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Abstract

In this report discrete time LQ control is treated. Necessary and sufficient conditions for the existence of a stabilizing LQ controller are given. The analysis is done using a state-space approach, and the proofs are constructive.

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1. Introduction

The LQ controller has a long history. The topic is treated in many text-books, among others [Kucera, 1991]. One of the interesting questions to answer is when there exists an optimal solution to the optimization problem. Optimal is in this case defined as minimizing the performance index and stabilizing the closed loop system. Other interesting questions are how to obtain the optimal controllers, and to find out when there is only one optimal controller, i.e. uniqueness of the solution. Most work considers only the case with no cross-coupling in the performance index. Non-necessary conditions of detectability and full normal rank are frequently assumed. There seems to be little previous coverage of the case with dynamically redundant inputs. This has many interesting applications, among others in air-craft control, where several rudder-surfaces cooperate, see e.g. [Adams and Banda, 1993]. In very recent papers, e.g. [Kucera, 1993], it is claimed that weaker conditions of existence are easier to obtain using the transfer function approach. In this report it will be shown how necessary and sufficient conditions can be obtained along the state-space approach. The results correspond to the ones obtained for the continuous time case in [Willems *et al.*, 1986].

The report is organized as follows. In Section 2 the LQ problem is stated. Then in Section 3 some examples are given that illuminates the need for theory that covers the case of dynamically redundant inputs. In Section 4 necessary and sufficient conditions are given for the existence of a stabilizing solution. The optimal controller can be obtained by first making some preliminary feedback to obtain a reduced problem, and then solving this problem using a standard Riccati-solver. In Section 5 the results are summarized. Finally, there is in the appendix an algorithm for obtaining all solutions to LQ-problems in the case of non-uniqueness.

2. Control Problem

Consider the following state space description

$$\begin{cases} x(k+1) = Ax(k) + Bu(k) + v(k) \\ z(k) = Cx(k) + Du(k) \end{cases}$$

where $v(k) \in R^n$ is a sequence of independent identically distributed zero mean Gaussian random variables with covariance R_1, which for simplicity will be assumed to be positive definite. The signal $u(k) \in R^m$ is the control signal, $x(k) \in R^n$ is the state, and $z(k) \in R^p$ is the signal to be controlled. It will be assumed that (A, B) is stabilizable. If this is not the case, there will obviously not exist any stabilizing controller. Let $u(k)$ be given by the feedback

$$u(k) = -Lx(k)$$

and introduce the stationary loss function

$$J(L) = \mathrm{E}\left\{z^T(k)z(k)\right\} = \mathrm{E}\left\{ \begin{pmatrix} x(k) \\ u(k) \end{pmatrix}^T Q \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \right\} \tag{1}$$

where $\mathrm{E}(\cdot)$ denotes expectation, and where

$$Q = \begin{pmatrix} Q_1 & Q_{12} \\ Q_{12}^T & Q_2 \end{pmatrix} = \begin{pmatrix} C^T \\ D^T \end{pmatrix} \begin{pmatrix} C & D \end{pmatrix}$$

Notice that $Q \geq 0$ if and only if it can be written as above. Consider the optimization problem

$$\min_{L \in \mathcal{L}_s} J(L) \tag{2}$$

where $\mathcal{L}_s$ is the set of L such that $A - BL$ is stable. This problem can be solved by introducing the so called algebraic Riccati equation defined by

$$\begin{cases} S = (A-BL)^T S(A-BL) + (C-DL)^T(C-DL) \\ G = D^T D + B^T SB \\ GL = D^T C + B^T SA \end{cases} \tag{3}$$

which is equivalent to

$$\begin{pmatrix} I & 0 \\ L & I \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & G \end{pmatrix} \begin{pmatrix} I & 0 \\ L & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{3a}$$

It will later be discussed under what conditions this equation has a solution S, and what properties these solutions have. Denote the set of positive semidefinite solutions S to the algebraic Riccati-equation (3) by $\mathcal{S}$. For the time being assume that it has at least one positive semidefinite solution S such that there exists a corresponding $L \in \mathcal{L}_s$, and denote the set of such solutions by $\mathcal{S}_s$.

