\omega_0))\| &\le \inf_{K\in RL_\infty} \|F_l(G,K)\|_\infty\\ &\le \sup_{\omega\in\mathbb{R}} \|F_l(G(j\omega),K_0(j\omega))\|.\end{aligned}$$

Moreover,

$$\inf_{K\in RL_\infty} \|F_l(G(j\omega_0),K(j\omega_0))\| \ge \inf_{C\in\mathbb{C}^{m\times n}} \|F_l(G(j\omega_0),C)\|$$

and by assumption

$$\inf_{C\in\mathbb{C}^{m\times n}} \|F_l(G(j\omega_0),C)\| = \sup_{\omega\in\mathbb{R}} \|F_l(G(j\omega),K_0(j\omega))\|.$$

Hence, it follows that

$$\inf_{K\in RL_\infty} \|F_l(G(j\omega_0),K(j\omega_0))\| \ge \sup_{\omega\in\mathbb{R}} \|F_l(G(j\omega),K_0(j\omega))\|.$$

Thus,

$$\inf_{K\in RL_\infty} \|F_l(G,K)\|_\infty = \|F_l(G(j\omega_0),K_0(j\omega_0))\|,$$

and the minimization is achieved by K_0. ■

In the remainder of this note, we will only consider K_0 in Theorem 1 being a static controller, i.e., $K_0 \in \mathbb{R}^{m\times n}$. Then, given (7) that has a bottleneck frequency that is not known, Theorem 1 suggests the following recipe for synthesis.

Recipe 1: Given $G(s) \in RL_\infty$ search over $\omega_0 \in \mathbb{R}$, to find

$$K_0 \in \arg\min_{C\in\mathbb{C}^{m\times n}} \|F_l(G(j\omega_0),C)\| \tag{12}$$

1) that is real-valued and stabilizing,
2) and such that $\omega_0 \in \arg\max_{\omega\in\mathbb{R}} \|F_l(G(j\omega),K_0)\|$.

The problem (12) can be implemented as

$$\begin{aligned}&\min\ t\\ &\text{subject to} \left.\begin{bmatrix} I & G_{zw} + G_{zu}QG_{yw}\\ G_{zw}^* + G_{yu}^*Q^*G_{zw}^* & tI\end{bmatrix}\right|_{s=j\omega_0} \succeq 0,\end{aligned}$$

where $Q \in \mathbb{C}^{m\times n}$ and $K_0 = (I + QG_{yu}(j\omega_0))^{-1}Q$. Given a state-space representation of the closed-loop system, stability can simply be checked through the solution of the Lyapunov equation while the frequency requirement can be tested through a bisection algorithm, see [2] for more details.

IV. Simple Analytical Solutions

In this section, we characterize a subclass of the systems for which a K_0, such that (12) holds, can be stated explicitly.

Theorem 2: Consider G with

$$\begin{bmatrix}G_{zw}(s) & G_{zu}(s)\\ G_{yw}(s) & G_{yu}(s)\end{bmatrix} = \left[\begin{array}{c|c} M_1(s) & M_2(s)\\ 0 & \alpha I\\ \hline M_1(s) & M_2(s)\end{array}\right] \tag{13}$$

where $M_1,\ M_2 \in RL_\infty$ and $\alpha > 0$. Then, for a given $\omega_0 \in \mathbb{R}$,

$$\min_{C\in\mathbb{C}^{m\times n}} \|F_l(G(j\omega_0),C)\| \tag{14}$$

is solved by $C_* = -\alpha^{-1}M_2(j\omega_0)^*$.

Remark 2: Note that α is a design variable that can be utilized to alter the gain of C_*. It determines the trade-off between minimizing state deviations and control effort. The system treated in Section II is of the form (13), however, with $\alpha = 1$. For a general $\alpha > 0$, the cost function is instead

$$\int_0^\infty \left(\sum_{i=1}^N x_i(t)^2 + \sum_{(i,j)\in\mathcal{E}} \alpha^2 u_{ij}(t)^2 + \alpha^2 u_{ji}(t)^2 \right) \mathrm{d}t,$$

from which the tradeoff effect becomes apparent.

Remark 3: It holds that $\|F_l(G,K)^T\|_\infty = \|F_l(G,K)\|_\infty$, where $F_l(G,K)^T$ is the dual system to $F_l(G,K)$, see [2, p. 34]. Hence, the synthesis for the dual to (13) is also covered by Theorem 2, see [30] for an interpretation of this problem. Moreover, it is possible to add alternative weight matrices on the regulated output z and still be able to explicitly solve the minimization problem, see [26, Rem. 1] as well as [30].