It is well-known and easily seen from (3) that if $\mathcal{S}_s$ is non-empty, then the optimal L is unique if and only if $G > 0$, which is implied by e.g. $Q_2 > 0$. It is actually true that the non-explicit condition $G > 0$ is equivalent to the explicit condition of (A, B, C, D) being left invertible, see [Kucera, 1991]. Alternative formulations of left invertibility of (A, B, C, D) are

$$\max_z \operatorname{rank} H(z) = m$$
$$\max_z \operatorname{rank} P(z) = m + n$$
$$\max_z \operatorname{rank} P_E(z) = m + 2n$$

where

$$H(z) = C(zI - A)^{-1}B + D$$
$$P(z) = \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}$$
$$P_E(z) = \begin{pmatrix} 0 & zI - A & -B \\ z^{-1}I - A^T & C^T C & C^T D \\ -B^T & D^T C & D^T D \end{pmatrix}$$

Here $H(z)$ is related to spectral factorization methods for solving LQ problems, see e.g. [Kucera, 1991], and $P_E(z)$ is related to deflating subspace methods for solving LQ problems, see e.g. [Laub, 1990]. In this report it will be seen that $P(z)$ is more fruitful to work with when existence results are of interest.

Under the condition of left invertibility, or equivalently one of the full normal rank conditions, it is possible to establish a necessary and sufficient condition for the existence of an element in $\mathcal{S}_s$. This condition is that $H(z)$, $P(z)$ or $P_E(z)$ should have full normal column rank on the unit circle. This will be proven in Lemma 3. In e.g. [Kucera, 1991; Kucera, 1993] strong detectability, equivalent to a minimum phase assumption, is used in the state-space approach when proving the non-emptiness of $\mathcal{S}_s$. One reason why this non-necessary condition is imposed there, is that all elements of $\mathcal{S}$ are considered and not merely the one in $\mathcal{S}_s$.

In Lemma 4 the condition of left invertibility will be relaxed. Then there will not be a unique feedback L, but the proof will be carried out in a constructive way, and hence all solutions are obtained. Further a necessary and sufficient condition for the existence of a solution, i.e. the non-emptiness of $\mathcal{S}_s$, is obtained. It says that $H(z)$, $P(z)$ or $P_E(z)$ should not lose column rank on the unit circle.

3. Examples

What happens if L is not unique? Does this correspond to any relevant non-trivial control problems? First a trivial example will be given.

EXAMPLE 1—Statically Redundant Inputs
Let the system dynamics be given by

$$x(k+1) = 0.5x(k) + bu_1(k) + u_2(k) + v(k)$$

and define the performance index as

$$\mathrm{E}\left\{x^2(k) + \rho[u_1(k) + u_2(k)]^2\right\}$$

By making an input signal transformation

$$\begin{cases} \bar{u}_1(k) = u_1(k) + u_2(k) \\ \bar{u}_2(k) = u_1(k) - u_2(k) \end{cases}$$

it follows that system dynamics are given by

$$x(k+1) = 0.5x(k) + \frac{b+1}{2}\bar{u}_1(k) + \frac{b-1}{2}\bar{u}_2(k)$$

and that the performance index becomes

$$\mathrm{E}\left\{x^2(k) + \rho\bar{u}_1^2(k)\right\}$$

If $b \neq 1$ then it is optimal to choose $\bar{u}_1(k) = 0$, since it is possible to control the system with only $\bar{u}_2(k)$ which is not visible in the performance index. If $b = 1$, then $\bar{u}_2(k)$ may be taken arbitrary, since it neither influences the system nor the performance index. In the case of $b = 1$ it is possible to rewrite the control problem as a simpler problem with

$$x(k+1) = 0.5x(k) + \bar{u}_1(k) + v(k)$$

□

This example illuminates what happens when there are redundant input signals. Now a more relevant problem including actuator dynamics will be considered. For this example it is not as trivial to eliminate the non-uniqueness.

EXAMPLE 2—Dynamically Redundant Inputs
Let the system dynamics be given by

$$x(k+1) = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ b_1 & b_2 & 0.5 \end{pmatrix} x(k) + \begin{pmatrix} 1-a_1 & 0 \\ 0 & 1-a_2 \\ 0 & 0 \end{pmatrix} u(k) + v(k)$$

where $b_1 b_2 \neq 0$, and define the performance index as

$$\mathrm{E}\left\{x_3^2(k)\right\}$$

By introducing a preliminary feedback

$$\begin{cases} \bar{u}_1(k) = (1-a_1)u_1(k) + a_1 x_1(k) \\ \bar{u}_2(k) = (1-a_2)u_2(k) + a_2 x_2(k) \end{cases}$$

and new states via

$$\begin{cases} \bar{x}_1(k) = b_1x_1(k) + b_2x_2(k) \\ \bar{x}_2(k) = b_1x_1(k) - b_2x_2(k) \\ \bar{x}_3(k) = x_3(k) \end{cases}$$

the dynamics can be written

$$\bar{x}(k+1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0.5 \end{pmatrix} \bar{x}(k) + \begin{pmatrix} b_1 & b_2 \\ b_1 & -b_2 \\ 0 & 0 \end{pmatrix} \bar{u}(k) + v(k)$$