Proof: For G defined by (13) it holds that

$$G_{zw} = \begin{bmatrix} G_{yw} \\ 0 \end{bmatrix} \text{ and } G_{zu} = \begin{bmatrix} G_{yu} \\ \alpha I \end{bmatrix}. \tag{15}$$

Hence,

$$\begin{aligned} F_l(G^0, C) &= \left(\begin{bmatrix} I \\ 0 \end{bmatrix} + \begin{bmatrix} G^0_{yu} \\ \alpha I \end{bmatrix} C(I - G^0_{yu}C)^{-1} \right) G^0_{yw} \\ &= \begin{bmatrix} I \\ \alpha C \end{bmatrix} (I - G^0_{yu}C)^{-1} G^0_{yw}, \end{aligned}$$

where the notation $G^0 := G(j\omega_0)$ has been used. Given $\gamma > 0$, $\|F_l(G^0, C)\| \le \gamma$ is equivalent to

$$|F_l(G^0, C)w|^2 \le \gamma^2 |w|^2$$

for all $w \in \mathbb{C}^l$. Moreover, it is equivalent to

$$|x|^2 + |\alpha C x|^2 \le \gamma^2 |w|^2$$

for all x, w such that $x = (I - G^0_{yu}C)^{-1} G^0_{yw} w$. Denote $u = \alpha C x$, and rewrite the latter equality as $x - \alpha^{-1} G^0_{yu} u = G^0_{yw} w$. Now, for fixed w, consider

$$\begin{aligned} &\text{minimize} \quad |x|^2 + |u|^2 \\ &\text{subject to} \quad x - \alpha^{-1} G^0_{yu} u = G^0_{yw} w, \end{aligned}$$

that lower bounds γ as the constraint $u = \alpha C x$ is excluded. By Lemma 1 in the Appendix, a standard least-squares problem included for completeness, it follows that

$$\gamma \ge \left\| \begin{bmatrix} I \\ -\alpha^{-1}(G^0_{yu})^* \end{bmatrix} \left(I + \alpha^{-2} G^0_{yu} (G^0_{yu})^* \right)^{-1} G^0_{yw} \right\|. \tag{16}$$

Hence,

$$(14) \ge \left\| \begin{bmatrix} I \\ -\alpha^{-1}(G^0_{yu})^* \end{bmatrix} \left(I + \alpha^{-2} G^0_{yu} (G^0_{yu})^* \right)^{-1} G^0_{yw} \right\|.$$

Finally, define $C_* = -\alpha^{-1}(G^0_{yu})^*$ and notice that $\|F_l(G^0, C_*)\|$ is equal to the lower bound stated above. Thus,

$$C_* = -\alpha^{-1}(G^0_{yu})^* = -\alpha^{-1} G_{yu}(j\omega_0)^* = -\alpha^{-1} M_2(j\omega_0)^*$$

solves (14) and the proof is complete. ∎

For (13), step (12) in Recipe 1 is given by

$$K_0 = -\alpha^{-1} M_2(j\omega_0)^*. \tag{17}$$

For completion, we will add the result illustrated in Section II next. It is a slight variation of [26, Theorem 1], although here given a completely new proof. It presents a class of systems for which the bottleneck frequency is the zero frequency and an optimal controller can be given directly as $K_0 = -\alpha^{-1} M_2(0)^*$. Notice that, in general, if C_* is complex, one would need to find a dynamic controller $K_0(s) \in RL_\infty$ that fulfils $K_0(j\omega_0) = C_*$.

Corollary 1 ([26, Theorem 1]): Consider G with

$$\begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|c} (sI - A)^{-1} H & (sI - A)^{-1} B \\ 0 & \alpha I \\ \hline (sI - A)^{-1} H & (sI - A)^{-1} B \end{array} \right]$$

where A, B and H are real-valued matrices of appropriate dimensions and $\alpha > 0$. Moreover, $A = A^T$ and A is Hurwitz. Then, $K_0 = \alpha^{-1} B^T A^{-1}$ minimizes $\|F_l(G,K)\|_\infty$ over RL_∞.