By defining the control-signal transformation

$$\begin{cases} \tilde{u}_1(k) = b_1\bar{u}_1(k) + b_2\bar{u}_2(k) \\ \tilde{u}_2(k) = b_1\bar{u}_1(k) - b_2\bar{u}_2(k) \end{cases}$$

it is obvious that the control signal $\tilde{u}_2(k)$ as well as the state $\bar{x}_2(k)$ will not influence the state $\bar{x}_3(k)$ the variance of which is to be minimized. Hence for this example there is not only a redundant control signal but also a redundant state. □

The examples given above suggest different types of non-uniqueness that can be encountered when solving LQ-problems. The proof of Lemma 4 and the appendix will give constructive algorithms for making control-signal and state transformations and preliminary feedbacks in order to obtain a reduced problem for which there exists a unique solution. Notice that the inputs and states which are removed in this procedure represent the full non-uniqueness of the original problem.

4. Solution

It will now be discussed how to obtain the solution of the optimization problem (2) under the weakest possible conditions.

LEMMA 1
The set $\mathcal{S}_s$ has at most one element.

Proof: Let S_1 and S_2 be two elements of $\mathcal{S}_s$. Then $A_1 = A - BL_1$ and $A_2 = A - BL_2$ are both stable. Further it follows from (3) that

$$A_2^T(S_1 - S_2)A_1 = S_1 - S_2$$

This implies that

$$S_1 - S_2 = \left(A_2^T\right)^k (S_1 - S_2)A_1^k \to 0, \quad k \to \infty$$

Hence $S_1 = S_2$. □

LEMMA 2
Let $S \in \mathcal{S}_s$. Then the optimal feedback is given by any $L \in \mathcal{L}_s$ that solves

$$GL = D^TC + B^TSA$$

and the corresponding optimal loss is given by

$$J = \mathrm{tr} SR_1$$

Proof: For any $S \in \mathcal{S}$ it holds by (3) that the loss function may be written

$$J = \mathrm{E}\left\{x^T(k)Sx(k) - x^T(k+1)Sx(k+1)\right\} \\ + \mathrm{E}\left\{[u(k) + Lx(k)]^T G[u(k) + Lx(k)] + v^T(k)Sv(k)\right\}$$

For any stabilizing control u the expectation is evaluated in stationarity and the first two terms cancel. By taking $u(k) = -Lx(k)$, which is stabilizing, it follows that $\mathrm{tr}SR_1$ is the minimal value of J. □

Now it will be shown when $\mathcal{S}_s$ is nonempty. The proof will be constructive, and thus one explicit way of obtaining the desired solution of the algebraic Riccati equation will be given. First an assumption will be made implying a unique solution. Remember the different equivalent formulations of this assumption given in Section 2. In the next lemma, the assumption will be relaxed.

LEMMA 3
Assume that (A, B) is stablizable, and that

$$\max_z \operatorname{rank} P(z) = n + m \tag{4}$$

Then $\mathcal{S}_s$ is nonempty if and only if

$$\operatorname{rank}_{|z|=1} P(z) = n + m$$

Furthermore the corresponding feedback matrix L is unique.

Proof: Define the following Kleinman-like recursion, e.g. [Kucera, 1991],

$$\begin{cases} S_i = (A - BL_i)^T S_i (A - BL_i) + (C - DL_i)^T (C - DL_i) \\ G_i = D^T D + B^T S_i B \\ G_i L_{i+1} = D^T C + B^T S_i A \end{cases} \tag{5}$$

for $i = 0, 1, \ldots$ with arbitrary initial value $L_0 \in \mathcal{L}_s$. Note that $\mathcal{L}_s$ is nonempty by the stabilizability of (A, B). It will first be shown that the sequence of L_i is well defined, and then the question about convergence will be investigated. Assume that $L_i \in \mathcal{L}_s$. Then there exists a unique $S_i \geq 0$ that solves the first equation in (5). This follows from the fact this equation is a Lyapunov-equation. Now there exists an L_{i+1} that solves the third equation in (5). This follows from the fact that

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0$$

If it can be concluded that $L_{i+1} \in \mathcal{L}_s$, it follows by induction that $L_i \in \mathcal{L}_s$ and $S_i \geq 0$ for all $i \geq 0$. Assume that $L_{i+1} \notin \mathcal{L}_s$. Then there exist λ and x such that $|\lambda| \geq 1$ and

$$(A - BL_{i+1})x = x\lambda \tag{6}$$

Now rewrite (5) to obtain

$$S_i = (A - BL_{i+1})^T S_i (A - BL_{i+1}) + (C - DL_{i+1})^T (C - DL_{i+1}) + \Delta_i \tag{7}$$

where

$$\Delta_i = (L_i - L_{i+1})^T G_i (L_i - L_{i+1})$$

Combining (6) and (7) gives

$$(1 - |\lambda|^2) x^* S_i x = x^* (C - DL_{i+1})^T (C - DL_{i+1}) x + x^* (L_i - L_{i+1})^T G_i (L_i - L_{i+1}) x$$

Since $|\lambda| \geq 1$ and $S_i \geq 0$ it follows that

$$x^* (L_i - L_{i+1})^T G_i (L_i - L_{i+1}) x = 0$$

If it can be shown that $G_i > 0$ it follows that $L_i x = L_{i+1}x$, and hence that λ is also an eigenvalue of $A - BL_i$, which is a contradiction. Thus it follows that $L_{i+1} \in \mathcal{L}_s$ provided $G_i > 0$. That this actually holds will now be shown. Rewrite (5) and (7) as

$$\begin{pmatrix} I & 0 \\ L_{i+1} & I \end{pmatrix}^T \begin{pmatrix} S_i - \Delta_i & 0 \\ 0 & G_i \end{pmatrix} \begin{pmatrix} I & 0 \\ L_{i+1} & I \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S_i & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{8}$$

Let $\Psi(z) = (zI - A)^{-1}B$, and let

$$H(z) = \begin{pmatrix} C & D \end{pmatrix} \begin{pmatrix} \Psi(z) \\ I \end{pmatrix}$$

Notice that

$$\begin{pmatrix} A & B \end{pmatrix} \begin{pmatrix} \Psi(z) \\ I \end{pmatrix} = z\Psi(z)$$

Thus by multiplying (8) by $\begin{pmatrix} \Psi(z) \\ I \end{pmatrix}$ from the right and its adjoint from the left the following equality is obtained

$$H^*(z)H(z) + \Psi^*(z)\Delta_i\Psi(z) = [I + L_{i+1}\Psi(z)]^* G_i [I + L_{i+1}\Psi(z)] \tag{9}$$

Now the rank condition (4) implies that there exists z such that rank $H(z) = m$, which by (9) and $\Delta_i \geq 0$ implies that $G_i > 0$. Thus it is proven that the sequence of L_i is well defined and that $L_i \in \mathcal{L}_s$ for all $i \geq 0$.

It will now be shown that the sequence S_i converges to some limit S. Further manipulations show that the following Lyapunov-equation holds

$$S_i - S_{i+1} = (A - BL_{i+1})^T (S_i - S_{i+1})(A - BL_{i+1}) + \Delta_i \tag{10}$$

Since $L_{i+1} \in \mathcal{L}_s$ and since $\Delta_i \geq 0$ it follows that $S_i - S_{i+1} \geq 0$. Thus it holds that $0 \leq S_{i+1} \leq S_i$, which implies that the sequence of S_i converges to some limit $S \geq 0$ as i goes to infinity. From (10) it also follows that $(A - BL_{i+1})^T(S_i - S_{i+1})(A - BL_{i+1}) \to 0$ and $\Delta_i \to 0$, since both matrices are positive semidefinite. The second equation in (5) implies that $G_i \to G = D^TD + B^TSB$. Since