Proof: By Theorem 2, $C_* = \alpha^{-1} B^T A^{-1}$ solves

$$\min_{C \in \mathbb{C}^{m\times n}} \|F_l(G(j\omega_0), C)\|$$

when $\omega_0 = 0$. Hence, given $K_0 = \alpha^{-1} B^T A^{-1}$, it holds that

$$K_0 \in \arg \min_{C \in \mathbb{C}^{m\times n}} \|F_l(G(0), C)\|.$$

We will now show that K_0 is stabilizing and

$$0 \in \arg\max_{\omega\in\mathbb{R}} \|F_l(G(j\omega), K_0)\|. \tag{18}$$

It follows from [2, Lemma 7.2, p 106] that the closed-loop system $F_l(G, K_0)$ is stable as for $P = -A^{-1} \succ 0$,

$$\begin{aligned} &\left(A + \alpha^{-1} B B^T A^{-T}\right)^T P + P\left(A + \alpha^{-1} B B^T A^{-T}\right) \\ &\qquad = -2I - 2\alpha^{-2} A^{-1} B B^T A^{-1} \prec 0. \end{aligned}$$

Further, (18) is equivalent to that the following inequality holds for all $\omega \in \mathbb{R}$,

$$H^T(\omega^2 A^T M^{-1} A + j\omega(A - A^T) + M)^{-1} H \preceq \|H^T M^{-1} H\| I, \tag{19}$$

where $M = AA^T + \alpha^{-2} B B^T$. As A is symmetric, the inequality above clearly holds. Hence, it follows from Theorem 1 that K_0 minimizes $\|F_l(G,K)\|_\infty$ over $K \in RL_\infty$. ∎

V. Applications to Large-Scale Systems

If a H_∞ optimal synthesis problem is solvable, it generally has several solutions. In other words, there exist several optimal control laws. Naturally, they will have different properties. In the design of controllers for large-scale systems, properties of sparsity in the structure of the control law is of importance. Hence, the choice of an optimal controller is crucial. In this section, we will illustrate how the controllers derived in Section IV can be applied to large-scale systems.

A. Temperature Control in Buildings

Consider a building with N rooms and denote the deviation in average temperature in room i, from some operating point, by T_i. The temperature dynamics in room i, as governed by Fourier's law of thermal conduction, is

$$m_i c\dot{T}_i = p_i(T_{out} - T_i) + \sum_{j\in\mathcal{E}_i} p_{ij}\,(T_j - T_i) + u_i + d_i, \quad (20)$$

where m_i is the air mass of room i and c is the specific heat capacity of air. Furthermore, $\mathcal{E}_i$ is the set of rooms that share a wall/floor/ceiling with room i. The heat conduction coefficients p_i and $p_{ij} = p_{ji}$ are constant, real-valued and strictly positive while T_{out} is the outdoor temperature. Inputs d_i and u_i are disturbances and control inputs, respectively. Disturbances occur, e.g., when a window is opened. Moreover, the temperatures can be controlled through heating and cooling devices modelled by the control inputs u_i, where it is assumed that the temperatures T_i are available for feedback. This example has previously been treated in [26, Ex. 1]. For a review of approaches to building temperature control, see [24]. It is important to point out that the approaches generally taken include integral action and/or predictive tools. However, the example given here illustrates what is achievable through proportional control.

The overall system can be written as $E\dot{T} = PT + u + w$ where $T = [T_1, \ldots, T_N]^T$, $u = [u_1, \ldots, u_N]^T$, E is a diagonal matrix with positive elements $E_{ii} = m_i c$, $P \prec 0$ and the i:th entry of w is equal to $d_i + p_i T_{out}$. Consider the variable transformation $x = E^{\frac{1}{2}}T$. Then, (20) for $i = 1, \ldots, N$ can be written as $\dot{x} = Ax + Bu + Hw$, where $A = E^{-1/2}PE^{-1/2} \prec 0$, $B = E^{-1/2}$ and $H = E^{-1/2}$. Notice that A is symmetric and Hurwitz. It follows from Corollary 1 that

$$u = \alpha^{-1}B^T A^{-1}x = \alpha^{-1}P^{-1}ET$$

minimizes the deviation in temperature with minimum control effort, as scaled by $\alpha > 0$, over the considered set of disturbances. The design variable α can be used to alter the gain of K_0, e.g., to compensate if P is close to singular.

In this example, the controller gain matrix $K_0 = \alpha^{-1}P^{-1}E$ will be full due to the inverse of P. However, at least for certain building structures, the controller can still be implemented in a distributed fashion, see [31, pp. 36-37] for an example. The approach builds on viewing the control law as the system of equations $Pu = ET$, where P is a sparse matrix. Further, given the LU decomposition of P, i.e., $P = LU$ where L and U are lower and upper triangular, respectively, the elements of the control signals vector u can be computed, sequentially, through the sparse systems of equations $L\nu = ET$ and $Uu = \nu$.