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^T \begin{pmatrix} S & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \geq 0$$

there exists L such that $GL = D^TC + B^TSA$. Thus $S \in \mathcal{S}$, and similarly to (9) it holds that

$$H^*(z)H(z) = [I + L\Psi(z)]^* G [I + L\Psi(z)] \tag{11}$$

Now the rank condition (4) implies that $G > 0$, and hence L is a unique solution. The sequence L_i therefore converges to L. Since the eigenvalues of $A - BL_i$ are inside the unit circle, it follows that in the limit the eigenvalues of $A - BL$ are inside or on the unit circle. Now these closed loop poles are the zeros of $I + L\Psi(z) = I + L(zI - A)^{-1}B$, and from (11) it follows that any closed loop pole on the unit circle would also be a zero of $H(z)$. Converse it also follows that any zero of $H(z)$ on the unit circle would show up as a closed loop pole. Therefore L is stablizing if and only if $H(z)$ has no zeros on the unit circle, or equivalently that

$$\text{rank}_{|z|=1} P(z) = n + m$$

This concludes the proof of Lemma 3. □

Remark. From the proof of the lemma it follows that there always exists $L \in \mathcal{L}_s$ giving a performance index that is arbitrarily close to the optimal value.

Now the full normal rank condition will be relaxed. The approach taken here is to find a reduced problem with lower dimensions, such that the full normal rank condition is fulfilled.

LEMMA 4
Assume that (A, B) is stabilizable. Then it holds that $\mathcal{S}_s$ is nonempty if and only if

$$\text{rank}_{|z|=1} P(z) = \max_z \text{rank}\, P(z)$$

Proof: The proof will be carried out by making input signal-transformations, state-transformations, preliminary feedbacks, removal of signals in the performance index which are identically zero, and removal of stabilized states which are not observable in the performance index. These operations do not influence whether $\mathcal{S}_s$ is nonempty or not.

Consider the following representation

$$\begin{pmatrix} x(k+1) \\ z(k) \end{pmatrix} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} x(k) \\ u(k) \end{pmatrix} \tag{12}$$

Use an input signal-transformation that brings (12) to the form

$$\begin{pmatrix} x(k+1) \\ z(k) \end{pmatrix} = \begin{pmatrix} A & B_1 & 0 \\ C & D_1 & D_2 \end{pmatrix} \begin{pmatrix} x(k) \\ u_1(k) \\ u_2(k) \end{pmatrix}$$

where B_1 has full column rank. Make a further transformation such that

$$\begin{pmatrix} x(k+1) \\ z_1(k) \\ z_2(k) \end{pmatrix} = \begin{pmatrix} A & B_1 & 0 & 0 \\ C_1 & D_{11} & 0 & 0 \\ C_2 & D_{21} & 0 & I \end{pmatrix} \begin{pmatrix} x(k) \\ u_1(k) \\ u_{21}(k) \\ u_{22}(k) \end{pmatrix}$$

It is now possible to use u_{21} arbitrary, while the optimal u_{22} is given by

$$u_{22}(k) = -C_2 x(k) - D_{21} u_1(k)$$

Hence the problem has been reduced to the case

$$\begin{pmatrix} x(k+1) \\ z_1(k) \end{pmatrix} = \begin{pmatrix} A & B_1 \\ C_1 & D_{11} \end{pmatrix} \begin{pmatrix} x(k) \\ u_1(k) \end{pmatrix}$$

where B_1 has full column rank. rank. Make further u- and z-transformations such that

$$\begin{pmatrix} x(k+1) \\ z_{11}(k) \\ z_{12}(k) \end{pmatrix} = \begin{pmatrix} A & B_{11} & B_{12} \\ C_{11} & I & 0 \\ C_{12} & 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u_{11}(k) \\ u_{12}(k) \end{pmatrix}$$

Introduce a preliminary feedback $u_{11}(k) = \bar{u}_{11}(k) - C_{11}x(k)$. This results in

$$\begin{pmatrix} x(k+1) \\ z_{11}(k) \\ z_{12}(k) \end{pmatrix} = \begin{pmatrix} A - B_{11}C_{11} & B_{11} & B_{12} \\ 0 & I & 0 \\ C_{12} & 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ \bar{u}_{11}(k) \\ u_{12}(k) \end{pmatrix}$$

Thus it is possible to assume from the start that the problem is given on the form

$$\begin{pmatrix} x(k+1) \\ z_1(k) \\ z_2(k) \end{pmatrix} = \begin{pmatrix} A & B_1 & B_2 \\ 0 & I & 0 \\ C_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} x(k) \\ u_1(k) \\ u_2(k) \end{pmatrix} \tag{13}$$

where $\begin{pmatrix} B_1 & B_2 \end{pmatrix}$ has full column rank.