B. Water Irrigation Networks and Asymmetry

Consider the water irrigation network in Figure 2. The water level in pool i, around some operating point, is denoted q_i and

$$\begin{aligned}\dot{q}_i &= \frac{1}{\alpha_i}\left(-\beta_i q_i + r_i - w_i - u_{i-1}\right),\\ \dot{r}_i &= \frac{1}{\tau_i}\left(-r_i + u_i\right).\end{aligned} \quad (21)$$

The variable r_i is a fictive entity that approximately models a time-delay in the inflow u_i. Moreover, u_{i-1} is the outflow of pool i. Note that $u_0 = 0$, i.e., there is no regulated outflow from pool 1. The signal w_i is an unknown disturbance or uncertainty in the load profile. Parameters α_i, β_i and τ_i are all positive and the total number of pools is N. This model is based on[1] the model included in [32, Section 4].

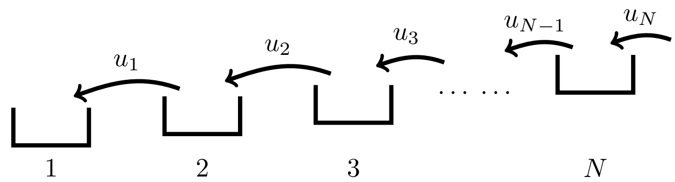

Fig. 2. Water irrigation network.

The overall system of N pools can be written as $\dot{x} = Ax + Bu + Hw$ where $x = [q_1, r_1, q_2, r_2, \ldots, q_N, r_N]^T$, $u = [u_1, u_2, \ldots, u_N]^T$ and $w = [w_1, w_2, \ldots, w_N]$. The matrix A is block-diagonal with N number of 2×2 elements, with element i given by

$$(A)_{ii} = \begin{bmatrix} -\beta_i/\alpha_i & 1/\alpha_i \\ 0 & -1/\tau_i \end{bmatrix}.$$

Moreover, column i of the control input matrix B is given by

$$(B)_{\cdot i} = \begin{cases} \begin{bmatrix} 0^{2N-1} & \dfrac{1}{\tau_N} \end{bmatrix}^T & \text{if } i = N,\\ \begin{bmatrix} 0^{2i-1} & \dfrac{1}{\tau_i} & -\dfrac{1}{\alpha_{i+1}} & 0^{2(N-i)-1} \end{bmatrix}^T & \text{otherwise,} \end{cases}$$

while H is of dimension $2N \times N$ with the $2(i-1)+1$th entry of the ith column being equal to $-1/\alpha_i$. For simplicity, we will now consider $\alpha_i, \tau_i, \beta_i = 1$ for all $i = 1, \ldots, N$. However, a similar statement can be made for a wide range of parameter values. Given $\alpha_i, \tau_i, \beta_i = 1$ for all $i = 1, \ldots, N$, the subsystems of the described model are of the type considered in Example 1 given in the Appendix. Through analysis similar to that performed there, it can be shown that the control law

$$u_i = \begin{cases} -q_N - r_N & \text{if } i = N,\\ -q_i - r_i + q_{i+1} & \text{otherwise,} \end{cases}$$

minimizes the H_∞ norm of the closed-loop system from w to $z = [x^T,\ u^T]^T$. This control law will keep the water level of each pool around its operating point with minimum control effort. Moreover, the control law is decentralized as each u_i is based only on measurements q_i, r_i and q_{i+1}.

We recognize that the proposed method is different from those presented in, see e.g., [25], [32], [33], and that it does not appreciate the full nature of the application at hand. Here, water irrigation networks have merely been used to illustrate our theoretical finding. It is in future works to fully characterize how well suited the controller we propose is for this particular problem. However, our solution gives insights into what is achievable, as it proposes an optimal controller. Further, it would be of interest to see what effects on transients the proposed controller has, similarly to how such behaviour is analyzed for vehicle platoons in [34].

[1]Time-delays are approximated and nonlinearities such as input capacity constraints are disregarded. Furthermore, the water of each pool is assumed to be utilized locally through the term $\beta_i q_i$.

C. Network of Synchronous Machines

In this section we will cover an application for which the optimal controller is defined at $\omega_0 \neq 0$. Consider the linearized model of the synchronous machine

$$\frac{1}{\omega_s^2}\ddot{\theta} + \frac{2\zeta}{\omega_s}\dot{\theta} + \theta = P_{\text{dist}} + P_{\text{gen}},$$

where P_{gen} is proportional to the mechanical power-input, P_{dist} is a disturbance in this input, i.e., a load, while θ is the rotor phase angle of the machine. Furthermore, $\omega_s,\ \zeta > 0$. Define

$$M(s) := \frac{\omega_0^2 s}{s^2 + 2\zeta\omega_0 s + \omega_0^2}$$

and the regulated output $z = [\dot{\theta}, P_{\text{gen}}]^T$. We would like to design a droop controller $P_{\text{gen}} = K\dot{\theta}$, where $K \in \mathbb{R}$, such that the H_∞ norm of the closed-loop system from P_{dist} to z is minimized. This problem can be written in terms of

$$G(s) = \begin{bmatrix} G_{zw}(s) & G_{zu}(s) \\ G_{yw}(s) & G_{yu}(s) \end{bmatrix} = \left[\begin{array}{c|c} M(s) & M(s) \\ 0 & 1 \\ \hline M(s) & M(s) \end{array}\right] \tag{22}$$

and K as the H_∞ optimal control problem defined in (8). We will now show that this system belongs to the class characterized by Theorem 1 through following Recipe 1.

Given $\omega_0 = \omega_s$, it follows from Theorem 2 that $C_* = -M(j\omega_0)^* = -\omega_s/2\zeta$ solves (12). Further, $K_0 = -\omega_s/2\zeta$ is real-valued, and stabilizing as

$$(1 - MK_0)^{-1}M = \frac{\omega_s^2 s}{s^2 + s(4\zeta^2 + \omega_s^2)\omega_s/2\zeta + \omega_s^2}$$

is stable. Finally,

$$\begin{aligned}
&\arg\max_{\omega\in\mathbb{R}} \|F_l(G(j\omega), K_0)\| \\
&= \arg\max_{\omega\in\mathbb{R}} \left| \frac{j\omega_s^2\omega}{-\omega^2 + j\omega(4\zeta^2 + \omega_s^2)\omega_s/2\zeta + \omega_s^2} \right| \\
&= \arg\max_{\omega\in\mathbb{R}} \frac{\omega_s^2\omega^2}{(\omega_s^2 - \omega^2)^2 + \omega^2(4\zeta^2 + \omega_s^2)^2\omega_s^2/4\zeta^2} \\
&\equiv \omega_s,
\end{aligned}$$

and we have shown that $K_0 = -\omega_s/2\zeta$ solves (8).

The network case needs to be treated through a slightly different approach. To begin, consider a network of N machines

$$m\ddot{\theta} + d\dot{\theta} + L\theta = P_{\text{dist}} + P_{\text{gen}},$$

where m and d are constants while L is the weighted Laplacian of the network's graph, see [35]. (Even more generality can be treated with $M\ddot{\theta} + D\dot{\theta} + L\theta = P_{\text{dist}} + P_{\text{gen}}$, where M and D are matrices, as long as there exists a uniform matrix that simultaneously diagonalizes M, D and L.) It follows that $M(s) = s(ms^2 I + dsI + L)^{-1}$ in (22). We will now analyze the lower-bound of the H_∞ optimal control problem that was utilized in Theorem 2, i.e., (16), in this example given by

$$\| \left(M(j\omega)^{-1} M(j\omega)^{-*} + I \right)^{-1} \|^{\frac{1}{2}}. \tag{23}$$

The maximum of (23) over $\omega \in \mathbb{R}$ is given by $1/\sqrt{d^2+1}$ and achieved at frequencies $\omega_0^i = \sqrt{\lambda_i/m}$, $i = 2, .., N$, where λ_i is the i:th smallest eigenvalue of L. Further, the controller gain $K_0 = -d^{-1}I$ can be shown to be stabilizing and to achieve the lower bound gain $1/\sqrt{d^2+1}$, hence it solves the H_∞ optimal control problem for the given system[2]. However, notice that the controller gain is not that given by Theorem 2, i.e., $K_0 = -M(j\omega_0)^*$, and this particular system has in fact several *bottleneck frequencies*.