By state coordinate change, state feedback, and u_2-transformations similar to [Kailath, 1980, Ch. 7.6] it is possible to split the state space, so that $\begin{pmatrix} x_2^T(k) & x_3^T(k) \end{pmatrix}^T$ represents the maximal unobservable subspace, while $x_3(k)$ represents the maximal controllability subspace. Thus $P(z)$ is strongly equivalent with

$$\begin{pmatrix} zI - A_{11} & 0 & 0 & -B_{11} & -B_{12} & 0 \\ -A_{21} & zI - A_{22} & 0 & -B_{21} & 0 & 0 \\ -A_{31} & -A_{32} & zI - A_{33} & -B_{31} & 0 & -B_{33} \\ 0 & 0 & 0 & I & 0 & 0 \\ C_{21} & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{14}$$

Only x_1 is observable in the loss, and only $\begin{pmatrix} u_1^T & u_2^T \end{pmatrix}^T$ gives any contribution to the controllability of x_1. The dynamics of x_2 and x_3 is not influenced by feedback from x_1 via $\begin{pmatrix} u_1^T & u_2^T \end{pmatrix}^T$, and u_3 can be used to place the eigenvalues corresponding to x_3 without any influence on the loss. Feedback from x_2 via u_1 maybe needed in case of unstable eigenvalues in A_{22}, making the corresponding modes observable, but A_{33} is still not influenced.

Now u_3 represents the arbitrariness of this LQ-problem, and it could therefore be solved as a reduced problem neglecting the x_3 and u_3 portions. There are optimal solutions if and only if there are optimal solutions to the reduced problem.

The loss of normal column rank in $\begin{pmatrix} zI - A_{33} & B_{33} \end{pmatrix}$ is equal to the dimension m_3 of u_3, and the same loss is found in the big pencil (14) and thus in $P(z)$.

The fourth block column in (14) has full rank and is linearly independent of the other columns. The first and fifth block columns have full normal rank by the construction. Otherwise the (A, B_2)-invariant subspace in $\mathrm{Ker}(C_2)$ could have been expanded. The second block column has full normal rank, and by the independence it thus follows that the normal rank of $P(z)$ is equal to $n + m - m_3$. The rank criterion for the reduced problem, characterizing no zeros on the unit circle, is thus applicable for the full problem, so

$$\mathrm{rank}_{|z|=1} P(z) = n + m - m_3$$

is hence a necessary and sufficient conditions for optimal LQ-controllers. □

Remark 1. Notice that the solution S to the non-reduced problem can easily be obtained by solving the Lyapunov-equation

$$S = (A - BL)^T S (A - BL) + (C - DL)^T (C - DL)$$

where L is one solution to the non-reduced problem. Then all solutions to the non-reduced problem can be obtained by solving

$$GL = D^T C + B^T SA$$

Remark 2. Similar approaches of removing states and input-signals have been taken in [Silverman, 1976; Clements and Anderson, 1978]. In [Clements and Anderson, 1978] all states and inputs corresponding to j-constant directions, i.e. states which corresponds to dead-beat control in optimality, are removed. With that approach more states are removed than with the approach taken here, and hence the removed states do not correspond to the non-uniqueness. However, by Remark 1 it is possible to generate all optimal solutions to the non-reduced problem also with the approach taken in [Clements and Anderson, 1978]. In the appendix a similar algorithm is presented. It is believed that this algorithm is easier to implement, although it may give less insight than the approach taken above.

The results obtained are now summarized

THEOREM 1
Assume that (A, B) is stabilizable. Then it holds that $\mathcal{S}_s$ contains exactly one element S if and only if

$$\text{rank}_{|z|=1} P(z) = \max_z \text{rank}\, P(z)$$

The optimal control is given by

$$u(k) = -Lx(k)$$

where L is any solution of

$$GL = D^T C + B^T SA$$

in $\mathcal{L}_s$. The corresponding loss is given by

$$J = \text{tr} S R_1$$

Under the assumption that $\mathcal{S}_s$ is nonempty, the feedback matrix L is unique if and only if

$$\max_z \text{rank}\, P(z) = n + m$$

Remark. Also when $\mathcal{S}_s$ is empty there exists a solution S to the algebraic Riccati equation, e.g. defined by the Kleinman algorithm, and some $L \in \mathcal{L}_s$ giving a loss arbitrarily close to the infimum $J = \text{tr} S R_1$.

5. Conclusions

In this report stationary discrete time LQ control has been studied. Especially necessary and sufficient conditions for existence and uniqueness have been given. The proofs were constructive, and hence implementable algorithms have been obtained.

Very little is new. However, there seems to be a lot of confusion and no good previous coverage of both the stability and uniqueness investigations. In [Clements and Anderson, 1978] the uniqueness is partly addressed, but more states and input-signals are removed than the ones corresponding to the non-uniqueness. The approach taken in this report is more close to the approach in [Willems *et al.*, 1986], where the continuous time case is investigated. Assuming left invertability the transfer function approach in [Kucera, 1991] gives the necessary and sufficient conditions for stability of the optimal controller. However, there seems to be no previous proof using the state-space approach. Frequently non-necessary conditions of strong detectability are imposed, equivalent to a minimum-phase condition.

The main contribution of the report is the necessary and sufficient condition of Lemma 4 stating when there exists a stabilizing optimal controller. Further it is believed that the simple algorithm in the appendix will remove unnecessary restrictions imposed when using most available software tools for solving LQ-problems.

6. References

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7. Appendix

Here an algorithm related to the one in [Clements and Anderson, 1978] will be presented. Consider

$$P(z) = \begin{pmatrix} zI - A & -B \\ C & D \end{pmatrix}$$

Make an input-signal transformation such that

$$\begin{pmatrix} -B \\ D \end{pmatrix} \sim \begin{pmatrix} -B_1 & 0 \\ D_1 & 0 \end{pmatrix}$$

with $\begin{pmatrix} -B_1 \\ D_1 \end{pmatrix}$ having full column rank equal to m_1. The inputs corresponding to the zero-column can be taken arbitrary. Now consider the reduced problem

$$P_1(z) = \begin{pmatrix} zI - A & -B_1 \\ C & D_1 \end{pmatrix}$$

If rank $P_1(0) = n + m_1$ then the algorithm is terminated, and the problem can be solved using Lemma 3. Otherwise form the null-space basis

$$\begin{pmatrix} X \\ U \end{pmatrix}$$

for $P_1(0)$, i.e.

$$\begin{pmatrix} A \\ C \end{pmatrix} X + \begin{pmatrix} B_1 \\ D_1 \end{pmatrix} U = 0 \tag{15}$$

It follows that X has full rank, i.e. $X\alpha = 0$ implies $\alpha = 0$, since otherwise it holds by (15) that

$$\begin{pmatrix} -B_1 \\ D_1 \end{pmatrix} U\alpha = 0$$

which is a contradiction either to $\begin{pmatrix} -B_1 \\ D_1 \end{pmatrix}$ having full rank or to $\begin{pmatrix} X \\ U \end{pmatrix}$ having full rank. From X having full rank it follows that there exists L such that $-LX = U$ giving

$$\left\{ \begin{pmatrix} A \\ C \end{pmatrix} - \begin{pmatrix} B_1 \\ D_1 \end{pmatrix} L \right\} X = \begin{pmatrix} A \\ C \end{pmatrix} X + \begin{pmatrix} B_1 \\ D_1 \end{pmatrix} U = 0 \tag{16}$$

where the last equality follows from (15). Introduce the feedback $u = -Lx + \bar{u}$. This corresponds to defining

$$\bar{P}_1(z) = P_1(z) \begin{pmatrix} I & 0 \\ -L & I \end{pmatrix} = \begin{pmatrix} zI - A + B_1 L & -B_1 \\ C - D_1 L & D_1 \end{pmatrix}$$

By the PBH rank test and (16) it now follows that the modes corresponding to X are unobservable in $C - D_1 L$. By defining the state-transformation

$$x = \begin{pmatrix} \bar{X} & X \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

it follows that

$$\bar{P}_1(z) \sim \begin{pmatrix} zI - A_1 & 0 & -B_{11} \\ -A_{21} & zI & -B_{12} \\ C_1 & 0 & D_1 \end{pmatrix}$$

Now the x_2-states are stable and unobservable and can be removed. Repeat the above procedure with state feedback now only from x_1 until rank $P_j(0) = n + m_j$. The procedure thus gives one optimal state feedback L, and to obtain S and all L apply Remark 1 of Lemma 4.