VI. Conclusions and Directions for Future Work

We have characterized a class of systems that have a so called *bottleneck frequency*. This property leads to a simplified H_∞ optimal control problem, and in certain cases sparse solutions can be explicitly stated. The latter is beneficial for the control of large-scale systems, as has been shown through examples covering temperature dynamics in buildings, water irrigation and electrical networks. Future works should investigate how time-delayed and nonlinear systems can fit in the presented framework.

For large-scale applications, the explicity stated control law is an example of a decentralized controller that is also globally optimal. It is inevitably so that one has to consider the interplay between the achieved performance and simplicity of the design procedure as a part of the design process. However, it is not always the case that optimal performance is unachievable with a decentralized control law, as was illustrated in this note. Moreover, the presented results show that structural constraints, or regularization techniques, are not always necessary for synthesis of decetralized or distributed controllers.

Appendix

Lemma 1 ([36, p. 161, Sec 6.10, Th. 1]): Consider $A \in \mathbb{C}^{n\times m}$ with full row-rank and $b \in \mathbb{C}^n$. Then, $z_0 = A^*(AA^*)^{-1}b$ solves

$$\begin{aligned} &\min \quad |z|^2 \\ &\text{subject to} \quad Az = b, \end{aligned}$$

and the optimal value is equal to $|A^*(AA^*)^{-1}b|^2$.

Example 1: Given $a > 0$, consider G as in (13) with

$$M_1(s) = (sI - A)^{-1}H,\ M_2 = (sI - A)^{-1}B \ \text{ and } \alpha = 1,$$

where

$$A = \begin{bmatrix} -a & a \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad H = I.$$

It follows from Theorem 2 that $C_* = -B^T(-j\omega_0 I - A^T)^{-1}$ solves the minimization problem in (12). We will now show that part 1) and 2) of Recipe 1 follows through for $\omega_0 = 0$, and that $K_0 = B^T A^{-T}$ is in fact a solution to (8). For part 1), we need to show that $A + BK_0$ is Hurwitz. It is clearly the case as the eigenvalues of

$$A + BB^T A^{-T} = \begin{bmatrix} -a - 1/a & a \\ 0 & -1 \end{bmatrix}$$

[2]Notice that although L has an eigenvalue which is zero, the input-output map of the closed-loop system will be stable. Asymptotic stability is only guaranteed for the controllable and observable part of the state-space, which excludes the direction associated with the zero eigenvalue.

have strictly negative real-part for $a > 0$. Moreover, part 2) is equivalent to that (19) holds for all $\omega \in \mathbb{R}$, just as in the proof of Corollary 1, which further can be written as

$$\begin{bmatrix} \omega^2 \dfrac{a^2}{a^2+1} + 2a^2 + 1 - g^{-1} & -a(1-j\omega) \\ -a(1+j\omega) & \omega^2 + 1 - g^{-1} \end{bmatrix} \succeq 0, \quad (24)$$

where

$$g := \frac{1}{a^2+1} \left\| \begin{bmatrix} 1 & a \\ a & 2a^2+1 \end{bmatrix} \right\|.$$

By the definition of g,

$$\begin{bmatrix} 2a^2 + 1 - g^{-1} & -a \\ -a & 1 - g^{-1} \end{bmatrix} \succeq 0,$$

which is equivalent to $2a^2 + 1 - g^{-1} \geq 0$, $1 - g^{-1} \geq 0$ and $(1 - g^{-1})(2a^2 + 1 - g^{-1}) - a^2 \geq 0$ by the Schur complement lemma. Inequality (24) holds for all ω in $\mathbb{R}$ if and only if

$$\begin{aligned} &0 \leq \omega^2 + 1 - g^{-1}, \quad 0 \leq \omega^2 \frac{a^2}{a^2+1} + 2a^2 + 1 - g^{-1}, \\ &0 \leq \omega^4 \frac{a^2}{a^2+1} + \omega^2 \left(3a^2 + (1 - g^{-1}) \left(1 + \frac{a^2}{a^2+1} \right) \right) \\ &\qquad\qquad + (2a^2 + 1 - g^{-1})(1 - g^{-1}) - a^2, \end{aligned} \quad (25)$$

holds for all $\omega \in \mathbb{R}$, again by the Schur complement lemma. It is easy to see that the first two inequalities hold by the definition of g and the terms ω^2 and $a^2/(a^2+1)$ being non-negative. The coefficients in the latter inequality are all non-negative, again by the definition of g, while ω appears as ω^2 and ω^4. Thus, it holds for all ω and $K_0 = B^T A^{-T}$ is in fact optimal.

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